Zero-divisor Semigroups of Star Graphs and Two-star Graphs

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Abstract

The zero-divisor graph of a commutative semigroup with zero is a graph whose vertices are the nonzero zero-divisors of the semigroup, with two distinct vertices joined by an edge in case their product in the semigroup is zero. In this paper, we give formulas to calculate the numbers of non-isomorphic zero-divisor semigroups corresponding to star graphs $K_{1,n}$, two- star graphs $T_{m,n}$ and windmill graphs respectively.

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Key Words: Zero-divisor graph; Zero-divisor semigroup; Star graph; Two-star graph; Windmill graph.

1 Introduction

For any commutative semigroup S with zero element 0, there is an undirected zero-divisor graph $\Gamma(S)$ associated to S. The vertex set of $\Gamma(S)$ is the set of all nonzero zero-divisors of S, and for distinct vertices x and y of $\Gamma(S)$, there is an edge connecting x and y if and only if $xy = 0$. Zero-divisor graphs of commutative semigroups have been studied in several literatures, such as [1, 2, 5, 6, 7].

For any semigroup S, let T be the set of all zero-divisors of S. Then T is an ideal of S and in particular, it is also a semigroup with the property that all of its elements are zero-divisors of the semigroup T. We call such semigroups *zero-divisor semigroups*. Obviously we have $\Gamma(S) \cong \Gamma(T)$. For

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a given connected simple graph G if there exists a zero-divisor semigroup S such that $\Gamma(S) \cong G$, then we say that G has a *corresponding semigroup*, and we call S a semigroup determined by the graph G.

A star graph (Fig. 1) is a special complete bipartite graph $K_{1,n}$, where one part is a single vertex. A two-star graph $T_{m,n}$ (Fig. 2) is the union of two star graphs $K_{1,m}$ and $K_{1,n}$ whose centers are connected by a single edge.

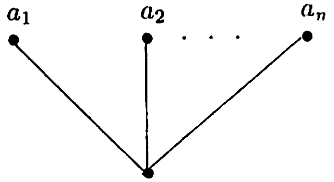

Fig. 1

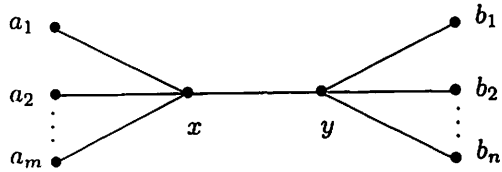

Fig. 2

In 2002 and 2005, DeMeyer et al. [1, 2] proved that complete graphs, star graphs and two-star graphs have corresponding semigroups. In [6], Wu and Cheng gave a formula to calculate the number of non-isomorphic zero-divisor semigroups corresponding to complete graphs K_n. In this paper, we give formulas to calculate the numbers of non-isomorphic zero-divisor semigroups corresponding to star graphs $K_{1,n}$, two-star graphs $T_{m,n}$ and windmill graphs respectively.

In this paper, all semigroups are multiplicative commutative zero-divisor semigroups with zero element 0, where $0x = 0$ for all $x \in S$, and all graphs in this paper are undirected, simple and connected. For other graph theoretical notions and notations adopted in this paper, please refer to [3].

2 Main Results

In this section, let $f(n)$ denote the number of non-isomorphic commutative semigroups with n elements. From [4] we know that $f(2) = 3$, $f(3) = 12$, $f(4) = 58$, $f(5) = 325$. $f(6) = 2143$, $f(7) = 17,291$, $f(8) = 221,805$, $f(9) = 11,545,843$.

Theorem 2.1 *For $n \geq 2$, the number of non-isomorphic zero-divisor semigroups corresponding to the star graph $K_{1,n}$ is*

$$S(n) = n + 2 + 2f(n-1) + 2f(n).$$

Proof. Let $S = \{0, a_1, \ldots, a_n, x\}$ be a semigroup corresponding to the star graph $K_{1,n}$, where x is the center vertex of $K_{1,n}$. We can decompose the set $A = \{a_i \mid 1 \leq i \leq n\}$ into a union of the following three pairwise disjoint subsets:

(1) $A_1 = \{a_i \in A \mid a_i^2 = x\}$;
(2) $A_2 = \{a_i \in A \mid a_i^2 = 0\}$;
(3) $A_3 = \{a_i \in A \mid a_i^2 \in A\}$.

We have four cases for our discussion.

Case 1. Assume $|A_1| = k \geq 1$. Without loss of generality, we let $a_1^2 = \cdots = a_k^2 = x$. In this case, $x^2 = a_1^2 x = a_1(a_1 x) = 0$. For any $i \geq 2$, if $a_1 a_i = a_s$ for some s, then $a_1 a_s = a_1^2 a_i = x a_i = 0$, a contradiction. So $a_1 a_i = x$ for all $i \geq 2$ and then $a_1 a_i^2 = (a_1 a_i) a_i = x a_i = 0$. Hence, $a_i^2 = 0$ for all $i \geq k+1$. For any $i, j \geq 2$ with $i \neq j$, if $a_i a_j = a_s$ for some s, then $a_1 a_s = a_1 a_i a_j = x a_j = 0$, a contradiction. So, $a_i a_j = x$ for any $i, j \geq 2$ with $i \neq j$. Thus, for each $1 \leq k \leq n$, we have a unique multiplicative table. Now we need to verify the associative law, namely, for any $u, v, w \in S - \{0\}$,

$$(uv)w = u(vw) \tag{$*$}$$

(1) If $u = v = w = x$, then the equality $(*)$ obviously holds.

(2) If exactly two of u, v, w are x, by commutativity, we can suppose $u = v = x$, then the left-hand side of $(*)$ is $(x^2)w = 0$, and the right-hand side of $(*)$ is $x(xw) = x0 = 0$. Thus, in this case the equality $(*)$ holds.

(3) If exactly one of u, v, w is x, by commutativity, we can suppose $u = x$, then $(xv)w = 0$ and $x(vw) = 0$. So the equality $(*)$ holds.

(4) If x does not occur in $\{u, v, w\}$, then the equality $(*)$ obviously holds.

Therefore, in this case, there are n mutually non-isomorphic semigroups corresponding to the graph $K_{1,n}$.

Case 2. Assume $|A_1| = 0$, $|A_2| = k \geq 2$. Without loss of generality, we let $a_1^2 = a_2^2 = \cdots = a_k^2 = 0$. First we conclude that $a_1 a_2 = x$. In fact, if $a_1 a_2 = a_1$, then $a_1 a_2 = a_1 a_2^2 = 0$, a contradiction. If $a_1 a_2 = a_i$ for some $i \geq 2$, then $a_1 a_i = a_1^2 a_2 = 0$, a contradiction. Thus, $a_1 a_2 = x$. Similarly, it is easy to check that $a_1 a_i = x$ and $a_2 a_i = x$ for all $3 \leq i \leq n$. Clearly, $x^2 = (a_1 a_2)x = a_1(a_2 x) = 0$ and $a_1 a_i^2 = (a_1 a_i) a_i = x a_i = 0$ for all $1 \leq i \leq n$. Hence, $a_1 a_j^2 = 0$ for all $j \geq k+1$, and then $a_j^2 = a_1$ for all $j \geq k+1$. If $k < n$, then $a_1 a_2 = a_j^2 a_1 = a_j(a_j a_1) = a_j x = 0$, a contradiction. Thus, $k = n$. For any $i, j \geq 2$ with $i \neq j$, if $a_i a_j = a_s$ for some $s \geq 2$, then $a_1 a_s = a_1 a_i a_j = x a_j = 0$, a contradiction. If $a_i a_j = a_1$, then $a_1 a_2 = a_2 a_i a_j = x a_j = 0$, a contradiction. So $a_i a_j = x$ for any $i \neq j$. Thus, in this case, we have a unique multiplicative table. It is easy to verify that the multiplicative table satisfies the associative law. Therefore, in this case, there is only one zero-divisor semigroup corresponding to the graph $K_{1,n}$.

Case 3. Assume $|A_1| = 0$ and $|A_2| = 1$. Without loss of generality, we suppose $a_1^2 = 0$. It is easy to check that $a_1 a_i = x$ or a_1 for all $i \geq 2$.

Subcase 3.1 Assume that there exists some $i \geq 2$ such that $a_1 a_i = x$.

Without loss of generality, we suppose $a_1a_2 = x$. In this case, we have $x^2 = 0$. For all $i \geq 3$, if $a_1a_i = a_1$, then $a_1a_2 = a_1a_2a_i = xa_i = 0$, a contradiction. So $a_1a_i = x$ for all $i \geq 2$. Hence, $a_1a_i^2 = 0$ for all $i \geq 2$, which implies that $a_i^2 = a_1$ for all $i \geq 2$. For any $i, j \geq 2$ with $i \neq j$, if $a_ia_j = x$, then $a_i^2a_j = 0$, i.e., $a_1a_j = 0$, a contradiction. If $a_ia_j = a_s$ for some $s \geq 2$, then $a_1a_s = a_1a_ia_j = xa_j = 0$, a contradiction. So $a_ia_j = a_1$. Thus, in this case, we have a unique multiplicative table. By direct verification, the multiplicative is associative. Hence, in this subcase, there is a unique corresponding semigroup to the graph $K_{1,n}$.

Subcase 3.2 Assume that $a_1a_i = a_1$ for all $i \geq 2$. In this case, $x^2 = 0$ or $x^2 = x$, and $\{a_2, \ldots, a_n\}$ is a sub-semigroup of S. It is need to verify the associative law (*).

(1) If $u, v, w \in \{a_2, \ldots, a_n\}$, then the equality (*) holds since $\{a_2, \ldots, a_n\}$ is a semigroup.

(2) If $u, v, w \notin \{a_2, \ldots, a_n\}$, it is easy to see that the equality (*) holds.

(3) If exactly two of u, v, w belong to $\{a_2, \ldots, a_n\}$, by commutativity, we can suppose $u, v \in \{a_2, \ldots, a_n\}$ and $w \notin \{a_2, \ldots, a_n\}$. If $w = a_1$, then $(uv)w = a_sa_1 = a_1$, $s \neq 1$, and $u(vw) = u(va_1) = ua_1 = a_1$. If $w = x$, then $(uv)w = u(vw) = 0$. So the equality (*) holds.

(4) If exactly one of u, v, w belongs to $\{a_2, \ldots, a_n\}$, by commutativity, we can suppose $u \in \{a_2, \ldots, a_n\}$ and $v, w \notin \{a_2, \ldots, a_n\}$, then $(uv)w = 0 = u(vw)$.

Therefore, in this subcase, there are $2f(n-1)$ non-isomorphic zero-divisor semigroups corresponding to the graph $K_{1,n}$.

Case 4. Assume $|A_1| = 0$ and $|A_2| = 0$. In this case, $x^2 = 0$ or $x^2 = x$, and A is a sub-semigroup of S. Similar to Subcase 3.2, we can show that, in this case, there are $2f(n)$ zero-divisor semigroups corresponding to the graph $K_{1,n}$.

By the above discussion, we have proved that there are $n + 2 + 2f(n-1) + 2f(n)$ non-isomorphic zero-divisor semigroups corresponding to the graph $K_{1,n}$, i.e. $S(n) = n + 2 + 2f(n-1) + 2f(n)$. □

Recall that a semigroup S with zero is nilpotent if $S^k = 0$ for some $k > 0$. From the above proof, we can see that the nilpotent semigroups corresponding to the star graph $K_{1,n}$ are exactly the semigroups determined in Case 1, Case 2 and Subcase 3.1. Thus, we have the following:

Corollary 2.2 ([9], Theorem 3.6) *Let $n \geq 2$. Then the number of non-isomorphic nilpotent semigroups corresponding to the star graph $K_{1,n}$ is $N(n) = n + 2$.*

Remark: If $n = 2$, there are $2+2+2f(1)+2f(2) = 12$ non-isomorphic zero-divisor semigroups corresponding to the graph $K_{1,2}$. This coincides with the result in [5, Theorem 2.1]. If $n = 3$, there are $3+2+2f(2)+2f(3) = 35$

non-isomorphic zero-divisor semigroups corresponding to the graph $K_{1,3}$, which coincides with the result in [5, Theorem 2.6]. By taking advantage of the known values of $f(n)$ for $1 \leq n \leq 9$ given in [4], we can list all values of $S(n)$ and $N(n)$ for $2 \leq n \leq 9$ as follows:

Table 1

graphs	$K_{1,2}$	$K_{1,3}$	$K_{1,4}$	$K_{1,5}$	$K_{1,6}$	$K_{1,7}$	$K_{1,8}$	$K_{1,9}$
$S(n)$	12	35	146	773	4944	38877	478202	23535307
$N(n)$	4	5	6	7	8	9	10	11

Theorem 2.3 *Let m and n be positive integers. Let $T(m,n)$ denote the number of non-isomorphic zero-divisor semigroups corresponding to the two-star graph $T_{m,n}$. Then*
(1) $T(m,n) = f(m) + f(n)$ if $m \neq n$;
(2) $T(m,n) = f(n)$ if $m = n$.

Proof. (1) Assume $m \neq n$. Let $S = \{0, a_1, \ldots, a_m, b_1, \ldots, b_n, x, y\}$ be a semigroup corresponding to the two-star graph $T_{m,n}$, where x and y are the centers of the two star graphs $\{0, a_1, \ldots, a_m, x\}$ and $\{0, b_1, \ldots, b_n, y\}$, respectively.

From [1, Theorem 4], , we know that $\{0, x, y\}$ is an ideal of S, and hence $x^2, y^2 \in \{0, x, y\}$ and $a_i y \in \{x, y\}$, $b_i x \in \{x, y\}$ for any i. If $a_i y = x$, then $b_1 x = b_1 a_i y = 0$, a contradiction. So $a_i y = y$ for all i. Similarly, $b_i x = x$ for all i. If $x^2 = y$, then $b_1 x^2 = b_1 y = 0$. On the other hand, $b_1 x^2 = (b_1 x)x = x^2 = y$, a contradiction. So $x^2 = x$ or $x^2 = 0$. Similarly, $y^2 = y$ or $y^2 = 0$.

Next, we consider the values of $a_i b_j$ for any i, j. If $a_i b_j = a_k$ for some k, then $a_k y = 0$, a contradiction. If $a_i b_j = b_k$ for some k, then $b_k x = 0$, a contradiction. Hence, $a_i b_j \in \{x, y\}$.

We can conclude that $a_i a_j \in \{a_1, \ldots, a_m, y\}$. In fact, if $a_i a_j = b_k$, for some k, then $a_i y = a_i(a_j y) = (a_i a_j)y = b_k y = 0$, a contradiction. If $a_i a_j = x$, then $a_i y = a_i(a_j y) = a_i a_j)y = xy = 0$, a contradiction. If $a_i^2 = 0$ for some i, then $a_i y = a_i a_i y = 0$, a contradiction. Hence $a_i a_j \in \{a_1, \ldots, a_m, y\}$ for all i, j. Similarly. $b_i b_j \in \{b_1, \ldots, b_n, x\}$ for all i, j.

Now we claim that exactly one of x^2, y^2 is zero. Since $a_i b_j \in \{x, y\}$, it is easy to see that at least one of x^2, y^2 is zero. On the other hand, if $x^2 = y^2 = 0$, let's consider a_1^2. If $a_1^2 = y$, then $a_1^2 y = y^2$, which implies $y^2 = y$, a contradiction. So $a_1^2 \in \{a_1, \ldots, a_m\}$. We can suppose $a_1^2 = a_k$ for some k. Similarly. we can get $b_1^2 = b_l$ for some l. Next, we consider $a_1 b_1$. If $a_1 b_1 = x$, then $a_k b_1 = a_1^2 b_1 = a_1 x = 0$, a contradiction. If $a_1 b_1 = y$, then $a_1 b_l = a_1 b_1^2 = b_1 y = 0$, a contradiction too. Therefore, we only have the following two cases for our discussion.

Case 1. Assume $x^2 = x$ and $y^2 = 0$. In this case, $a_ib_j = y$ for alli, j. For any i, j, if $b_ib_j = b_k$, for some k, then $b_ka_1 = b_ib_ja_1 = b_iy = 0$, a contradiction. So $b_ib_j = x$ for all i, j. Finally, $a_ia_j \in \{a_1, \ldots, a_m\}$ and then $\{a_1, \ldots, a_m\}$ is a sub-semigroup of S. So we obtain the following multiplicative table (Table 2), where $\diamond$ represents an element in $A = \{a_1, \ldots, a_m\}$.

Table 2

$\cdot$	a_i	a_j	b_i	b_j	x	y
a_i	$\diamond$	$\diamond$	y	y	0	y
a_j	$\diamond$	$\diamond$	y	y	0	y
b_i	y	y	x	x	x	0
b_j	y	y	x	x	x	0
x	0	0	x	x	x	0
y	y	y	0	0	0	0

Now we need to verify the associativity (*).

(1) If $u, v, w \in A$, then the equality (*) obviously holds.

(2) If $u, v, w \notin A$, then the equality (*) also obviously holds.

(3) If exactly two of u, v, w belong to A, by commutativity, we can suppose $u, v \in A$ and $w \in \{x, y, b_1, \ldots, b_n\}$. Then $(uv)w = u(vw) = 0$ if $w = x$, and $(uv)w = u(vw) = y$ if $w \in \{y, b_1, \ldots, b_n\}$. So, in this case, the equality (*) holds.

(4) If exactly one of u, v, w belongs to A, by commutativity, we can suppose $u \in A$. In this case, it is easy to verify that the equality (*) holds. Hence, in Case 1, there are $f(m)$ non-isomorphic zero-divisor semigroups corresponding to the two-star graph $T_{m,n}$.

Case 2. Assume $x^2 = 0$ and $y^2 = y$. Similar to Case 1, we have $f(n)$ non-isomorphic zero-divisor semigroups corresponding to the two-star graph $T_{m,n}$.

Therefore, $T(m, n) = f(m) + f(n)$ in case $m \neq n$.

(2) Assume $m = n$. By the symmetry of the graph $T_{n,n}$, we know that the semigroups in Case 1 and the semigroups in Case 2 in the above discussion are isomorphic. Hence, $T(n, n) = f(n)$. □

By taking advantage of the known values of $f(n)$ for $1 \leq n \leq 9$ given in [4], we can list all values of $T(m, n)$ for $m, n \leq 9$:

Table 3

$T_{m,n}$	1	2	3	4	5	6	7	8	9
1	1	4	13	59	326	2144	17292	221806	11545844
2		3	15	61	328	2146	17294	221808	11545846
3			12	70	337	2155	17303	221817	11545855
4				58	383	2201	17349	221863	11545901
5					325	2468	17616	222130	11546168
6						2143	19434	223948	11547986
7							17291	239096	11563134
8								221805	11767648
9									11545843

From the proof of the above theorem, we can get the following:

Corollary 2.4 *There exist no nilpotent semigroups corresponding to the two-star graph $T_{m,n}$ for any $m, n \geq 1$.*

Remark: This result was contained in Corollary 2.13 of [8]. In fact, by Corollary 2.13 and 3.7(3) of the mentioned paper, no finite or infinite two-star graph has either nilpotent semigroups or semilattices.

We use W_n to denote the windmill graph with n triangles (Fig. 4), and use $F_{n,m}$ to denote the graph of Fig. 3, a special refinement of the star graph $K_{1,2n+m}$. In [9], Wu, Liu and Chen obtained the counting formulas of the numbers of non-isomorphic zero-divisor semigroups of the graph W_n and $F_{n,m}$. Here, we get these two formulas by another different method. This method is more direct and simple.

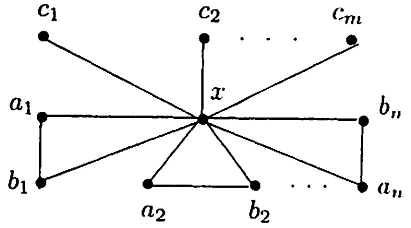

Fig. 3

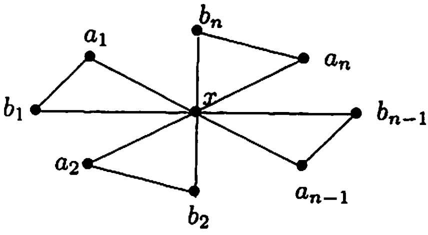

Fig. 4

Theorem 2.5 **([9], Theorem 3.2)** *Let $n \geq 2$, $m \geq 0$ and let $F_{n,m}$ denote the graph of Fig. 3, a special refinement of the star graph $K_{1,2n+m}$. Then there are $\frac{1}{2}(m+1)(n+1)(n+2)$ non-isomorphic zero-divisor semigroups corresponding to the graph $F_{n,m}$ and all of these zero-divisor semigroups are nilpotent.*

Proof. First, let us consider a_1a_2. If $a_1a_2 = a_1$, then $a_1b_2 = a_1a_2b_2 = 0$, a contradiction. If $a_1a_2 = a_k$ for some $k \geq 2$, then $a_kb_1 = a_1a_2b_1 = 0$, a contradiction. If $a_1a_2 = b_k$ for some k, then $b_1b_k = a_1a_2b_1 = 0$, a contradiction. If $a_1a_2 = c_k$ for some k, then $b_1c_k = a_1a_2b_1 = 0$, a contradiction. So $a_1a_2 = x$. Similarly, $a_ia_j = x$, $a_ib_j = x$, $b_ib_j = x$ for all $i \neq j$, and $a_ic_j = x$, $b_ic_j = x$ for any i, j. This implies $x^2 = 0$. On the other hand, it is easy to verify that $a_i^2, b_i^2, c_i^2 \in \{0, x\}$.

Let $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$, $C = \{c_1, \ldots, c_m\}$ and $A_1 = \{a_i \,|\, a_i^2 = 0\}$, $B_1 = \{b_i \,|\, b_i^2 = 0\}$, $C_1 = \{c_i \,|\, c_i^2 = 0\}$. We suppose $|A_1| = k$, $|B_1| = l$ and $|C_1| = r$. By symmetry of the a_i and b_i, it is not difficult to see that there are $1+2+\cdots+(n+1) = \frac{1}{2}(n+1)(n+2)$ different cases on A_1 and B_1. Clearly, C_1 has $m+1$ different cases. So there are $\frac{1}{2}(m+1)(n+1)(n+2)$ distinct multiplicative tables on S. It is easy to check that these multiplicative tables are all associative.
Since $a_i^2, b_i^2, c_i^2 \in \{0, x\}$ and $x^2 = 0$, it is easy to verify that all of these zero-divisor semigroups are nilpotent. This completes our proof. □

When $m = 0$, the graph Fig. 3 becomes a windmill graph Fig. 4. In this special case, we have:

Corollary 2.6 **([9], Lemma 3.1)** *Let $n \geq 2$. Then there are $\frac{1}{2}(n+1)(n+2)$ non-isomorphic zero-semigroups corresponding to the windmill graph W_n.*

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