Estimates of the choice numbers and the Ohba numbers of some complete multipartite graphs

Julian Allagan[1a] Peter Johnson[2b]
[a] School of Science Technology Engineering and Mathematics, Gainesville State College, Watkinsville, GA- 30677, USA
[b]Department of Mathematics and Statistics
221 Parker Hall, Auburn University, AL - 36849, USA

Abstract

Estimates of the choice numbers and the Ohba numbers of the complete multipartite graphs $K(m, n, 1 \ldots, 1)$ and $K(m, n, 2 \ldots, 2)$ are given for various values of $m \geq n \geq 1$. The Ohba number of a graph G is the smallest integer n such that $ch(G \vee K_n) = \chi(G \vee K_n)$.

1 The choice number

Throughout this paper, the graph $G = (V, E)$ will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We also use the notation $K(\underbrace{m_1, m_2, \ldots, m_k}_{k})$ to denote a complete k–partite graph ($k \geq 2$) in which the parts have sizes $m_1, m_2, \ldots, m_k$.

A *list assignment* to the graph G is a function L which assigns a finite set (list) $L(v)$ to each vertex $v \in V(G)$. A *proper L–coloring* of G is a function $\psi : V(G) \to \bigcup_{v \in V(G)} L(v)$ satisfying, for every $u, v \in V(G)$,

(i) $\psi(v) \in L(v)$,

(ii) $uv \in E(G) \to \psi(v) \neq \psi(u)$.

The *choice number* or *list–chromatic number of G*, denoted by $ch(G)$, is the smallest integer k such that there is always a proper L–coloring of G if L satisfies $|L(v)| \geq k$ for every $v \in V(G)$. We define G to be *k–choosable* if it admits a proper L–coloring whenever $|L(v)| \geq k$ for all $v \in V(G)$; then $ch(G)$ is the smallest integer k such that G is k–choosable.

Since the chromatic number $\chi(G)$ is similarly defined with the restriction that the list assignment is to be constant, it is clear that for all G, $\chi(G) \leq ch(G)$. There are many graphs whose choice number exceeds (sometimes greatly) their chromatic number. Figure 1 depicts the smallest graph G whose choice number exceeds its chromatic number.

[1]Email: aallagan@gmail.com
[2]Email: johnspd@auburn.edu

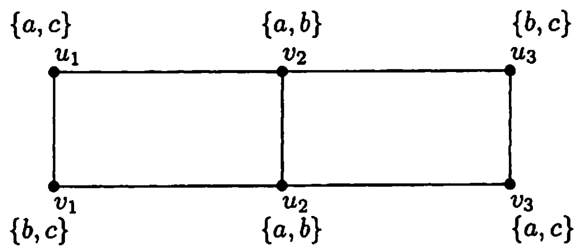

Figure 1: K(3,3) minus two independent edges with a list assignment L.

It is easy to see that G is not properly $L-$colorable, so $ch(G) > 2 = \chi(G)$. Since G is connected, and neither a complete graph nor an odd cycle, by Brooks' theorem for the choice number [2], $ch(G) \leq \Delta(G) = 3$. Thus, $ch(G) = 3$.

Any graph G for which the extremal equality $\chi(G) = ch(G)$ holds is said to be *chromatic–choosable.* It is not hard to see that cycles, cliques and trees are all chromatic–choosable. (The case of even cycles requires a little work. See [2].)

The following are some known results on the choice numbers of some complete multipartite graphs.

Theorem A.(Erdös, Rubin and Taylor [2]) *The complete $k-$partite graph $K(2,2,\ldots,2)$ is chromatic–choosable.*

Theorem B.(Gravier and Maffray [4]) *If $k > 2$, then the complete $k-$partite graph $K(3,3,2,\ldots,2)$ is chromatic–choosable.*

This result does not hold for $k = 2$ since $K(3,3)$ contains the graph in Figure 1, whose choice number is bigger than 2.

Corollary B. *The complete $k-$partite graph $K(3,2\ldots,2)$ is chromatic–choosable.*

Proof. Since $K(3,2\ldots,2)$ is a complete $k-$partite graph, $k = \chi(K(3,2\ldots,2)) \leq ch(K(3,2\ldots,2))$. Further, $K(3,2\ldots,2)$ is a subgraph of the complete $k-$partite graph $K(3,3,2,\ldots,2)$. Therefore $ch(K(3,2\ldots,2)) \leq k$ if $k > 2$. Thus, $ch(K(3,2\ldots,2)) = k$ if $k > 2$. When $k = 2$, we have $K(3,2)$, of which it is well known that the choice number is 2. See [5], for instance. □

Theorem C.(Kierstead [6]) *Let G denote the complete $k-$partite graph $K(3,3,3,\ldots,3)$. Then $ch(G) = \lceil\frac{(4k-1)}{3}\rceil$.*

Observe that this result implies that $ch(G) = k+1$ when $2 \leq k \leq 4$ and $k+1 < ch(G) < \frac{3k}{2}$ when $k \geq 5$.

Theorem D. (Enomoto et al.[1]2002)
Let G_k denote the complete $k-$partite graph $K(4,2,\ldots,2)$. Then

$$ch(G_k) = \begin{cases} k & if\ k\ is\ odd \\ k+1 & if\ k\ is\ even. \end{cases}$$

Theorem E.(Enomoto et al.[1]) *Suppose $k \geq 2$ and let G denote the complete $k-$partite graph $K(5,2,\ldots,2)$. Then $ch(G) = k+1$.*

Corollary E. *Let G denote the complete $k-$partite graph $K(6,2,\ldots,2)$. Then $ch(G) = k+1$ if $k \geq 2$.*

Proof. Since $k+1 = ch(K(5,2,\ldots,2)) \leq ch(K(6,2,\ldots,2))$, it is clear that $ch(G) \geq k+1$. Further, G is a subgraph of the complete $(k+1)-$partite graph $K(3,3,2,\ldots,2)$ which has choice number $k+1$ by Theorem B. Thus, $ch(G) \leq k+1$. So, $ch(G) = k+1$. □

2 The Ohba number

In 2002, Ohba [7] proved that for any given graph G, there exists an integer n_0 such that for any $n \geq n_0$, the join $G \vee K_n$ satisfies $ch(G \vee K_n) = \chi(G \vee K_n)$.

The **Ohba number** of G is the number $\phi(G)$ defined to be the smallest integer n for which $ch(G \vee K_n) = \chi(G \vee K_n)$. In particular when G is chromatic–choosable, $\phi(G) = 0$.

Observe that $|V(G \vee K_n)| \leq 2\chi(G \vee K_n) + 1$ if and only if $n \geq |V(G)| - 2\chi(G) - 1$. Now, Ohba's conjecture [7] states that if $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic–choosable. Thus, Ohba's conjecture would imply that $\phi(G) \leq max(0, |V(G)| - 2\chi(G) - 1) \leq max(0, |V(G)| - 5)$ for every graph G with an edge.

Conversely, if $\phi(G) \leq max(0, |V(G)| - 2\chi(G) - 1)$ for all G then Ohba's conjecture is true. It is further clear that Ohba's conjecture is true for every graph of order at most 5, since the graph in Figure 1 is known to be the smallest graph that is not chromatic–choosable, and it is of order 6.

Proposition 2.1. *For any graph G, $\phi(G) \geq ch(G) - \chi(G)$.*

Proof. If G is chromatic–choosable, by the definition $\phi(G) = ch(G) - \chi(G) = 0$.

Suppose G is not chromatic–choosable. Then $ch(G) > \chi(G)$. Let s be the smallest positive integer such that $ch(G \vee K_s) = \chi(G \vee K_s)$. Since $\chi(G \vee K_s) = \chi(G) + s$, this implies that $s = ch(G \vee K_s) - \chi(G)$. Further, $ch(G) \leq ch(G \vee K_s)$ for all $s \geq 1$. So, $s \geq ch(G) - \chi(G)$. Thus, $\phi(G) \geq ch(G) - \chi(G)$. □

3 Results

3.1 Estimates of $ch(K(m, n, 1, \ldots, 1))$

We present here the choice numbers of the complete k–partite graphs $K(m, n, 1, \ldots, 1)$ for various values of $m \geq n \geq 1$ and their corresponding Ohba numbers. Pretty clearly, if $k - 2 \leq \phi(K(m, n))$ then $\phi(K(m, n, 1, \ldots, 1)) = \phi(K(m, n)) - (k - 2)$. So, $\phi(K(m, n, 1, \ldots, 1)) = max\{0, \phi(K(m, n)) - (k - 2)\}$. Consequently, we just need $\phi(K(m, n))$.

Throughout this section, we denote the parts of the complete k– partite graph $K(m, n, 1, \ldots, 1)$ by $V_1, V_2, \ldots, V_s$ where $V_1 = \{x_1, \ldots, x_m\}$, $V_2 = \{y_1, y_2, \ldots, y_n\}$ and $V_s = \{v_s\}$ for $s = 3, \ldots, k$.

Theorem 3.1. *Let G denote the complete k–partite graph $K(m, n, 1, \ldots, 1)$. Then $ch(G) \leq n + k - 1$ for all $1 \leq n \leq m$.*

Proof. When $k = 2$, it is shown in [5] that $ch(G) \leq n + 1$ for all $m \geq n$. The proof for arbitrary $k \geq 2$ will be similar. Let $G' = G - V_1$, where V_1 is the part of G of size m, and let L be a list assignment to G with $|L(v)| \geq n + k - 1$ for each $v \in V(G)$. Since $|V(G')| = n + k - 2$, G' can be L–colored using at most $n+k-2$ distinct colors, say $\alpha_1, \ldots, \alpha_{n+k-2}$. Thus, for each $v \in V_1$, $|L(v) - \{\alpha_1, \ldots, \alpha_{n+k-2}\}| \geq 1$, and so G is L–colorable. Hence $ch(G) \leq n + k - 1$.

□

Lemma 3.1. *Let H denote the complete $(k-1)$–partite graph $K(2, 1, \ldots, 1)$ with parts $V_1 = \{y_1, y_2\}$, $V_s = \{v_s\}$, for each $s = 2, \ldots, k - 1$. Let L be a list assignment to H satisfying that $L(y_1) = A$ and $L(y_2) = B$ for some disjoint k–sets of colors A and B, and $|L(w)| \geq k$ for each $w \in V(H)$. Then the number of different color sets arising from proper L–colorings of H is at least $\dfrac{k^2 + 3k}{2}$.*

Proof. Let $K_{k-2} \cong H - V_1$ and $C_{i,j} = \{$color sets from proper L–colorings of K_{k-2} with i element(s) from A, j element(s) from $B\}$, with $0 \leq i, j \leq k - 2$ and $i + j \leq k - 2$. Let $c_{i,j} = |C_{i,j}|$.

Claim 1. $\sum_{\substack{0\le i,j\le k-2\\ i+j\le k-2}} c_{i,j} \ \ge\ \binom{k}{2}$.

Proof: The number of proper $L-$colorings of K_{k-2} is at least $k(k-1)\dots(k-(k-3)) = k(k-1)\dots 3 = \frac{k!}{2}$. Further, since each color set appears at most $(k-2)!$ times, the number of distinct color sets arising from the proper $L-$colorings is at least $\frac{k!}{2(k-2)!}$, meaning $\sum_{\substack{0\le i,j\le k-2\\ i+j\le k-2}} c_{i,j} \ge \frac{k!}{2(k-2)!} = \binom{k}{2}$.

Define $\mathcal{D}_{p,q}$ = {color sets from proper $L-$colorings of H with p element(s) from A, q element(s) from B}, with $1 \le p,q \le k$, $p+q \le k$ and let $d_{p,q} = |\mathcal{D}_{p,q}|$. Then the total number of color sets from proper $L-$colorings of H is $\sum_{\substack{1\le p,q\le k\\ p+q\le k}} d_{p,q}$. Since any coloring of H uses exactly one color from A on y_1 and one color from B on y_2, every color set in $\mathcal{D}_{p,q}$ is of the form $D = C \cup \{a,b\}$ for some $a \in A \setminus C$ and $b \in B \setminus C$ and $C \in \mathcal{C}_{p-1,q-1}$. For each pair p,q such that $1 \le p,q \le k$, $p+q \le k$, consider the bipartite graph with bipartition $\mathcal{D}_{p,q}$, $\mathcal{C}_{p-1,q-1}$ with $D \in \mathcal{D}_{p,q}$, $C \in \mathcal{C}_{p-1,q-1}$ adjacent if and only if $C \subseteq D$. Now each $C \in \mathcal{C}_{p-1,q-1}$ has degree $(k-(p-1))(k-(q-1))$ and each $D \in \mathcal{D}_{p,q}$ has degree at most pq in this bipartite graph. Therefore $pqd_{p,q} \ge \sum_{D\in\mathcal{D}_{p,q}} deg(D) = \sum_{C\in\mathcal{C}_{p-1,q-1}} deg(C) = (k-p+1)(k-q+1)c_{p-1,q-1}$. Thus, the total number of proper $L-$coloring sets satisfies

$$\sum_{\substack{1\le p,q\le k\\ p+q\le k}} d_{p,q} \ \ge\ \sum_{\substack{1\le p,q\le k\\ p+q\le k}} \frac{(k-p+1)(k-q+1)}{pq} c_{p-1,q-1}. \qquad (1)$$

Claim 2. $f(p,q) \ge \frac{(k+2)^2}{k^2}$ where $f(p,q) = \frac{(k-p+1)(k-q+1)}{pq}$, $1 \le p,q \le k$ and $p+q \le k$.

Proof: Fix $s \in \{2,\dots,k\}$ and consider values of p and q such that $p+q=s$. Then $p = s-q$, and $1 \le q \le s-1$.

Now, $f(p,q) = f(s-q,q) = g(q) = \frac{(k+1-s+q)(k+1-q)}{(s-q)q}$. Also, we note that $g(1) = g(s-1) = \frac{k(k+2-s)}{(s-1)}$, and $g'(q) = \frac{h(q)}{[(s-q)q]^2}$ where $h(q) = -(k+1)(k+1-s)[s-2q]$. Therefore, g achieves a minimum on $[1,s-1]$ at $q = s/2$. We have for all $q \in [1,s-1]$, $f(s-q,q) \ge g(s/2) =$

$f(s/2, s/2) = \frac{(k+1-s/2)^2}{s^2/4}$.

Clearly this minimum decreases as s increases. Therefore, for all $p, q \in \{1, \ldots, k-1\}$, $p+q \leq k$, $f(p,q) \geq f(k/2, k/2) = \frac{(k/2+1)^2}{k^2/4} = \frac{(k+2)^2}{k^2}$.

From Claim 2 and the inequality 1,

$$\sum_{\substack{1 \leq p,q \leq k \\ p+q \leq k}} d_{p,q} \geq \frac{(k+2)^2}{k^2} \cdot \sum_{\substack{0 \leq i,j \leq k-2 \\ i+j \leq k-2}} c_{i,j} \geq \frac{(k+2)^2}{k^2} \cdot \frac{k!}{2(k-2)!} = \frac{k^2+3k}{2} - \frac{2}{k}.$$

Hence for all $k \geq 3$, the number of different color sets arising from proper L–colorings of H is at least $\frac{k^2+3k}{2}$.

□

Theorem 3.2. *Let G denote the complete k–partite graph $K(m, 2, 1, \ldots, 1)$, $k \geq 3$. Then*

$$ch(G) = \begin{cases} k & \text{if } m < \frac{k^2+3k}{2} \\ k+1 & \text{if } m \geq k^2. \end{cases}$$

Proof. Let L be a list assignment to G with $|L(v)| = k$ for each $v \in V(G)$. Suppose G has no proper L–coloring.

Observe that $L(y_1) \cap L(y_2) = \emptyset$. Otherwise there is a color $c \in L(y_1) \cap L(y_2)$. Then we can color y_1 and y_2 with c and the remaining subgraph $G - V_2 = K(m, 1, \ldots, 1)$ can be colored from $L - \{c\}$ because $ch(G - V_2) = k - 1$.

Let $H = G - V_1$. Since $L(y_1) \cap L(y_2) = \emptyset$, by Lemma 3.1, the number of distinct sets arising from the proper L–colorings of the subgraph H is at least $\frac{k^2+3k}{2}$.

Further, G is not L–colorable if and only if the set of colors, which will be of size k, of each of the proper colorings of H occurs as a list in V_1. Therefore for $m < \frac{k^2+3k}{2}$, G is L– colorable. Thus, if $m < \frac{k^2+3k}{2}$, $ch(G) \leq k$. Also $k = \chi(G) \leq ch(G)$, so $ch(G) = k$ if $m < \frac{k^2+3k}{2}$.

When $m = k^2$, we provide the following list assignment L' to $V(G)$ such that there is no proper L'–coloring of G.

Let A and B be disjoint sets of colors of size k, say $A = \{\alpha_1, \ldots, \alpha_k\}$ and $B = \{\beta_1, \ldots, \beta_k\}$. Let $L'(y_1) = L'(v_3) = \ldots = L'(v_k) = A$, $L'(y_2) = B$.

Any coloring of $H = K(2, 1, \ldots, 1)$ requires exactly $k-1$ colors from A and one color from B, and there are exactly k^2 color sets from such colorings. Let $m = k^2$ lists on V_1 be the k^2 different sets $(A \setminus \{\alpha_i\}) \cup \{\beta_i\}$, $1 \le i, j \le k$. Since each of the proper colorings of H occurs as a list in V_1, $ch(K(m, 2, 1, \ldots, 1)) > k$ for $m = k^2$.

Further, by Theorem 3.1, $ch(K(m, 2, 1, \ldots, 1)) \le k + 1$ for all m. This concludes the proof.

□

Corollary 3.2.1. $\lfloor\sqrt{m}\rfloor - 1 \le \phi(K(m,2)) \le \lceil \frac{-7+\sqrt{8m+17}}{2} \rceil$ *for* $m \ge 5$.

Proof. If $k \le \lfloor\sqrt{m}\rfloor$, then $k^2 \le m$, so by Theorem 3.2, $k+1 = ch(K(m, 2, 1 \ldots, 1)) > \chi(K(m, 2, 1 \ldots, 1)) = k$. Thus, if $k \le \lfloor\sqrt{m}\rfloor$, $\phi(K(m, 2)) \ge (k-2)+1 = k-1$. Consequently, $\phi(K(m, 2)) \ge \lfloor\sqrt{m}\rfloor - 1$, for all $m \ge 1$. Further, by Theorem 3.2, if $m \le \dfrac{k^2+3k}{2} - 1$ and $k \ge 3$, then $\phi(K(m, 2)) \le k - 2$. The smallest positive value of k for which $m \le \dfrac{k^2+3k}{2} - 1$ is the positive solution of $k^2 + 3k - 2(m+1) = 0$, so the smallest integer value of k satisfying that inequality is $k_0 = \lceil \frac{-3+\sqrt{8m+17}}{2} \rceil$; we have $\phi(K(m, 2)) \le k_0 - 2 = \lceil \frac{-7+\sqrt{8m+17}}{2} \rceil$. The requirement $m \ge 5$ ensures that $k_0 \ge 3$. □

Remark:

$\phi(K(m, 2)) = 1$ for $4 \le m \le 8$, by Theorem 3.2 and the fact, proven in [5], that $ch(K(m, 2)) = 3$ for all $m \ge 4$.

Theorem 3.3. *Let G denote the complete k-partite graph $K(m, n, 1, \ldots, 1)$, and $2 \le n \le m$.*
Then $ch(G) = n + k - 1$ if $m \ge \binom{n+k-2}{k-1}(n+k-2)^{n-1}$.

Proof. Let $C_1, C_2, \ldots, C_n$ be pairwise disjoint $(n+k-2)$-sets of colors.

We provide the following list assignment L to G, with $|L(v)| = n+k-2$ for each $v \in V(G)$ as follows: $L(y_1) = L(v_3) = \ldots = L(v_k) = C_1$ and $L(y_j) = C_j$ for each $2 \le j \le n$. L on V_1 will be described shortly.

Any proper L-coloring of $G' = G - V_1 \cong K(n, 1, \ldots, 1)$ requires exactly $k-1$ colors from C_1 and exactly one color from each C_j for $2 \le j \le n$, giving $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ distinct sets of colors from proper L-colorings, each set of size $n+k-2$. Let $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ lists on V_1 be the $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ different sets of colors from such proper L-colorings of G', and if $m > \binom{n+k-2}{k-1}(n+k-2)^{n-1}$, let the remaining vertices in V_1 be supplied with any lists whatever of size $n+k-2$. Since each of the $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ sets of proper colorings of G' occurs as a list in V_1, G cannot be properly L–colored, so $ch(K(m,n,1,\ldots,1)) > n+k-2$ for $m \geq \binom{n+k-2}{k-1}(n+k-2)^{n-1}$. Further, from Theorem 3.1, $ch(G) \leq n+k-1$ for all $m \geq 2$. Thus, for $m \geq \binom{n+k-2}{k-1}(n+k-2)^{n-1}$, $ch(G) = n+k-1$. □

Corollary 3.3.1. *With G, m, n and k as in the hypothesis of Theorem 3.3, if $2 \leq r \leq n-1$ and $m \geq \binom{r+k-2}{k-1}(r+k-2)^{r-1}$, then $ch(G) \geq r+k-1$.*

Proof. When $m \geq \binom{r+k-2}{k-1}(r+k-2)^{r-1}$, $ch(K(m,r,1,\ldots,1)) = r+k-1$ by Theorem 3.3. Further, with $2 \leq r \leq n-1 < m$, $K(m,r,1,\ldots,1)$ is a subgraph of the graph $G = K(m,n,1,\ldots,1)$.

□

Corollary 3.3.2. *With G, m, n and k as in the hypothesis of Theorem 3.3, if $2 \leq r \leq n$ and $m \geq \binom{r+k-2}{k-1}(r+k-2)^{n-1}$, then $\phi(G) \geq r-1$.*

Proof. By Proposition 2.1, $\phi(G) \geq ch(G) - \chi(G)$. Therefore $\phi(G) \geq r+k-1-(k) = r-1$.

□

3.2 An estimate of $ch(K(m,2,\ldots,2))$

Throughout this section, $[n] = \{1,\ldots,n\}$ and $\binom{[n]}{t} = \{t-\textit{subsets of } [n]\}$.

Proposition 3.1. *Let G denote the complete k–partite graph $K(m,2,2,\ldots,2)$. Then $ch(G) \leq 2k-1$.*

Proof. We may assume that $k \geq 2$. Let L be a list assignment to G such that $|L(v)| \geq 2k-1$ for each $v \in V(G)$. Since $G - V_1 \cong K(2,\ldots,2)$ has choice number $k \leq 2k-1$, there is a proper L–coloring of $G - V_1$, and it will use at most $2(k-1)$ distinct colors, say $\alpha_1,\ldots,\alpha_{2k-2}$. For each $v \in V_1$, $|L(v) - \{\alpha_1,\ldots,\alpha_{2k-2}\}| \geq 1$, and so G is L–colorable. Hence $ch(G) \leq 2k-1$.

□

For $n \geq m \geq t \geq 0$, **the covering number** $C(n,m,t)$ is defined by $C(n,m,t) = min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \binom{[n]}{m}$ and $\forall\, B \in \binom{[n]}{t}, \exists\, A \in \mathcal{F}$ *such that* $B \subseteq A\}$.

Lemma 3.2. *$C(n,m,t)$ is also the smallest size of a collection $\mathcal{F}'$ of $n-m$ subsets of $[n]$ (or any other fixed $n-$set) such that for every $(n-t)-$set $B' \in \binom{[n]}{n-t}$, some $A' \in \mathcal{F}'$ is contained in B'.*

Proof. Given $\mathcal{F}$, as in the original definition of $C(n,m,t)$, form $\mathcal{F}' = \{[n] \setminus A \mid A \in \mathcal{F}\}$, the collection of complements of sets in $\mathcal{F}$. Similarly, given $\mathcal{F}' \subseteq \binom{[n]}{n-m}$, form $\mathcal{F} = \{[n] \setminus A' \mid A' \in \mathcal{F}'\}$, the collection of complements of sets in $\mathcal{F}'$. Because $|\mathcal{F}| = |\mathcal{F}'|$, in each case, and because complementation reverses inclusion, verification of the lemma's claim is straightforward.

□

Theorem 3.4. *Let G denote the complete $k-$partite graph $K(m,2,\ldots,2)$ and $k \leq r \leq 2k-2$. If $m \geq C(r,\lceil r/2\rceil,r-k+1).C(r,\lfloor r/2\rfloor,r-k+1)$ then $ch(K(m,2,\ldots,2)) \geq r+1$.*

Proof. Let A, B be disjoint $r-$sets. Denote by $V_1, V_2, \ldots, V_k$ the parts of G, with $V_1 = \{x_1,\ldots,x_m\}$, $V_i = \{u_i,v_i\}$, $i = 2,\ldots,k$. Start defining a list assignment to G by assigning A to each u_i and B to each v_i. By Lemma 3.2, we can find a family $\mathcal{F}_1$ of $r - \lfloor r/2\rfloor = \lceil r/2\rceil-$subsets of A and a family $\mathcal{F}_2$ of $r - \lceil r/2\rceil = \lfloor r/2\rfloor-$subsets of B such that every $r-(r-k+1) = (k-1)-$subset of A contains some set in $\mathcal{F}_1$, and every $(k-1)-$subset of B contains some set in $\mathcal{F}_2$, and $|\mathcal{F}_1| = C(r,\lceil r/2\rceil,r-k+1)$, $|\mathcal{F}_2| = C(r,\lfloor r/2\rfloor,r-k+1)$. Make $|\mathcal{F}_1|.|\mathcal{F}_2|$ lists of length r by forming the unions $F_1 \cup F_2$, $F_1 \subseteq \mathcal{F}_1$, $F_2 \subseteq \mathcal{F}_2$. If $m \geq |\mathcal{F}_1|.|\mathcal{F}_2|$ then we can endow V_1 with these lists. Then for every proper coloring of $G \setminus V_1$, some list on V_1 is in the set of colors used. Hence $ch(K(m,2,\ldots,2)) > r$ for $m \geq C(r,\lceil r/2\rceil,r-k+1).C(r,\lfloor r/2\rfloor,r-k+1)$.

□

Corollary 3.4.1. *If $m \geq \binom{2k-2}{k-1}^2$ then $ch(K(m,2,\ldots,2)) = 2k-1$.*

Proof. For $r = 2(k-1)$, if $m \geq C(2k-2,k-1,k-1)^2 = \binom{2k-2}{k-1}^2$ then $ch(K(m,2,\ldots,2)) \geq 2k-1$ by Theorem 3.4. Further, using Proposition 3.1, we establish that $ch(K(m,2,\ldots,2)) = 2k-1$.

□

Corollary 3.4.2. *If* $m \geq \binom{2k-2}{k-1}^2$ *then* $\phi(K(m,2,\ldots,2)) \geq k-1$.

Proof. By Proposition 2.1 and Corollary 3.4.1, $\phi(G) \geq 2k-1-k = k-1$. □

Remark: Taking $r = k$ in Theorem 3.4 and using the easily seen fact that $C(n,m,1) = \lceil \frac{n}{m} \rceil$, for $n \geq m \geq 1$, we obtain that if $k \geq 2$ is even and $m \geq 4$ then $ch(K(m,2,\ldots,2)) \geq k+1$, which, by Corollary B and Theorem D, is a tight result. When k is odd, $k \geq 3$, we obtain that $ch(K(m,2,\ldots,2)) \geq k+1$ for $m \geq 6$, which is not quite tight, by Theorem E.

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