MINIMAL FORBIDDEN SUBGRAPHS FOR THE KLEIN BOTTLE WITH LOW CONNECTIVITY

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ABSTRACT. Kuratowski proved that a finite graph embeds in the plane if it does not contain a subdivision of either K_5 or $K_{3,3}$, called Kuratowski subgraphs. Glover asked if a finite minimal forbidden subgraph for the Klein bottle can be written as the union of 3 Kuratowski subgraphs such that the union of each pair of these fails to embed in the projective plane. We show that this is true for all finite minimal forbidden graphs for the Klein bottle with connectivity < 3.

1. INTRODUCTION

We say that a graph G without vertices of degree two is a minimal forbidden subgraph or an irreducible graph for a surface S if G does not embed in S, but any proper subgraph of G embeds in S.

Kuratowski [7] showed that minimal forbidden subgraphs for the plane are K_5 and $K_{3,3}$. Given a graph G, any subgraph of G that is a subdivision of K_5 or $K_{3,3}$ is called a *Kuratowski subgraph* of G.

Then one might ask if Kuratowski's result can be extended to higher genus surfaces in terms of Kuratowski subgraphs. Glover has asked if a finite graph G is a minimal forbidden subgraph for the nonorientable surface $\mathbb{N}_g$, then G can be written as the union of $g+1$ Kuratowski subgraphs such that the union of each pair of these fails to embed in the projective plane, the union of each triple of these fails to embed in the Klein bottle if $g \geq 2$, and the union of each triple of these fails to embed in the torus if $g \geq 3$. We call this conjecture the *Kuratowski covering conjecture* and prove the following partial result.

Theorem 1.1. *A finite minimal forbidden subgraph for the Klein bottle with connectivity < 3 can be written as the union of 3 Kuratowski subgraphs such that the union of each pair of these fails to embed in the projective plane.*

In the following, we mean finite graphs by graphs. A list of minimal forbidden subgraphs for the projective plane has been found by Glover, Huneke, and Wang [4] and Archdeacon [1] proved that this list is complete.

Brunet, Richter, and Širáň [2] showed that every minimal forbidden subgraph for a nonorientable surface is a union of Kuratowski subgraphs but this is not true for orientable surfaces. Decker [3] showed the latter result as well. For the projective plane, the following fact is known and we keep this result for future use.

Lemma 1.2. [6] *Every minimal forbidden subgraph for the projective plane is a union of two Kuratowski subgraphs.*

Moreover, it has been shown that the Kuratowski covering conjecture about arbitrary nonorientable surfaces is true for all minimal forbidden subgraphs of order < 10 [5], [6].

The remainder of this paper is organized as follows. We present preliminaries in Section 2 and prove Theorem 1.1 in Section 3.

Remark. A strengthened form of the Kuratowski covering conjecture analogous to the complete Kuratowski theorem for the plane says that a finite graph G fails to embed in $\mathbb{N}_g$ if and only if there are $g+1$ Kuratowski subgraphs in G satisfying the conditions of the Kuratowski covering conjecture.

2. Preliminaries

A cycle in a surface is said to be *null* if it can be contracted to a point in the surface, and *essential* otherwise. It is known [4] that the projective plane contains no disjoint essential cycles.

We define S_vG to be a graph obtained by splitting v into two vertices v' and v'' with connecting edge so that some edges incident to v in G are adjacent to v' and the other edges incident to v in G are adjacent to v''. A subgraph A of G is called a k_4 in G if there exists a graph B such that $A \subset B \subset G$, and A is a subdivision of K_4, B is a subdivision of K_5, or S_vK_5, with a degree four vertex or the cubic vertices of B not in A. A subgraph A of G is called a $k_{2,3}$ in G if there exists a graph B such that $A \subset B \subset G$, and A is a subdivision of $K_{2,3}$, B is a subdivision of $K_{3,3}$, with one of the cubic vertices of B not in A. For other basic definitions, we refer the reader to [8].

k_4 and $k_{2,3}$ are called *k-graphs* and the existence of two disjoint k-graphs in a graph G implies that G does not embed in the projective plane.

Lemma 2.1. [4] *If a graph G contains two disjoint k-graphs, then G is nonprojective planar.*

Lemma 2.2. [9] *If there is an embedding Γ of a graph G in $\mathbb{N}_{g+1}$, but G does not embed in $\mathbb{N}_g$, then Γ is an open 2-cell embedding.*

We define $G + uv$ as follows.

$$G + uv = \begin{cases} G \cup uv & \text{if } uv \notin \mathrm{E(G)} \\ G & \text{otherwise} \end{cases}$$

Lemma 2.3. *Let $G = H_1 \cup H_2$ and $V(H_1 \cap H_2) = \{u, v\}$. If $H_1 + uv$ embeds in $\mathbb{N}_h$ and $H_2 + uv$ embeds in $\mathbb{N}_k$, then $G + uv$ embeds in $\mathbb{N}_{h+k}$ for $h, k \geq 0$.*

Proof. Consider an embedding Γ_1 of $H_1 + uv$ in $\mathbb{N}_h$ and an embedding Γ_2 of $H_2 + uv$ in $\mathbb{N}_k$. If we remove an open disc from each of the two surfaces $\mathbb{N}_h$ and $\mathbb{N}_k$ and identify the two boundary components of the resulting manifolds so that $\Gamma_1(uv)$ and $\Gamma_2(uv)$ are identified, we obtain an embedding of $G + uv$ in $\mathbb{N}_{h+k}$. □

Lemma 2.4. *Let $G = H_1 \cup H_2$ and $V(H_1 \cap H_2) = \{u, v\}$ where each H_i contains a path connecting u and v for $i = 1, 2$. Suppose $H_1 + uv$ embeds in $\mathbb{N}_{g+1}$, but does not embed in $\mathbb{N}_g$. Then G embeds in $\mathbb{N}_{g+1}$ if and only if $H_2 + uv$ is planar.*

Proof. Sufficiency follows from Lemma 2.3 since a graph embeds in the plane if and only if it embeds in the sphere, that is, $\mathbb{N}_0$. For necessity, suppose that there is an embedding Γ of G in $\mathbb{N}_{g+1}$. Since H_2 contains a path P connecting u and v, G contains a subdivision H' of $H_1 + uv$. The embedding Γ of G is an extension of an embedding of H', which is an open 2-cell embedding by Lemma 2.2. Then $\Gamma(P)$ is adjacent to precisely two open 2-cells in this embedding. Thus there is an embedding of a subdivision of $H_2 + uv$ in the union of these two open 2-cells and $\Gamma(P)$, hence in the plane. □

Lemma 2.4 implies the following two corollaries.

Corollary 2.5. *Let $G = H_1 \cup H_2$ and $V(H_1 \cap H_2) = \{u, v\}$ where each H_i contains a path connecting u and v and $|V(H_i)| \geq 3$ for $i = 1, 2$. If G is a minimal forbidden subgraph for the Klein bottle, then both $H_1 + uv$ and $H_2 + uv$ are nonplanar.*

Proof. Suppose that $H_2 + uv$ is planar. Since $|V(H_2)| \geq 3$ and G does not have a vertex of degree two by definition of minimal forbidden subgraphs, the union of H_1 and a path connecting u and v in H_2 is a proper subgraph of G, hence $H_1 + uv$ embeds in the Klein bottle. If $H_1 + uv$ is planar, then G would embed in the plane by Lemma 2.3, a contradiction. So $H_1 + uv$ cannot be planar and there is a nonorientable surface $\mathbb{N}_g$ such that $H_1 + uv$ embeds in $\mathbb{N}_{g+1}$, but does not embed in $\mathbb{N}_g$ where g is either 0 or 1. But, then, $H_2 + uv$ has to be nonplanar by Lemma 2.4 since G does not embed in $\mathbb{N}_2$. Similarly, it can be shown that $H_1 + uv$ is nonplanar. □

Corollary 2.6. *Let $G = H_1 \cup H_2$ and $V(H_1 \cap H_2) = \{u, v\}$ where each H_i contains a path connecting u and v for $i = 1, 2$. If $H_1 + uv$ embeds in the Klein bottle, but does not embed in the projective plane, and $H_2 + uv$ is nonplanar, then G does not embed in the Klein bottle.*

3. The Proof of Theorem 1.1

3.1. **Connectivity $\leq$ 1.** Suppose that G is a minimal forbidden subgraph for the Klein bottle and let $G = H_1 \cup H_2$ where either H_1 and H_2 are vertex disjoint or $V(H_1 \cap H_2) = \{u\}$. The following argument applies to both of these cases. By the minimality of G, each H_i, $i = 1, 2$ is nonplanar. On the other hand, if both H_1 and H_2 are projective planar, then $H_1 \cup H_2$ would embed in the Klein bottle. Suppose that H_1 is nonprojective planar and consider a subdivision H' of a minimal forbidden subgraph for the projective plane in H_1. Since H' is a proper subgraph of G, H' embeds in the Klein bottle, but does not embed in the projective plane, hence every embedding of H' in the Klein bottle is an open 2-cell embedding by Lemma 2.2. So the union G' of H' and any Kuratowski subgraph contained in H_2 does not embed in the Klein bottle. But the minimality of G implies $G' = G$, so H_1 is a subdivision of a minimal forbidden subgraph for the projective plane and H_2 is a Kuratowski subgraph. Moreover, any subgraph of H_1 share at most one vertex with H_2. Thus G satisfies the Kuratowski covering conjecture by Lemma 1.2 and Lemma 2.1.

3.2. **Connectivity 2.** Suppose that G is a 2-connected minimal forbidden subgraph for the Klein bottle and let $G = H_1 \cup H_2$ where $V(H_1 \cap H_2) = \{u, v\}$ and $|V(H_i)| \geq 3$ for $i = 1, 2$. We note that both H_1 and H_2 are connected since G is 2-connected. Thus each H_i contains a path P_i connecting u and v for $i = 1, 2$. We also note that both $H_1 + uv$ and $H_2 + uv$ are nonplanar by Corollary 2.5.

If both $H_1 + uv$ and $H_2 + uv$ are projective planar, G would embed in the Klein bottle by Lemma 2.3, so at least one of $H_1 + uv$ and $H_2 + uv$ must be nonprojective planar. Suppose that $H_1 + uv$, say, is nonprojective planar. Then $H_1 \cup P_2$ contains a subdivision H' of a minimal forbidden subgraph for the projective plane. As noted above, $H_2 + uv$ is nonplanar, which means that $H_2 \cup P_1$ contains a Kuratowski subgraph H''. We may assume that H' and H'' do not contain uv because of P_1 and P_2. Therefore both H' and H'' are subgraphs of G.

Let $G' = H' \cup H''$ and let P_i^o, $i = 1, 2$ be the union of all the edges and all the vertices in P_i except for u and v. P_2 and P_1 may or may not be contained in H' and H'', respectively, so let us consider the following four cases.

(i) $P_2 \subseteq H'$ and $P_1 \subseteq H''$
(ii) $P_2 \subseteq H'$ and $P_1 \not\subseteq H''$

(iii) $P_2 \not\subseteq H'$ and $P_1 \subseteq H''$
(iv) $P_2 \not\subseteq H'$ and $P_1 \not\subseteq H''$

In case (i), $G' = ((H' \backslash P_2^o) \cup P_1) \cup ((H'' \backslash P_1^o) \cup P_2)$ and G' satisfies all the conditions of Corollary 2.6, so it does not embed in the Klein bottle. But $G' = G$ by the minimality of G. By Lemma 1.2, H' is a union of two Kuratowski subgraphs H_1' and H_2'. $H'' \cap H_i' \subseteq P_1 \cup P_2$ for $i = 1, 2$, so there are two disjoint k-graphs in $H'' \cup H_i'$ since $H'' \backslash u$ contains a k-graph disjoint from P_1^o and $H_i' \backslash v$ contains a k-graph disjoint from P_2^o. So $H'' \cup H_i'$ is nonprojective planar for $i = 1, 2$ by Lemma 2.1. That is, the Kuratowski covering conjecture is true for this case.

It is easy to show that G' does not embed in the Klein bottle for the other three cases using the fact that every embedding of H' in the Klein bottle is an open 2-cell embedding. This implies that $G' = G$ and it can be shown that the Kuratowski covering conjecture is true in these cases using three Kuratowski subgraphs H_1', H_2', and H'' of G' where $H' = H_1' \cup H_2'$ in a similar way as in case (i). This completes the proof of Theorem 1.1.

References

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