The partition theorem of connected digraphs and its enumerative applications*

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Abstract

The partition theorem of connected graphs was established in 1985 and it is very useful in graphical enumeration. In this paper, we generalize the partition theorem from connected graphs to weakly connected digraphs. Applying these two partition theorems, we obtain the recursive formulas for enumerations of labeled connected (even) digraphs, labeled rooted connected (even) digraphs whose roots have a given number of blocks, and labeled connected d-cyclic ($d \geq 0$) (directed) graphs, etc. Moreover, a new proof of the counting formula for labeled trees (Cayley formula) is given.

Keyword: Graph, Digraph, Connected, Partition, Recursive formula

1 The theorem on partition of connected digraphs

A (directed) graph G of order p consists of a finite nonempty set V of p vertices together with a specified set X of q unordered (ordered) pairs of distinct vertices; this automatically excludes loops and multiple lines (arcs). A connected graph is a graph in which every two vertices are joined by a path. A weakly connected digraph (connected digraph for short) is a digraph whose underlying graph is connected. For all other definitions, not given here, see e.g. [4].

A (directed) graph of order p labeled with integers 1, 2, ... , p is called a labeled (directed) graph. Labeled enumeration problems are classical and very interesting. Usually, completely explicit closed formulas and recurrences are used frequently in graphical enumeration.

In [5], a theorem on the partition of connected graphs was introduced. Using this theorem, the proofs of some enumerative formulas were simplified, and some new recursive formulas for graphical enumeration were obtained (see [5]).

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In this paper, we generalize the partition theorem from connected graphs to connected digraphs. In other words, a theorem on the partition of connected digraphs is introduced. Applying these two partition theorems, the recursive formulas for enumerations of labeled connected (even) digraphs, and labeled rooted connected (even) digraphs whose roots have a given number of blocks are obtained in Section 2, respectively.

Moreover, enumeration of labeled connected d-cyclic ($d \geq 0$) (directed) graphs is an interesting problem ([3]), and several famous counting formulas related to it such as Cayley formula [1] and Renyi formula [6] were obtained by different mathematicians. In Section 3, some recurrences for the number of labeled connected d-cyclic (directed) graphs are obtained. Note that connected 0, 1 and 2-cyclic (directed) graphs are (oriented) trees, connected unicyclic and bicyclic (directed) graphs, respectively. Then we also study the counting formulas for these special cases. In addition, combining the recursive formula for enumeration of labeled trees with Abel Identity, a new and simple proof of the counting formula for labeled trees (namely, Cayley formula) is given.

Now the theorem on partition of connected digraphs is introduced.

Theorem 1.1 *Let $D = (V, A)$ be a connected digraph and $u, v \in V$. Then there exists a partition $V(D) = \Upsilon \cup \Omega$ and $\Upsilon \cap \Omega = \emptyset$ such that*

(1) $u \in \Upsilon$ and $v \in \Omega$;

(2) the subdigraphs of D induced by Υ (resp. Ω) is connected;

(3) for each arc $\vec{a} \in A$ joining Υ and Ω, $v \in \Omega$ is the tail or head of $\vec{a}$.

In addition, when u, v are two fixed vertices, the partition is unique.

Proof. **(Existence)** To begin with, we show that there exists a partition $V(D) = \Upsilon \cup \Omega$ and $\Upsilon \cap \Omega = \emptyset$ satisfying the conditions (1), (2), (3) above.

(1) For a connected digraph D, choose two vertices at random, denoted by u and v. Let $\tilde{D}$ denote the underlying graph of D. Now divide the vertices set $V(D)$ into two subsets Υ and Ω such that (i) $u \in \Upsilon$, $v \in \Omega$; (ii) For $w \in V(D)$ but $w \neq u$, $w \neq v$, if there exists a path P in $\tilde{D}$ joining w and u such that $v \notin P$, then $w \in \Upsilon$; Otherwise, $w \in \Omega$. Hence $V(D) = \Upsilon \cup \Omega$ and $\Upsilon \cap \Omega = \emptyset$.

(2) Let x, y be any two vertices of the subdigraph induced by Ω (denoted by $D[\Omega]$). Note that every path P in $\tilde{D}$ joining x and u contains v, and every path Q in $\tilde{D}$ joining y and u contains v. Let $P(x, t_1) \subseteq P$ be the path joining x and $t_1 \in P$, and let $Q(y, t_2) \subseteq Q$ be the path joining y and $t_2 \in Q$.

Obviously, $P(x, v) + Q(y, v)$ is a path in $\tilde{D}$ joining x and y. It suffice to show that $z \in \Omega$ for $z \in P(x, v) + Q(y, v)$ and $z \neq x$, $z \neq y$, $z \neq v$. By contradiction, suppose $z \in \Upsilon$. It leads to there exists a path W in $\tilde{D}$ between z and u such that $v \notin W$. However, if $z \in P(x, v)$, then $P(x, z) + W$ is a path in $\tilde{D}$ joining x and u such that $v \notin P(x, z) + W$; if $z \in Q(y, v)$, then $Q(y, z) + W$ is a path in $\tilde{D}$ joining y and u such that $v \notin Q(y, z) + W$, which is a contradiction to $x, y \in \Omega$. Therefore, $D[\Omega]$ is connected.

Let $m, n \in V(D[\Upsilon])$. Then there exists a path M (resp. N) in $\tilde{D}$ joining m (resp. n) and u such that $v \notin M$ (*resp.* N). Clearly, $M + N$ is a path joining m and n in $\tilde{D}$, and all vertices of $M + N$ belong to Υ. Hence $D[\Upsilon]$ is also connected.

(3) Let $\vec{a}$ be an arc joining Υ and Ω. We claim that $v \in \Omega$ is the tail or head of $\vec{a}$. Otherwise, there exists some vertex $b \in \Omega$ and $b \neq v$ such that there is an arc

joining b and some vertex in Υ, then there exists a path in $\widetilde{D}$ between b and u not including v. Hence $b \in \Upsilon$, a contradiction.

(Uniqueness) Suppose there exist two partitions satisfying (1), (2), (3):

$$V(D) = \Upsilon_1 \cup \Omega_1,\ \Upsilon_1 \cap \Omega_1 = \emptyset,\ u \in \Upsilon_1,\ v \in \Omega_1;$$
$$V(D) = \Upsilon_2 \cup \Omega_2,\ \Upsilon_2 \cap \Omega_2 = \emptyset,\ u \in \Upsilon_2,\ v \in \Omega_2.$$

Let $c \in \Upsilon_1$. Then there exists a path in $\widetilde{D}$ joining c and u not including v. If $c \notin \Upsilon_2$, then all paths in $\widetilde{D}$ joining c and u must include v, which is a contradiction. Hence $c \in \Upsilon_2$, and then $\Upsilon_1 \subseteq \Upsilon_2$. Similarly, it can be proved that $\Upsilon_2 \subseteq \Upsilon_1$. Therefore, $\Upsilon_1 = \Upsilon_2$, and the proof of $\Omega_1 = \Omega_2$ is analogous. □

The partition theorem of connected graphs is needed in this paper.

Theorem 1.2 *([5]) Let $G = (V, E)$ be a connected graph and $u, v \in V$. Then there exists a partition $V(G) = \Upsilon \cup \Omega$ and $\Upsilon \cap \Omega = \emptyset$ such that*

(1) $u \in \Upsilon$ and $v \in \Omega$;

(2) $G[\Upsilon]$ and $G[\Omega]$ are both connected;

(3) For each edge $e \in E$ joining Υ and Ω, $v \in \Omega$ is the end of e.

In addition, when the vertices u, v are fixed, the partition is unique.

2 Recurrences for enumerations of labeled connected (even) digraphs and labeled rooted connected (even) digraphs

Let C_p and $\overrightarrow{C}_p$ denote the numbers of labeled connected graphs and digraphs of order p, respectively. It is known that Mallows and Riordan [8] obtained the following recurrence for C_p, but their proof was complicated.

$$C_p = \sum_{k=1}^{p-1}\left[\binom{p-2}{k-1} \cdot (2^k - 1) \cdot C_k \cdot C_{p-k}\right] \ (p \geq 2), \text{ where } C_1 = 1.$$

In [5], by using the partition theorem of connected graphs, the proof of the recurrence given above was simplified. Now taking a consideration on $\overrightarrow{C}_p$, by Theorem 1.1, a recursive formula is obtained as follows.

Theorem 2.1 $\overrightarrow{C}_1 = 1$ *and* $\overrightarrow{C}_p = \sum_{k=1}^{p-1}[\binom{p-2}{k-1} \cdot (4^k - 1) \cdot \overrightarrow{C}_k \cdot \overrightarrow{C}_{p-k}]$ $(p \geq 2)$.

Proof. Clearly, $\overrightarrow{C}_1 = 1$. When $p \geq 2$, we take two vertices, denoted by u and v, from all labeled vertices of a labeled connected digraph D of order p. Moreover, choose $k-1$ $(1 \leq k \leq p-1)$ vertices from other $p-2$ labeled vertices, and together with u form the subset Υ (there are $\binom{p-2}{k-1}$ ways). Besides, the rest of $p-k$ vertices (including v) form the other subset Ω.

Note that there are $\overrightarrow{C}_k$ (resp. $\overrightarrow{C}_{p-k}$) ways for the subset Υ (resp. Ω) to construct different labeled connected digraphs. What's more, by Theorem 1.1, there are $4^k - 1$ possible combinations of arcs joining the k vertices of Υ and v. (It may be two arcs, one arcs (two directions for choosing), or no arcs joining each vertex of

Υ and v. However, for connectivity requirement, the case of no arcs joining all vertices of Υ and v is impossible.)

Multiply these factors to get the number of labeled connected digraphs satisfying Υ contains k vertices, that is, $\binom{p-2}{k-1} \cdot (4^k - 1) \cdot \vec{C}_k \cdot \vec{C}_{p-k}$. By summing these equalities from $k = 1$ to $p - 1$, we arrive again at the number of $\vec{C}_p$. □

From the recurrences above, the values of C_p and $\vec{C}_p$ are listed as follows.

p	1	2	3	4	5	6	...
C_p	1	1	4	38	728	26704	...
$\vec{C}_p$	1	3	54	3834	1027080	1067308488	...

Remark 1 *Let $C_{p,q}$ be the number of labeled connected graphs with p vertices and q edges. Note that we can get a labeled connected digraph by replacing each edge of a labeled connected graph with either an arc (two directions for choosing) or a directed 2-cycle. Hence there is another formula for $\vec{C}_p$.*

$$\vec{C}_p = \sum_{k=p-1}^{\binom{p}{2}} 3^k C_{p,k}, \text{ where } p \geq 2$$

However, computing $C_{p,q}$ is not so easy, although a formula for $C_{p,q}$ is obtained by inclusion-exclusion principle (see [1]): $C_{p,q} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k} \sum \frac{p!}{\prod_{i=1}^{k} p_i!} \binom{m}{q}$, where $m = \sum_{i=1}^{k} \binom{p_i}{2}$, and the second sum is over all compositions (namely, ordered partitions) $p_1 + p_2 + \cdots + p_k = p$ of p with k parts. Because such formula has something to do with all compositions of p, calculating $C_{p,q}$ is complicated. Therefore, the recurrence in Theorem 2.1 is simpler.

For a digraph D, denote the degree of a vertex $v \in V(D)$ by the sum of the indegree and outdegree of v, that is, $d(v) = d^+(v) + d^-(v)$. If every vertex of a graph G (or a digraph D) has even degree, then G (or D) is called even.

Let E_p and $\vec{E}_p$ denote the numbers of connected labeled even graphs and digraphs of order p respectively. For the number E_p (see [5]), we have

$$E_p = \sum_{k=1}^{p-1} [\binom{p-2}{k-1} \cdot E_{p-k} \cdot (C_k - E_k)] \ (p \geq 2), \text{ where } E_1 = 1.$$

The recursive formula of $\vec{E}_p$ is obtained in the following theorem.

Theorem 2.2 $\vec{E}_p = \sum_{k=1}^{p-1} [\binom{p-2}{k-1} \cdot \vec{E}_{p-k} \cdot (2^k \cdot \vec{C}_k - \vec{E}_k)]$ $(p \geq 2)$, *where* $\vec{E}_1 = 1$.

Proof. It is obvious that $\vec{E}_1 = 1$. When $p \geq 2$, we take two vertices u and v from all labeled vertices of a connected labeled even digraph D of order p. Moreover, choose $k - 1$ $(1 \leq k \leq p - 1)$ vertices from the other $p - 2$ labeled vertices, and together with u form the subset Υ (there are $\binom{p-2}{k-1}$ ways). Besides, the rest of $p-k$ vertices (including v) form the other subset Ω.

Since D is even, we consider the following two cases:

Case 1. The subsets Υ and Ω both form labeled connected even subdigraphs (there are $\vec{E}_k$ and $\vec{E}_{p-k}$ ways) respectively. By Theorem 1.1, there are $2^k - 1$ possible combinations of arcs joining the k vertices of Υ and v. (It may be two arcs or no arcs joining each vertex of Υ and v. However, for connectivity requirement, the case of no arcs joining all vertices of Υ and v is impossible.)

Case 2. Υ forms a labeled connected uneven digraph (namely, not an even digraph) and Ω forms a labeled connected even subdigraph (there are $\vec{C}_k - \vec{E}_k$ and $\vec{E}_{p-k}$ ways) respectively. Suppose $D[\Upsilon]$ has t vertices with odd degree, where $2 \le t \le k$. Clearly, t is an even number. Then by Theorem 1.1, there is an arc (two directions for choosing) joining each of these t vertices and v, and there are two arcs or no arcs joining each of the other $k-t$ vertices with even degree in D and v. Hence there are $2^t \cdot 2^{k-t} = 2^k$ ways joining the k vertices of Υ and v.

Multiply these factors to get the number of labeled connected even digraphs satisfying Υ contains k vertices, that is,

$$\binom{p-2}{k-1} \cdot \vec{E}_{p-k} \cdot [(2^k - 1)\vec{E}_k + 2^k(\vec{C}_k - \vec{E}_k)] = \binom{p-2}{k-1} \cdot \vec{E}_{p-k} \cdot (2^k \cdot \vec{C}_k - \vec{E}_k).$$

By summing these equalities from $k = 1$ to $p-1$, we arrive again at the number of $\vec{E}_p$. This completes the proof. □

According to the values of C_k and $\vec{C}_k$ $(1 \le k < p)$ with the recursive formulas of E_p and $\vec{E}_p$, it is not difficult to obtain that

p	1	2	3	4	5	6	$\cdots$
E_p	1	0	1	3	38	720	$\cdots$
$\vec{E}_p$	1	1	12	454	63000	33160336	$\cdots$

A cutpoint of a graph is one whose removal increases the number of components. A block is a connected, nontrivial graph containing no cutpoints. Fixing a root vertex r for a connected graph G (or a connected digraph D), the number of blocks containing r in G (or in the underlying graph of D) is called the number of blocks of r. Let $R_{l,\,p}$ and $\vec{R}_{l,\,p}$ denote the numbers of rooted labeled connected graphs and digraphs of order p whose roots have l blocks, respectively. The recurrence of $R_{l,\,p}$ is obtained in [5]:

$$R_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1} \cdot (2^k - 1) \cdot R_{l-1,\,p-k} \cdot C_k] \ (l \ge 1,\ p \ge 2),$$

where $R_{0,\,1} = 1$, $R_{0,\,t_1} = R_{t_2,\,1} = 0$ $(t_1 \ge 2,\ t_2 \ge 1)$.

Now the recursive formula of $\vec{R}_{l,\,p}$ is studied.

Theorem 2.3 $\overrightarrow{R}_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot(4^k-1)\cdot\overrightarrow{R}_{l-1,\,p-k}\cdot\overrightarrow{C}_k]$ $(l \ge 1,\ p \ge 2)$, *where* $\overrightarrow{R}_{0,\,1} = 1,\ \overrightarrow{R}_{0,\,t_1} = \overrightarrow{R}_{t_2,\,1} = 0\ (t_1 \ge 2,\ t_2 \ge 1)$.

Proof. Let D be a labeled rooted connected digraph of order p whose root have l blocks. Denote the root of D by v and choose another vertex u. By Theorem 1.1, we give a partition of $V(D)$ such that $V(D) = \Upsilon \cup \Omega$, $\Upsilon \cap \Omega = \emptyset$, $u \in \Upsilon$, $v \in \Omega$, $|\Upsilon| = k$, $|\Omega| = p-k$; Besides, there are $\overrightarrow{R}_{l-1,\,p-k}$ ways for $D[\Omega]$ to form a labeled rooted connected digraph of order $p-k$ whose root v has $l-1$ blocks. And there are $\overrightarrow{C}_k$ ways for $D[\Upsilon]$ to form a labeled connected digraph of order k.

Since $D[\Upsilon]$ is connected, the number of blocks of the root v only add one by joining v and some vertex of Υ (for connectivity requirement, there are 4^k-1 possible combinations of arcs joining the k vertices of Υ and v). Then it forms a labeled rooted connected digraph of order p whose root v have l blocks.

Multiplying these factors and summing these equalities from $k = 1$ to $p-1$, we have $\overrightarrow{R}_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot(4^k-1)\cdot\overrightarrow{R}_{l-1,\,p-k}\cdot\overrightarrow{C}_k]$ $(l \ge 1,\ p \ge 2)$. □

By Theorem 2.3, the values of $\overrightarrow{R}_{l,\,p}$ are listed as following.

$p\backslash l$	0	1	2	3	4	⋯
1	1	0	0	0	0	⋯
2	0	3	0	0	0	⋯
3	0	45	9	0	0	⋯
4	0	3402	405	27	0	⋯
5	0	977670	46899	2430	81	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Remark 2 *Let* $R_{l,\,p,\,q}$ *be the number of rooted labeled connected graphs with* p *vertices and* q *edges whose roots have* l *blocks. Similarly as Remark 1, it is not difficult to obtain another formula for* $\overrightarrow{R}_{l,\,p}$ *, that is,* $\overrightarrow{R}_{l,\,p} = \sum_{k=p-1}^{\binom{p}{2}} 3^k R_{l,\,p,\,k}$. *Nevertheless, enumeration of* $R_{l,\,p,\,q}$ *is unknown and not easy. Consequently, the recurrence of* $\overrightarrow{R}_{l,\,p}$ *in Theorem 2.3 is simple and interesting.*

Let $W_{l,\,p}$ and $\overrightarrow{W}_{l,\,p}$ denote the numbers of rooted labeled connected even graphs and digraphs of order p whose roots have l blocks, respectively.

Theorem 2.4 $W_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot W_{l-1,\,p-k}\cdot(C_k - E_k)]$ $(l \ge 1,\ p \ge 2)$, *where* $W_{0,\,1} = 1,\ W_{0,\,t_1} = W_{t_2,\,1} = 0\ (t_1 \ge 2,\ t_2 \ge 1)$.

Proof. Considering a rooted labeled connected even graph G of order p whose root have l blocks, and applying Theorem 1.2: denote the root of G by v and choose another vertex u, and we get a partition of $V(G)$ such that $V(G) = \Upsilon \cup \Omega$, $\Upsilon \cap \Omega = \emptyset$, $u \in \Upsilon$, $v \in \Omega$, $|\Upsilon| = k$, $|\Omega| = p-k$; Besides, there are $W_{l-1,\,p-k}$ ways for $G[\Omega]$ to form a rooted labeled connected even graph of order $p-k$ whose root v has $l-1$ blocks; In addition, there are $C_k - E_k$ ways for $G[\Upsilon]$ to form a labeled connected uneven graph of order k.

Since $G[\Upsilon]$ is connected and uneven, there are t vertices with odd degree in $G[\Upsilon]$, where t is an even number. Hence there is only one possible combination of edges between the k vertices of Υ and v, that is, joining the vertex v and each vertex with odd degree of $G[\Upsilon]$. It forms a rooted labeled connected even graph of order p whose root v have l blocks. Multiplying these factors and summing the equalities from $k = 1$ to $p - 1$, we obtain the result as desired. □

It follows from Theorem 2.4 that the values of $W_{l,\,p}$ are listed as follows.

$p \backslash l$	0	1	2	3	4	$\cdots$
1	1	0	0	0	0	$\cdots$
2	0	0	0	0	0	$\cdots$
3	0	1	0	0	0	$\cdots$
4	0	3	0	0	0	$\cdots$
5	0	35	3	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem 2.5 $\vec{W}_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1} \cdot \vec{W}_{l-1,\,p-k} \cdot (2^k \cdot \vec{C}_k - \vec{E}_k)]$ $(l \geq 1,\ p \geq 2)$, *where* $\vec{W}_{0,\,1} = 1$, $\vec{W}_{0,\,t_1} = \vec{W}_{t_2,\,1} = 0$ $(t_1 \geq 2,\ t_2 \geq 1)$.

Proof. Let D be a rooted labeled connected even digraph of order p whose root have l blocks. Denote the root of D by v and choose another vertex u. By Theorem 1.1, we give a partition of $V(D)$ such that $V(D) = \Upsilon \cup \Omega$, $\Upsilon \cap \Omega = \emptyset$, $u \in \Upsilon$, $v \in \Omega$, $|\Upsilon| = k$, $|\Omega| = p - k$. Besides, there are $\vec{W}_{l-1,\,p-k}$ ways for $D[\Omega]$ to form a labeled rooted connected even digraph of order $p - k$ whose root v has $l - 1$ blocks. For $D[\Upsilon]$ and the arcs between Υ and Ω, we have the following two cases:

Case 1. There are $\vec{E}_k$ ways for $D[\Upsilon]$ to form a labeled connected even digraph of order k. Since $D[\Upsilon]$ is connected, there are $2^k - 1$ possible combinations of arcs joining Υ and v (there are two arcs or no arcs between v and each vertex of Υ, but it can not be no arcs between v and all vertices of Υ for connectivity requirement), and it forms a rooted labeled connected even digraph of order p whose root v has l blocks.

Case 2. There are $\vec{C}_k - \vec{E}_k$ ways for $D[\Upsilon]$ to form a labeled connected uneven digraph of order k. Since $D[\Upsilon]$ is connected and uneven, suppose $D[\Upsilon]$ has $2t$ $(1 \leq t \leq \lfloor \frac{k}{2} \rfloor)$ vertices with odd degree. There is an arc (two directions for choosing) joining each of these $2t$ vertices and v, and there are two arcs or no arcs joining each of the other $k - 2t$ vertices with even degree in D and v. Hence there are $2^{2t} \cdot 2^{k-2t} = 2^k$ ways for joining the k vertices of Υ and v, and it also forms a labeled rooted connected even digraph of order p whose root v have l blocks.

Multiplying these factors and summing the equalities from $k = 1$ to $p - 1$,

$$\vec{W}_{l,\,p} = \sum_{k=1}^{p-1}[\binom{p-2}{k-1} \cdot \vec{W}_{l-1,\,p-k} \cdot (2^k \cdot \vec{C}_k - \vec{E}_k)]\ (l \geq 1,\ p \geq 2).$$

The proof of Theorem 2.5 is finished. □

According to the values of $\overrightarrow{C}_k$, $\overrightarrow{E}_k$ $(k \geq 1)$ and the recursive formula of $\overrightarrow{W}_{l,p}$ in Theorem 2.5, the values of $\overrightarrow{W}_{l,p}$ $(l \geq 0,\ p \geq 1)$ are obtained.

$p \backslash l$	0	1	2	3	4	$\cdots$
1	1	0	0	0	0	$\cdots$
2	0	1	0	0	0	$\cdots$
3	0	11	1	0	0	$\cdots$
4	0	420	33	1	0	$\cdots$
5	0	60890	2043	66	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Directly computing or observing the tables above, it is easy to see that

Corollary 2.6 *When* $l \geq p$, $R_{l,p} = \overrightarrow{R}_{l,p} = W_{l,p} = \overrightarrow{W}_{l,p} = 0$.

When $l = p-1$, $R_{l,p} = \overrightarrow{W}_{l,p} = 1$, $\overrightarrow{R}_{l,p} = 3^l$, $W_{l,p} = \begin{cases} 1 & (l=0); \\ 0 & (l>0). \end{cases}$

3 The number of labeled (directed) connected d-cyclic graphs

Let d be a nonnegative integer. Define the connected d-cyclic (directed) graph of order p as the connected (directed) graph with p vertices and $p+d-1$ edges (or arcs). Especially, connected 0, 1 and 2-cyclic (directed) graph are also called (oriented) tree, connected (directed) unicyclic and bicyclic graph respectively, where an oriented tree is a digraph obtained from a tree by specifying, for each edge, an order on its ends (so it contains no directed 2-cycles).

Enumeration of labeled connected d-cyclic (directed) graphs is an interesting problem. Let $Q_p^{(d)}$ and $\overrightarrow{Q}_p^{(d)}$ $(d \geq 0)$ denote the numbers of labeled connected d-cyclic graphs and digraphs of order p, respectively.

Obviously, $Q_1^{(d)} = \overrightarrow{Q}_1^{(d)} = \begin{cases} 1 & (d=0) \\ 0 & (d>0) \end{cases}$. Moreover, denote $Q_p^{(0)} = T_p$, $\overrightarrow{Q}_p^{(0)} = \overrightarrow{T}_p$, $Q_p^{(1)} = U_p$, $\overrightarrow{Q}_p^{(1)} = \overrightarrow{U}_p$, $Q_p^{(2)} = B_p$, $\overrightarrow{Q}_p^{(2)} = \overrightarrow{B}_p$.

A recursive formula for $Q_p^{(d)}$ is showed in the following theorem.

Theorem 3.1 $Q_p^{(d)} = \sum_{k=1}^{p-1}\{\binom{p-2}{k-1} \cdot \sum_{i=0}^{d}[\sum_{j=0}^{d-i}\binom{k}{i+1}Q_{p-k}^{(d-i-j)}Q_k^{(j)}]\}$ $(d \geq 0,\ p \geq 2)$.

Proof. Take two vertices, denoted by u and v, from p labeled vertices of G, where G is a labeled connected d-cyclic graph. Moreover, choose $k-1$ $(1 \leq k \leq p-1)$ vertices from the other $p-2$ labeled vertices, and together with u form the subset Υ (there are $\binom{p-2}{k-1}$ ways). Besides, the rest of $p-k$ vertices (including v) form the other subset Ω.

To construct the labeled connected d-cyclic graph G, by Theorem 1.2, $G[\Upsilon]$ forms a labeled connected j–cyclic graph and $G[\Omega]$ forms a labeled connected

$(d-i-j)$-cyclic graph (there are $Q_k^{(j)}$ and $Q_{p-k}^{(d-i-j)}$ ways respectively), where $i = 0, 1, \ldots, d$, and $j = 0, 1, \ldots, d-i$. Moreover, there are $i+1$ edges joining the vertex v and some $i+1$ vertices of Υ, which counts $\binom{k}{i+1}$ ways.

Multiplying these factors and summing the equalities from $k = 1$ to $p-1$, we arrive again at the number of $Q_p^{(d)}$. This completes the proof. □

By Theorem 3.1, a table for the values of $Q_p^{(d)}$ is as follows.

$d\backslash p$	1	2	3	4	5	6	7	8	$\cdots$
0	1	1	3	16	125	1296	16807	262144	$\cdots$
1	0	0	1	15	222	3660	68295	1436568	$\cdots$
2	0	0	0	6	205	5700	156555	4483360	$\cdots$
3	0	0	0	1	120	6165	258125	10230360	$\cdots$
4	0	0	0	0	45	4945	331506	18602136	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Taking a consideration on $\overrightarrow{Q}_p^{(d)}$, two formulas are showed as follows.

Theorem 3.2 *(1)* $\overrightarrow{Q}_p^{(d)} = \sum_{k=1}^{p-1}\{\binom{p-2}{k-1}\cdot\sum_{i=0}^{d}[\sum_{j=0}^{d-i}\binom{2k}{i+1}\overrightarrow{Q}_{p-k}^{d-i-j}\overrightarrow{Q}_k^{j}]\}$ $(d \geq 0,\ p \geq 2)$.

(2) $\overrightarrow{Q}_p^{(d)} = \sum_{k=0}^{d}[\binom{p+k-1}{d-k}\cdot 2^{p-d-1+2k}\cdot Q_p^{(k)}]$ $(d \geq 0,\ p \geq 2)$.

Proof. (1) By the theorem on the partition of connected digraphs (Theorem 1.1), a similar discussion as the proof of Theorem 3.1 leads to the desired result.

(2) Let k be an integer with $0 \leq k \leq d$. Given a labeled connected k-cyclic graph G of order p (the number of such graphs is $Q_p^{(k)}$), we can construct a labeled connected d-cyclic digraphs of order p as follows:

Choose $d-k$ edges from all $(p+k-1)$ edges of G (there are $\binom{p+k-1}{d-k}$ ways), and replace each of these $d-k$ edges with a directed 2-cycle. Besides, give each of the other $p-d-1+2k$ edges of G an orientation (there are $2^{p-d-1+2k}$ ways).

Multiplying these factors and summing the equalities from $k = 0$ to d, we get the number of $\overrightarrow{Q}_p^{(d)}$, and the theorem is proved. □

As corollaries, we study the special cases for labeled connected d-cyclic graphs (resp. digraphs), where $d = 0, 1, 2$.

In the year of 1897, *Cayley* obtained a counting formula for T_p, that is, $T_p = p^{p-2}$ ([1]). From then on, quite a few mathematicians such as Prüfer, Pólya and Kirchhoff proved Cayley formula by using different methods.

By Theorems 3.1 and 3.2, the recurrences for T_p (resp. $\overrightarrow{T}_p$) can be deduced. Combining the recursive formula for T_p with Abel's identity, we give a new and simple proof of the counting formula for T_p by using mathematical induction.

Lemma 3.3 *([2])(Abel identity)* $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x(x-iz)^{i-1}(y+iz)^{n-i}$, *where* x, y *and* z *are real numbers.*

Replacing n by $n-2$, and let $x=1, z=-1, y=n-1$, we have

$$n^{n-2}=\sum_{i=0}^{n-2}\binom{n-2}{i}(i+1)^{i-1}(n-i-1)^{n-i-2}.$$

Let $n-i-1=k$. It follows that

$$n^{n-2}=\sum_{k=1}^{n-1}\binom{n-2}{k-1}\cdot k^{k-1}\cdot(n-k)^{n-k-2}\ (*).$$

Corollary 3.4 *Let p be an integer with $p\geq 2$. Then*
(1) $T_1=1,\ T_p=\sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot k\cdot T_k\cdot T_{p-k}]=p^{p-2}$;
(2) $\vec{T}_1=1,\ \vec{T}_p=\sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot 2k\cdot\vec{T}_k\cdot\vec{T}_{p-k}]=2^{p-1}\cdot p^{p-2}$.

Proof. (1) Obviously, $T_1=1$ and $T_p=Q_p^0=\sum_{k=1}^{p-1}[\binom{p-2}{k-1}\cdot k\cdot T_k\cdot T_{p-k}]$ $(p\geq 2)$ by Theorem 3.1. Now we show that $T_p=p^{p-2}$ by induction on p (≥ 2).

When $p=2$, $T_2=T_1\cdot T_1=1=2^0$, hence the statement holds in this case. Suppose the statement holds for $p\leq n-1$. Now we consider the case of $p=n$. Combining the recursive formula of T_n $(n\geq 2)$ with the inductional assumption,

$$T_n=\sum_{k=1}^{n-1}[\binom{n-2}{k-1}\cdot k\cdot T_k\cdot T_{n-k}]=\sum_{k=1}^{n-1}[\binom{n-2}{k-1}\cdot k^{k-1}\cdot(n-k)^{n-k-2}].$$

By Equality $(*)$, it follows that $T_n=n^{n-2}$ and this completes the proof.

(2) By Theorem 3.2 and note that $T_p=p^{p-2}$, the result follows. □

It is known that a counting formula for U_p is given by Renyi (see [6, 7]), that is, $U_p=\frac{(p-1)!}{2}\sum_{k=3}^{p}\frac{p^{p-k}}{(p-k)!}$ $(p\geq 3)$. It follows from Theorems 3.1 and 3.2 that the counting formula for U_p, $\vec{U}_p$, B_p, $\vec{B}_p$ as obtained, respectively.

Corollary 3.5 $U_p=\sum_{k=1}^{p-1}\binom{p-2}{k-1}[kU_{p-k}\cdot T_k+kT_{p-k}\cdot U_k+\binom{k}{2}\cdot T_{p-k}\cdot T_k]$ $(p\geq 2)$.
$\vec{U}_p=\sum_{k=1}^{p-1}\binom{p-2}{k-1}[2k\vec{U}_{p-k}\cdot\vec{T}_k+2k\vec{T}_{p-k}\cdot\vec{U}_k+\binom{2k}{2}\cdot\vec{T}_{p-k}\cdot\vec{T}_k]$ $(p\geq 2)$.
In addition, $\vec{U}_p=(p-1)\cdot(2p)^{p-2}+2^{p-1}\cdot(p-1)!\cdot\sum_{k=3}^{p}\frac{p^{p-k}}{(p-k)!}$ $(p\geq 3)$.

Corollary 3.6 *(1)* $B_p=\sum_{k=1}^{p-1}\binom{p-2}{k-1}[kB_{p-k}\cdot T_k+kT_{p-k}\cdot B_k+kU_{p-k}\cdot U_k+\binom{k}{2}\cdot U_{p-k}\cdot T_k+\binom{k}{2}\cdot T_{p-k}\cdot U_k+\binom{k}{3}\cdot T_{p-k}\cdot T_k]$ $(p\geq 2)$, *where* $B_1=0$.

(2) $\vec{B}_p=\sum_{k=1}^{p-1}\binom{p-2}{k-1}[2k\vec{B}_{p-k}\cdot\vec{T}_k+2k\vec{T}_{p-k}\cdot\vec{B}_k+2k\vec{U}_{p-k}\cdot\vec{U}_k+\binom{2k}{2}\cdot\vec{U}_{p-k}\cdot\vec{T}_k+\binom{2k}{2}\cdot\vec{T}_{p-k}\cdot\vec{U}_k+\binom{2k}{3}\cdot\vec{T}_{p-k}\cdot\vec{T}_k]$ $(p\geq 2)$, *where* $\vec{B}_1=0$.

(3) $\vec{B}_p=2^{p+1}B_p+2^{p-2}\cdot\sum_{k=3}^{p}\frac{p!\cdot p^{p-k}}{(p-k)!}+2^{p-4}\cdot(p-1)(p-2)\cdot p^{p-2}$ $(p\geq 3)$.

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