Bounds on the size of super edge-magic graphs depending on the girth

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Abstract

Let $G = (V, E)$ be a graph of order p and size q. It is known that if G is super edge-magic graph then $q \leq 2p - 3$. Furthermore, if G is super edge-magic and $q = 2p - 3$, then the girth of G is 3. It is also known that if the girth of G is at least 4 and G is super edge-magic then $q \leq 2p - 5$. In this paper we show that there are infinitely many graphs which are super edge-magic, have girth 5, and $q = 2p - 5$. Hence we conclude that for super edge-magic graphs of girths 4 and 5, the size is upper bounded by two times the order of the graph minus 5, and this bound is tight.

Keywords: Super edge-magic graph, girth.

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1 Introduction.

For the undefined concepts and notation used in this paper, the reader is referred to [5].

We will use the notation $G = (V, E)$ in order to denote a graph with vertex set V and edge set E. The order and size of G will be denoted by p and q respectively. Also the girth of G will be denoted with the notation $g(G)$. By the notation $[a, b]$ where $a, b \in \mathbb{Z}$; $a < b$ we mean the set $\{a, a+1, a+2, \ldots, b\}$. Furthermore, the symbol $+_n$ means the sum in $\mathbb{Z}_n$.

In 1998, Enomoto, Lladó, Nakamigawa and Ringel [2] defined a graph $G = (V, E)$ of order p and size q to be super edge-magic if there exists a bijective function $f : V \cup E \longrightarrow \{1, 2, \ldots, p+q\}$ such that

1. $f(V) = \{1, 2, \ldots, p\}$.
2. $f(u) + f(uv) + f(v) = k \quad \forall uv \in E$.

The function f is called a super edge-magic labeling of G.

The following Lemma found in [3], provides us with an alternative definition of super edge-magic graphs which is sometimes very useful.

Lemma 1.1 *A graph $G = (V, E)$ of order p and size q is super edge-magic if and only if there exists a bijective function*

$$\bar{f} : V \longrightarrow \{1, 2, \ldots, p\}$$

such that the set

$$S = \{\bar{f}(u) + \bar{f}(v) \ : \ uv \in E\}$$

consists of q consecutive integers.

Note that Lemma (1.1) allows us to describe super edge-magic labelings only by means of the vertex labels and this is what will be done in the rest of the paper.

It is worthwhile mentioning that in 1991 Acharya and Hegde,[1], defined the concept of strongly indexable graph that turns out to be equivalent to the concept of super edge-magic graph.

In [2], Enomoto et al. established the following upper bound for the size of super edge-magic graphs.

Theorem 1.1 *If $G = (V, E)$ is a super edge-magic graph of order p and size q then*

$$q \leq 2p - 3.$$

In [4] Figueroa-Centeno et al. improved the result as follows.

Theorem 1.2 *Let $G = (V, E)$ be a super edge-magic graph of order p and size q, where $p \geq 4$ and $q \geq 2p - 4$. Then G contains triangles.*

Thus, in light of Theorems (1.1) and (1.2), we know that the girth of any super edge-magic graph of order $p \geq 4$ and size $q \geq 2p - 4$ is necessarily 3. Therefore we get the following corollary.

Corollary 1.1 *Let $G = (V, E)$ be a super edge-magic graph of order $p \geq 4$ and size q such that $g(G) \geq 4$. Then*

$$q \leq 2p - 5.$$

The bound established in Corollary (1.1) is tight since it is not hard to find bipartite graphs of order $p \geq 8$ and size $q = 2p - 5$ which are super edge-magic. Also it is easy to find graphs with girth 3 that attain the bound established in Theorem (1.1).

In this paper we prove that at least for graphs of girth 5, the bound obtained in Corollary (1.1) cannot be improved. We show this by establishing an infinite family of super edge-magic graphs with girth 5, for which their size is exactly equal to two times the order minus 5.

2 The family and the labeling.

Consider the following family: $\mathcal{P} = \{P_n : n \in \mathbb{N} \setminus \{1\}\}$ of graphs where each graph P_n has order $5n$ and size $10n - 5$. Next we describe the graphs of this family. The vertex set of P_n is the set $V(P_n) = [0, 5n - 1]$. The graph P_n consist of n cycles, each of them called level L_k for every $k \in [1, n]$. The vertices of the level L_k are $V(L_k) = [5k - 5, 5k - 1]$. Each vertex of L_k is joined with exactly one vertex of L_{k-1} and with exactly one vertex of L_{k+1} for every $k \in [2, n - 1]$. Therefore, the vertices of $L_2, \ldots, L_{n-1}$ are all of degree 4 and the vertices of L_1 and L_n have degree 3. At this point, let us define the adjacencies.

Let $a, b \in V(F_k)$; $k \in [1, n]$. We denote by $\bar{a}$ and $\bar{b}$ the remainders of a and b modulo 5. Then $ab \in E(P_n)$ if and only if either $\bar{a} = \bar{b} +_5 2$ or $\bar{b} = \bar{a} +_5 2$.

Next, $ab \in E(P_n)$ if and only if $\bar{b} = \pi(\bar{a})$ when k is odd or $\bar{b} = \pi^{-1}(\bar{a})$ when k is even, where π is the following permutation of elements the of $\mathbb{Z}_5$ written in cycle notation: $(0,4,1,2)(3)$.

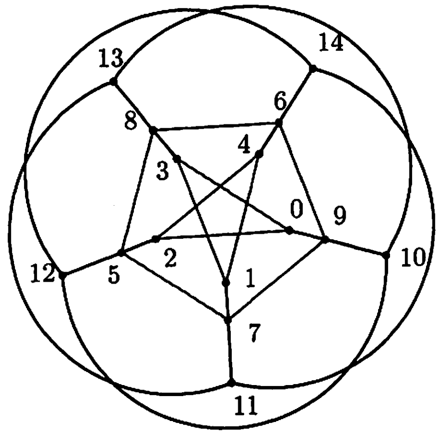

Figure 1: The graph P_3

Let $P_n \in \mathcal{P}$. We observe that the subgraphs of P_n induced by two consecutive levels are all isomorphic to the Petersen graph. Hence any cycle of order strictly smaller than 5, must contain vertices of at least three distinct level of P_n . It is easy to see that with vertices of at least three levels we can only construct cycles of order at least 6. Therefore we have that $g(P_n) = 5\ \forall P_n \in \mathcal{P}$. Next we introduce the following result regarding the super edge-magicness of the graphs in $\mathcal{P}$.

Theorem 2.1 *The graph $P_n \in \mathcal{P}$ is super edge-magic for all $n \in \mathbb{N} \setminus \{1\}$.*

Proof.
Let $f \longrightarrow [0, 5n-1]$ be the function defined by the rule $f(i) = i\ \forall i \in V(P_n)$. Then

$$\{f(a) + f(b) : \ ab \in E(F_k) : k \in [1,n]\} = [10k - 8, 10k - 4]$$

and if $k \in [1, n-1]$ we have that

$$\{f(a) + f(b) : \ ab \in E(P_n) : \ a \in V(F_k),\ b \in V(F_{k+1})\} = [10k - 3, 10k + 1].$$

Thus

$$\{f(a) + f(b) : \ ab \in E(P_n)\} = [2, 10n - 4] \text{ and } |[2, 10n - 4]| = |E(P_n)|.$$

Therefore the function $g : V(P_n) \longrightarrow [1, 5n]$ defined by the rule $g(i) = f(i) + 1 = i + 1 \ \forall i \in V(P_n)$ is a super edge-magic labeling of P_n. □

3 Conclusions and further research.

In this paper we have shown that if $G = (V, E)$ is a super edge-magic graph of order p, size q and girth 5, then $q \leq 2p - 5$, and that this bound is tight. For further research we propose to find tight upper bounds for the size of super edge-magic graphs of girth $g \geq 6$, or at least to improve the bound established in this paper.

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