On cyclic edge connectivity of regular graphs with two orbits*

Weihua Yang[a†] Huiqiu lin[b] Wei Cai[c] Xiaofeng Guo[d]

[a]Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

[b]Department of Mathematics, School of Science, East China University of Science and Technology, Shanghai 200237, China

[c]The First Aeronautical Institute of Air Force, Xinyang Henan 464000, China

[d]School of Mathematical Science, Xiamen University, Xiamen Fujian 361005, China

Abstract A cyclic edge-cut of a graph G is an edge set, the removal of which separates two cycles. If G has a cyclic edge-cut, then it is said to be cyclically separable. For a cyclically separable graph G, the cyclic edge-connectivity $c\lambda(G)$ is the cardinality of a minimum cyclic edge-cut of G. Let $\zeta(G)=\min\{\omega(X)|X$ induces a shortest cycle in $G\}$, where $\omega(X)$ is the number of edges with one end in X and the other end in $V(G)-X$. A cyclically separable graph G with $c\lambda(G) = \zeta(G)$ is said to be cyclically optimal. In this work, we discuss the cyclic edge connectivity of regular double-orbit graphs. Furthermore, as a corollary, we obtain a sufficient condition for mixed Cayley graphs which was introduced by Chen and Meng [3] to be cyclically optimal.

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†E-mail:ywh222@163.com(W. Yang).

1 Introduction

In a network, traditional connectivity is an important measure since it can correctly reflect the fault tolerance of network systems with few processors. However, it always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures is highly unlikely. Thus the disruption envisaged occurs in a worst case scenario. To overcome the shortcoming, Latifi et al. [4] proposed a kind of conditional edge-connectivity, denoted by $\lambda^k(G)$, which is the minimum size of an edge-cut S such that each vertex has degree at least k in $G - S$.

Let G be a simple graph and F be a set of edges in G. Call F a *cyclic edge-cut* if $G - F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge cut if and only if it has two disjoint cycles. Lováz [6] characterized all multigraphs without two disjoint cycles. We call those graphs which do have cyclic edge cuts *cyclically separable.* Following [9], we define the *cyclic edge- connectivity* of G, denoted by $c\lambda(G)$, as follows: if G is not connected, then $c\lambda(G) = 0$; if G is connected but does not have two disjoint cycles, then $c\lambda(G) = \infty$; otherwise, $c\lambda(G)$ is the minimum cardinality over all cyclic edge-cuts of G.

For any graph G with minimum degree $\delta(G) \geq 3$, it can be seen that $\lambda^2(G) = c\lambda$. In fact, since every subgraph of G with minimum degree at least 2 has a cycle, we see that $\lambda^2(G) \geq c\lambda$. On the other hand, let S be a minimum cyclic edge-cut of G. By the minimality of S, $G - S$ has exactly two connected components. If a component of $G - S$ has a degree-one vertex, then moving it to the other component decreases the number of edges in the minimum cyclic edge-cut, a contradiction. Hence S is also a λ^2-cut, and thus $\lambda^2(G) \leq c\lambda$.

A graph G is said to be *vertex transitive* if $Aut(G)$ acts transitively on $V(G)$. A bipartite graph G with bipartition $X_1 \cup X_2$ is called *half vertex transitive* [14] if $Aut(G)$ acts transitively both on X_1 and X_2. Let $x \in V(G)$, we call the set $\{x^g : g \in Aut(G)\}$ an *orbit* of $Aut(G)$. Clearly, $Aut(G)$ acts transitively on each orbit of $Aut(G)$. Clearly, the half vertex transitive graph, except for the vertex transitive bipartite graphs, has two orbits.

For two vertex sets $X, Y \subset V(G)$, $[X, Y]$ is the set of edges with one end in X and the other end in Y. $G[X]$ is the subgraph of G induced by vertex set X, $\overline{X}$ is the complement of X, $\omega(X) = |[X, \overline{X}]|$ is the number of edges between X and $\overline{X}$ in G. If $[X, \overline{X}]$ is a minimum cyclic edge-cut, then both $G[X]$ and $G[\overline{X}]$ are connected. Define

$$\zeta(G) = \min\{\omega(X) | X \text{ induces a cycle in } G\}.$$

In [9], Wang and Zhang shown that $\zeta(G) \geq c\lambda$ for any cyclically separable graph. A cyclically separable graph G is called *cyclically optimal* if $c\lambda = \zeta(G)$. That is, for the cyclically separable graph G, if G is not cyclically optimal, then $c\lambda < \zeta(G)$.

Some previous studies in this line include [5, 8, 13]. In [8], Nedela and Skoviera studied the existence of the cyclic edge cut in cubic multigraphs, showing that a connected cubic graph G has no cyclic edge cut if and only if it is isomorphic to one of K_4, $K_{3,3}$ or θ_2 (the multigraph with two vertices and three edges between them). Furthermore, $c\lambda \leq \zeta(G)$ in this case. Xu and Liu [13] shown that a k-regular simple graph G with $k \geq 3$ which is not K_4; K_5, and $K_{3,3}$ is cyclically separable, and $c\lambda \leq \zeta(G)$. Furthermore, they proved that a connected k-regular vertex transitive graph G with $k \geq 4$, $k \neq 5$ and girth $g(G) \geq 5$ is cyclically optimal. Wang and Zhang [9] shown that any vertex transitive graph with regularity degree $k \geq 4$ and girth $g \geq 5$ is cyclically optimal. Tian and Meng [11] reported that any half vertex transitive regular graph with $g(G) \geq 6$ is cyclically optimal. In this work, we study the cyclic edge-connectivity of regular graph with two orbits V_1 and V_2. We show that any $k(\geq 4)$-regular graph with two orbits and girth $g(G) \geq 5$ is cyclically optimal if $G[V_1]$ and $G[V_2]$ are connected. We refer to [3] for the detail of the double-orbits graph.

2 Preliminaries

A vertex set X is a *cyclic edge-fragment*, if $[X, \overline{X}]$ is a minimum cyclic edge-cut. A cyclic edge-fragment with the minimum cardinality is called a *cyclic edge-atom*. If no confusion, fragment and atom will stand for cyclic edge-fragment and cyclic edge-atom respectively. Clearly, if X is a fragment, then $\overline{X}$ is also a fragment, and both $G[X]$ and $G[\overline{X}]$ are connected. The following observation will be used frequently in the proofs: If X is an atom, and X' is a proper subset of X such that $[X', \overline{X'}]$ is a cyclic edge-cut, then $\omega(X') > \omega(X)$. The concepts of fragment and atom were first proposed by Mader [7] and Watkins [10], and their variations play an important role in studying various kinds of connectivity. An atom is said to be *trivial*, if it induces a cycle of G, otherwise it is *non-trivial*. For a vertex u, $N_G(u)$ denotes the set of neighbors of u in G. Denote by $d_G(u) = |N_G(u)|$ the degree of u in G. If no confusion, we use $d_X(u)$ to denote $d_{G[X]}(u)$ for a subset X of $V(G)$.

Observation 2.1. *Let G be a connected graph with $\delta(G) \geq 3$ and girth*

$g \geq 5$, *then* G *is cyclically separable.*

Proof. Let C be a shortest cycle in G. Since $g(G) \geq 5$, no two vertices in $V(C)$ have a common neighbor in $\overline{V(C)}$, that is, $\delta(G - C) \geq 2$. Clearly, $[V(C), \overline{V(C)}]$ is a cyclic edge cut of G. Hence, G is cyclically separable. □

Let $T_{m,n}$ denote the complete m-partite graph on n vertices in which all part are as equal in size as possible.

Lemma 2.2. *(Turán Theorem [1]). If G is simple and contains no K_{m+1}, then $\varepsilon(G) \leq \varepsilon(T_{m,n})$. Moreover, $\varepsilon(G) = \varepsilon(T_{m,n})$ only if $G \cong T_{m,n}$.*

Lemma 2.3. *[9]. Let G be a connected graph with $\delta(G) \geq 3$ and X be a fragment. Then*

(i) $\delta(G[X]) \geq 2$;

(ii) If $\delta(G[X]) \geq 3$, then $d_X(v) \geq d_{\overline{X}}(v)$ holds for any $v \in X$;

(iii) If $\delta(G) \geq 4$, and X is a non-trivial atom of G, then $\delta(G[X]) \geq 3$. Furthermore, $d_X(v) > d_{\overline{X}}(v)$ holds for any $v \in X$.

Lemma 2.4. *[11]. Let G be a k-regular graph with $k \geq 3$ and girth g, and X, Y be two distinct atoms with $X \cap Y \neq \emptyset$. If G is not cyclically optimal, then $|X \cap Y| \leq g - 1$ and $|X| = |Y| \leq 2(g - 1)$.*

Lemma 2.5. *Let G be a $k(\geq 4)$-regular graph with girth $g \geq 4$. If G is not cyclically optimal, then for any two distinct atoms X and Y of G, $X \cap Y = \emptyset$.*

Proof. By contradiction. Let X, Y be two non-trivial atoms and suppose $X \cap Y \neq \emptyset$. By Lemma 2.4, we have $|X| = |Y| \leq 2(g-1)$ and $|X \cap Y| \leq g-1$. By Lemma 2.3 (iii), we have $\delta(G[X]) \geq 3$ and $\delta(G[Y]) \geq 3$. We prove the lemma by considering two cases.

Case1. $g(G) = 4$. Then $|X| = |Y| \leq 2(g - 1) = 6$ and $|X \cap Y| \leq 3$. It is easy to see that $|X| = |Y| \neq 4$ or 5. In fact, if $|X| = 4$, then $G[X] \cong K_4$ by $\delta(G[X]) \geq 3$. But the girth g=4, a contradiction. If $|X| = |Y| = 5$, $\delta(G[X]) \geq 3$, then $|E(G[X])| \geq \frac{3|X|}{2} = \frac{15}{2}$. Note that $g(G[X]) \geq 4$, thus, $G[X]$ contains bo K_3. By Turán Theorem, we have that $|E(G[X])| \leq |E(T_{3,5})| = 6 < \frac{15}{2}$, a contradiction. If $|X| = |Y| = 6$, by Turán Theorem, $G[X] \cong K_{3,3}$ is the graph with the maximum number of edges since $G[X]$ contains no triangle. By $|X \cap Y| \leq 3$, we can see that $G[X \cap Y]$ contains a vertex u with $d_{G[X \cap Y]}(u) \leq 1$. Hence $k \geq d_X(u) + d_Y(u) - d_{G[X \cap Y]}(u) \geq 5$. We thus have:

$$\omega(X) \geq k|X| - 2|E(G[X])| \geq 6k - 18 \geq 4(k - 2) = g(k - 2) = \zeta(G) > c\lambda.$$

a contradiction.

Case2. $g(G) \geq 5$. By Observation 2.1, $G[X]$ and $G[Y]$ contain two disjoint cycle, which implies $|X| = |Y| \geq 2g > 2(g-1)$, a contradiction. □

3 Cyclically optimal regular double-orbit graphs

An *imprimitive block* of G is a proper nonempty subset A of $V(G)$ such that for any automorphism $\phi \in Aut(G)$, either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$. In [12], Tindell reported the following theorem.

Theorem 3.1. *[12]. Let G be a graph and let Y be the subgraph of G induced by an imprimitive block A of G. If G is vertex-transitive, then so is Y.*

Lemma 3.2. *Let G be a $k(\geq 4)$-regular graph with two orbits V_1 and V_2, and $g(G) \geq 4$. Suppose that X is an atom of G and G is not cyclically optimal.*

(i) If $X \subseteq V_1$ (or V_2), then V_1 (or V_2) is a disjoint union of distinct atoms and $G[X]$ is a t-regular vertex transitive graph, where $3 \leq t$.

(ii) If $X \cap V_1 \neq \emptyset$ and $X \cap V_2 \neq \emptyset$, then $V(G)$ is a disjoint union of distinct atoms and $G[X]$ is a double orbits graph.

Proof. Clearly, X is an imprimitive block of G by Lemma 2.5.

(i). If $X \subseteq V_1$ (or V_2), then V_1 is a disjoint union of distinct atoms by Theorem 3.1. By Lemma 2.3, $G[X]$ is a t-regular vertex transitive graph, where $3 \leq t$.

(ii). Assume $X_1 = X \cap V_1, X_2 = X \cap V_2$. By Lemma 2.5, X_1, X_2 are the imprimitive block of V_1 and V_2, respectively. So X_1 and X_2 are two orbits of X by Theorem 3.1. □

Theorem 3.3. *Let G be a $k(\geq 4)$-regular double-orbit graph with $g(G) \geq 5$ and $G[V_1]$ and $G[V_2]$ are connected, where V_1 and V_2 are two orbits of G. Then G is cyclically optimal.*

Proof. By contradiction. By Observation 2.1, G is cyclically separable. Assume that G is not cyclically optimal, then $c\lambda < \zeta(G)$. Let X be an atom of G. Then $G[X]$ contains two disjoint cycle and a shortest cycle C of $G[X]$ induce a cyclic edge-cut $[C, G[X]-C]$, then $|X-V(C)| \geq |V(C)|$ by Observation 2.1. Clearly, $\omega_{G[X]}(C) \leq |X-V(C)|$ since no two vertices of C have common neighbor in $X-V(C)$. Assume that two orbits of G are V_1 and V_2, and $X_1 = X \cap V_1, X_2 = X \cap V_2$.

If $X \subseteq V_1$ (or V_2) and assume that t is the regular degree of $G[x]$, we have $\omega_G(C) = \omega_{G[X]}(C) + (k-t)|V(C)| \leq |X-V(C)| + (k-t)|V(C)| \leq (k-t)|X-V(C)| + (k-t)|V(C)| = (k-t)|X| = \omega(X)$. That is, $\omega_G(C)$ is cyclic edge cut of G and $\omega_G(C) \leq \omega(X)$, this contradict that X is an atom.

Assume $X_1 = X \cap V_1 \neq \emptyset, X_2 = X \cap V_2 \neq \emptyset$, $V_1(C) = V_1 \cap V(C)$ and $V_2(C) = V_2 \cap V(C)$. By Lemma 3.2, all vertices in X_1(resp. X_2) have the same degree in $G[X]$, say t_1(resp. t_2). Since $G[V_1]$ and $G[V_2]$ are connected, it is easy to see that $t_i < k, i = 1, 2$. Hence, $\omega_G(C) = \omega_{G[X]}(C) + (k-t_1)|V_1(C)| + (k-t_2)|V_2(C)| \leq |X-V(C)| + (k-t_1)|V_1(C)| + (k-t_2)|V_2(C)| = |X_1 - V_1(C)| + |X_2 - V_2(C)| + (k-t_1)|V_1(C)| + (k-t_2)|V_2(C)| \leq (k-t_1)|X_1 - V_1(C)| + (k-t_2)|X_2 - V_2(C)| + (k-t_1)|V_1(C)| + (k-t_2)|V_2(C)| = (k-t_1)|X_1| + (k-t_2)|X_2| = \omega(X)$. We have that $\omega_G(C)$ is cyclic edge-cut of G and $\omega_G(C) \leq \omega(X)$, this contradict that X is an atom. □

Remark 3.4. *By the proof of Theorem 3.3, under the same conditions of Theorem 3.3 we have that if G is not cyclically optimal, at least one of $G[V_1]$ and $G[V_2]$ is disconnected. Furthermore, for the atom X, either the vertex of X_1 or the vertex of X_2 has degree k.*

In [2], Chen and Meng defined the mixed Cayley graph as follows. Let G be a finite group, S_0, S_1, S_2 be the subset of G and $1_G \notin S_i, i = 0, 1$. The mixed Cayley graph $X = MC(G, S_0, S_1, S_2)$ has vertex set $V(X) = G \times \{0, 1\}$ and edge set $E(X) = E_0 \cup E_1 \cup E_2$ where $E_i = \{\{(g, i), (s_i g, i)\} : g \in G, s_i \in S_i\}$ for $i = 0, 1$ and $E_2 = \{\{(g, 0), (s_2 g, 1)\} : g \in G, s_2 \in S_2\}$. Clearly, $MC(G, S_0, S_1, S_2)$ has at most two orbits. Combining the result of Wang and Zhang[9], we have the following corollary.

Corollary 3.5. *$MC(G, S_0, S_1, S_2)$ is cyclically optimal if $g(MC(G, S_0, S_1, S_2)) \geq 5$ and $Cay(G, S_i), i = 0, 1$ are connected.*

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