Graphs with unique minimum paired-dominating set*

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Abstract Let $G = (V, E)$ be a graph without isolated vertices. A set $D \subseteq V$ is a paired-dominating set if D is a dominating set of G and the induced subgraph $G[D]$ has a perfect matching. In this paper, a characterization is given for block graphs with a unique minimum paired-dominating set. Furthermore, a constructive characterization is also given for trees with a unique minimum paired-dominating set.

Keywords: Block graph; Domination; Paired-dominating set; Tree

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1 Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices. For a vertex $v \in V$, the *open neighborhood* of v is defined as $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is defined as $N[v] = N(v) \cup \{v\}$. For

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$A \subseteq V$, $N(A) = \bigcup_{x \in A} N(x)$ and $N[A] = N(A) \cup A$. The distance between u and v, denoted by $d(u, v)$, is the minimum length of a path between u and v. For a subset $S \subseteq V$, the subgraph of G induced by the vertices in S is denoted by $G[S]$. A *matching* in a graph G is a set of pairwise nonadjacent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M. For a subset $D \subseteq V$ and a vertex $x \in D$, the set $P(x, D) = N[x] - N[D - \{x\}]$ is the *private neighborhood* of x with regard to D and a vertex $y \in P(x, D)$ is called a *private neighbor* of x with regard to D. Some other notation and terminology not introduced here can be found in [18].

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [10, 11]. A *dominating set* of $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum size of a dominating set of G. A dominating set D of G with size $\gamma(G)$ is called a *γ-set*. A set $D \subseteq V$ is a *paired-dominating set* of G if D is a dominating set of G and $G[D]$ has a perfect matching. The *paired-domination number* $\gamma_p(G)$ is the minimum size of a paired-dominating set of G. A paired-dominating set D of G with size $\gamma_p(G)$ is called a *γ_p-set* of G. Let D be a paired-dominating set of G and M be a perfect matching in $G[D]$, we say u, v are paired in D if $u, v \in D$ and $uv \in M$. We also say u(v, respectively) is the paired vertex of v(u, respectively) in D. Paired-domination was introduced by Haynes and Slater [12, 13] and there are lots of results on this problem [3, 4, 5, 6, 7, 8, 14, 15, 16, 17].

Chordal graphs are raised in the theory of perfect graphs and the subclasses of chordal graphs are of most interesting in the study of many graphs optimization problem. Block graphs, which contains trees, is an important subclass of chordal graphs and there are many results on variations of domination in block graphs. (see [1, 2, 9, 19]). In this paper, we will give a characterization for block graphs with a unique γ_p-set and a constructive characterization for trees with a unique γ_p-set.

2 Characterization of block graphs with unique γ_p-set

In a graph $G = (V, E)$, a vertex x is a *cut-vertex* if there are more connected components in $G - x$ than that in G. A *block* of G is a maximal connected subgraph of G without cut-vertices. A *block graph* is a connected graph whose blocks are complete graphs. If every block is K_2, then it is a tree. Every block graph not isomorphic to complete graph has at least two *end blocks*, which are blocks with only one cut-vertex.

Proposition 1 *Any paired-dominating set of a block graph G contains at least one vertex in each block. Any γ_p-set of G contains at most two vertices in each end block.*

Proposition 2 *If D is a paired-dominating set of a block graph G such that $P(x, D) \neq \emptyset$ for each vertex $x \in D$, then each vertex in D is a cut-vertex of G. Furthermore, for each end block B, $D \cap V(B) = \{x\}$, where x is the cut-vertex in B.*

Proof Suppose to the contrary that there is a vertex $x \in D$ which is not a cut-vertex and x is contained in block B. Let M be a perfect matching of $G[D]$ and y be the paired vertex of x in D, then $y \in V(B)$. Since x is not a cut-vertex, we have $N[x] = V(B)$. On the other hand, $N[y] = V(B)$ if y is not a cut-vertex, and $V(B) \subset N[y]$ if y is a cut-vertex. In any case, $N[x] = V(B) \subseteq N[y] \subseteq N[D - \{x\}]$. Hence $P(x, D) = N[x] - N[D - \{x\}] = \emptyset$, a contradiction. By Proposition 1, D only contains the cut-vertex in each end block. □

Theorem 3 *Let $G = (V, E)$ be a block graph of order $n \geq 3$. D is a unique γ_p-set of G if and only if D is a paired-dominating set of G such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$.*

Proof $\Rightarrow$: Suppose D is a γ_p-set of G and there is a vertex $u_1 \in D$ with $P(u_1, D) = \emptyset$. Let M be a perfect matching in $G[D]$ and $v_1 \in D$ be the paired vertex of u_1. If $N(v_1) - D \neq \emptyset$, let $w \in N(v_1) - D$, then $(D - \{u_1\}) \cup \{w\}$ is also a γ_p-set of G as $P(u_1, D) = \emptyset$. It is a contraction to the uniqueness of D. Hence, we assume that $N(v_1) \subseteq D$. Let

$u_1, v_1, u_2, v_2, \cdots, u_k, v_k$ $(k \geq 2)$ be a maximal vertex sequence such that: (1) u_i, v_i are paired in D for $1 \leq i \leq k$; (2) $v_i u_{i+1} \in E$ for $1 \leq i \leq k-1$; (3) $N(v_i) \subseteq D$ for $1 \leq i \leq k-1$. So, either $N(v_k)$ is a subset of $\{u_1, v_1, u_2, v_2, \cdots, u_k\}$ or there is a vertex $w \in N(v_k)$ with $w \notin D$. For the former case, $D - \{u_1, v_k\}$ is a smaller paired-dominating set of G, a contradiction. For the later case, $(D - \{u_1\}) \cup \{w\}$ is also a γ_p-set of G, a contraction.

$\Leftarrow$: Let b be the number of blocks and c be the number of cut-vertices in G. By Proposition 2, we know that $b \geq 3$ and $c \geq 2$. We use induction on b. If $b = 3$, then there are two cut-vertices, say v_1, v_2, in G. It is easy to check that $D = \{v_1, v_2\}$ is a unique γ_p-set of G and $P(x, D) \neq \emptyset$ for every vertex $x \in D$.

Suppose G is a block graph with $b \geq 4$ blocks and D is a paired-dominating set of G such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$. Assume now that the assertion holds for smaller value of b. We first give the following claim.

Claim 1 *We may assume that any cut-vertex in G is contained in at most one end block.*

Proof Suppose there are two end blocks B and B' containing a cut-vertex u. Let $G' = G - (V(B') - \{u\})$ be a block graph with $b-1$ blocks. By Proposition 2, D only contains the vertex u in blocks B and B'. Hence, D is also a paired-dominating set of G' such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$. By inductive hypothesis, D is a unique γ_p-set of G'. Let D' be a γ_p-set of G. If $D' \cap (V(B') - \{u\}) = \emptyset$, then D' is also a γ_p-set of G'. Hence, $D' = D$. Assume that $D' \cap (V(B') - \{u\}) \neq \emptyset$. Let v $(\neq u)$ be a vertex in B and $w \in D' \cap (V(B') - \{u\})$. If $u \notin D'$, then $|D' \cap (V(B) \cup V(B'))| \geq 4$, hence $D'' = (D' - (V(B) \cup V(B'))) \cup \{u, v\}$ is a smaller paired-dominating set of G, a contradiction. If $u \in D'$, then u is paired with w. Moreover, $D' \cap (V(B) - \{u\}) = \emptyset$. Thus $D'' = (D' - \{w\}) \cup \{v\}$ is also a γ_p-set of G'. Hence, $D'' = D$. However, v is not a cut-vertex in G, a contradiction to Proposition 2. □

Let k be the diameter of G and $d(u, v) = k$. Suppose P : $u = v_0, v_1, \cdots, v_k = v$ is a path in G with length k. Then v_0, v_1 are in end blocks and v_i $(2 \le i \le k-1)$ are cut-vertices. Furthermore, any block contains at most two consecutive vertices in P. Let B_i $(0 \le i \le k-1)$ be the block containing v_i and v_{i+1}. Then B_0 and B_{k-1} are end blocks with cut-vertices v_1 and v_{k-1}, respectively. It is obvious that $k \ge 3$ as $c \ge 2$. If $k = 3$, then $D = \{v_1, v_2\}$ is a unique paired-dominating set of G and $P(x, D) \ne \emptyset$ for every vertex $x \in D$. Thus in the following proof, we assume that $k \ge 4$. By Claim 1, we may assume there are exactly two blocks B_0 and B_1 containing v_1.

Case 1: There is exactly one cut-vertex u_1 in B_1 except v_1 and v_2.

Let B be the end block containing u_1 except B_1. In this case, D contains exactly u_1, v_1 in blocks B and B_0. Hence, we can assume that u_1 and v_1 are paired in D.

If v_2 is only contained in B_1 and B_2, then v_2 is not in D since otherwise either v_2 or its paired vertex of has no private neighbor with regard to D. Let $G' = G - (V(B_0) \cup V(B_1) \cup V(B))$ and $D' = D - \{u_1, v_1\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x, D') \ne \emptyset$ for every vertex $x \in D'$. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Next we prove that D is a unique γ_p-set of G. Suppose D_0 is a γ_p-set of G. If D_0 does not contain v_2, then $D_0 \cap (V(B_0) \cup V(B_1) \cup V(B)) = \{v_1, u_1\}$ since D_0 is a γ_p-set. $D_0' = D_0 - \{v_1, u_1\}$ is also a γ_p-set of G', and hence $D_0' = D'$. It implies that $D_0 = D$. Assume now that $v_2 \in D_0$. By Proposition 1, $|D_0 - V(G')| = 3$ or 4 since D_0 is a γ_p-set of G. $|D_0 - V(G')| = 3$ implies that $D_0 - V(G') = \{u_1, v_1, v_2\}$. Hence, the paired vertex of v_2 in D_0, say w, is in block B_2. Obviously, w has a neighbor $y \notin D_0$. For otherwise, both v_2 and w have no private neighbor with regard to D_0 hence $D_0 - \{v_2, w\}$ is a smaller paired-dominating set of G, a contradiction. Now let $D_0' = (D_0 - \{v_1, u_1, v_2\}) \cup \{y\}$. Then, D_0' is a γ_p-set of G' and $P(y, D_0') = \emptyset$. Since D' is a unique γ_p-set of G', we know that $D_0' = D'$, and hence each vertex in D_0' has a private neighbor with regard to D_0'. It is a contradiction. If $|D_0 - V(G')| = 4$, without loss of generality, we assume that $D_0 - V(G') = \{v_1, u_1, u, v_2\}$, where $u \in V(B_1)$ $(V(B_0), V(B))$ and its paired vertex is v_2 (v_1, u_1). Hence, v_2 has a neighbor $w \in V(B_2) - D_0$. For otherwise, both v_2 and u have no private neighbor with regard to D_0.

Hence $D_0-\{v_2,u\}$ is a smaller paired-dominating set of G, a contradiction. Let $D_1=(D_0-\{u\})\cup\{w\}$. Now D_1 is a γ_p-set of G with $|D_1-V(G')|=3$. With the same arguments, we will get a contradiction. Therefore, D is a unique γ_p-set of G.

Suppose that there is a block B' containing v_2 except B_1 and B_2. If B' is an end block, then $v_2\in D$ by Proposition 2. If B' is not an end block, let B'' be an end block such that $V(B')\cap V(B'')=\{w\}$, then $w\in D$ by Proposition 2. In any case, v_2 is dominated by $D-\{u_1,v_1\}$. Let $G'=G-(V(B_0)\cup V(B_1)\cup V(B)-\{v_2\})$ and $D'=D-\{u_1,v_1\}$. Then D' is a paired-dominating set of G' such that $P(x,D')\neq\emptyset$ for each vertex $x\in D'$ and G' is a block graph with less than b blocks. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. By Proposition 1, $|D_0-V(G')|=2$ or 3. $|D_0-V(G')|=2$ implies that $D_0-V(G')=\{v_1,u_1\}$. So, $D_0'=D_0-\{v_1,u_1\}$ is also a γ_p-set of G', and hence $D_0'=D'$. It implies that $D_0=D$. If $|D_0-V(G')|=3$, then $v_2\in D_0$ and let $D_0-V(G')=\{v_1,u_1,u\}$, where $u\in V(B_1)$ $(V(B_0),V(B))$ and its paired vertex is v_2 (v_1,u_1). With the same arguments as above, we know that v_2 has a neighbor $w\in V(G')-D_0$. Let $D_0'=(D_0-\{u,v_1,u_1\})\cup\{w\}$. Similarly, we have D_0' is also a γ_p-set of G' and hence $D_0'=D'$. However, w has no private neighbor with regard to D_0', a contradiction. Therefore, D is a unique γ_p-set of G.

Case 2: There are more than one cut-vertices in B_1 except v_1 and v_2.

Take two cut-vertices u_1, u_2 and let $B(B'$, respectively) be the end block containing $u_1(u_2$, respectively). Let $G'=G-(V(B_0)\cup V(B))$ and $D'=D-\{u_1,v_1\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x,D')\neq\emptyset$ for every vertex $x\in D'$. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. We know that $D_0-V(G')$ has two or three vertices. If $|D_0-V(G')|=2$, then $D_0-V(G')=\{u_1,v_1\}$. Let $D_0'=D_0-\{u_1,v_1\}$. Then D_0' is a γ_p-set of G', and hence $D_0'=D'$. This implies that $D_0=D$.

If $|D_0-V(G')|=3$, without loss of generality, we assume that $|D_0\cap V(B_0)|=2$ and hence $D_0\cap V(B)=\{u_1\}$. We claim that $D_0\cap V(B')=\{u_2\}$. Suppose to the contrary that $|D_0\cap V(B')|=2$, it is easy to check $(D_0-(V(B_0)\cup V(B')))\cup\{v_1,u_2\}$ is a smaller paired-dominating set of G, a contradiction. Now let $D_1 = (D_0 - V(B_0)) \cup \{v_1, w\}$, where $w(\neq u_2)$ is a vertex in B'. So, D_1 is also a γ_p-set of G and $D_1' = D_1 - \{u_1, v_1\}$ is a γ_p-set of G'. Hence, $D_1' = D'$ and each vertex in D_1' is a cut-vertex by Proposition 2. This contradicts that $w \in D_1'$ is not a cut-vertex. Therefore, $D_0 = D$.

Case 3: There is no cut-vertex in B_1 except v_1 and v_2.
In this case, $v_1, v_2 \in D$ and they are paired in D by Proposition 2.

Case 3.1: There is another block B containing v_2 except B_1 and B_2.

If there is exactly one cut-vertex in B except v_2, then it is impossible to find D such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$. If there are more than one cut-vertices in B except v_2, then we can look B as B_1 and discuss it same as Case 1 or Case 2. So we can assume that every block containing v_2 except B_1 and B_2 is an end block. By Claim 1, we may assume there is exactly one end block B containing v_2.

If there exists cut-vertex in B_2 except v_2 and v_3. Let $w = w_1, w_2, \cdots, w_a$ be cut-vertices in B_2 except v_2 and v_3. By Proposition 2, it is easy to see that w_i $(1 \leq i \leq a)$ is dominated by $D - \{v_1, v_2\}$. Let $G' = G - (V(B_0) \cup V(B_1) \cup V(B) \cup (V(B_2) - \{w_1, \cdots, w_a, v_3\}))$ and $D' = D - \{v_1, v_2\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. Then $|D_0 - V(G')| = 2$ or 3. If $|D_0 - V(G')| = 2$, then $D_0 - V(G') = \{v_1, v_2\}$. Let $D_0' = D_0 - \{v_1, v_2\}$. Then D_0' is a γ_p-set of G', and hence $D_0' = D'$. This implies $D_0 = D$. If $|D_0 - V(G')| = 3$, then there is a vertex $x \in D_0 \cap (V(B_2) - V(G'))$ such that its paired vertex $y \in V(G') \cap V(B_2)$. Since D_0 is a γ_p-set of G, there exists a neighbor y' of y such that $y' \in V(G') - D_0$, as otherwise, $(D_0 - (V(G) - V(G')) - \{y\}) \cup \{v_1, v_2\}$ is a smaller paired-dominating set of G. Let $D_0' = (D_0 - (V(G) - V(G'))) \cup \{y'\}$. Then D_0' is a γ_p-set of G'. Hence we have $D_0' = D'$. However, $P(y', D') = \emptyset$. It is a contradiction. Therefore, $D_0 = D$.

Suppose that there is no cut-vertex in B_2 except v_2 and v_3. If there is no block containing v_3 except B_2 and B_3, then let $G' = G - (V(B_0) \cup V(B_1) \cup V(B_2) \cup V(B))$ and let $D' = D - \{v_1, v_2\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. If $v_3 \notin D_0$, then $D_0 - V(G') = \{v_1, v_2\}$. $D_0' = D_0 - \{v_1, v_2\}$ is also a γ_p-set of G'. Hence, $D_0' = D'$, this implies $D_0 = D$. If $v_3 \in D_0$, then $|D_0 - V(G')| = 3$ or 4. $|D_0 - V(G')| = 3$ implies that $D_0 - V(G') = \{v_1, v_2, v_3\}$. Let w be the paired vertex of v_3 in D_0, then $w \in V(B_3)$. Since D_0 is a γ_p-set of G, there is a neighbor w' of w such that $w' \notin D_0$. Then $D_0' = (D_0 - \{v_1, v_2, v_3\}) \cup \{w'\}$ is a γ_p-set of G'. Thus $D_0' = D'$. However, $P(w', D_0') = \emptyset$. It is a contradiction. If $|D_0 - V(G')| = 4$, without loss of generality, we assume $D_0 - V(G') = \{v_1, v_2, u, v_3\}$. Also, there is a neighbor w' of v_3 such that $w' \in V(G') - D_0$. Let $D_1 = (D_0 - \{u\}) \cup \{w'\}$ is a γ_p-set of G. In this case, $|D_1 - V(G')| = 3$. With the same argument, we can also obtain a contradiction. Therefore, D is a unique γ_p-set of G. Next, we assume that there is a block containing v_3 except B_2 and B_3. Let A be any of them. If A is an end block, then $v_3 \in D$ by proposition 2. If there is an end block A' such that $V(A) \cap V(A') = \{w\}$ $(w \neq v_3)$, then $w \in D$ by Proposition 2. Assume that A' is a block such that $V(A) \cap V(A') = \{w\}$ $(w \neq v_3)$ and A' is not an end block. If there are more than one cut-vertex in A' except w, we can discuss it same as Case 1 or Case 2. Thus we may assume there is exactly one cut-vertex w' in A' except w. Then $w, w' \in D$ by Proposition 2. In any case, v_3 is dominated by $D - \{v_1, v_2\}$. Let $G' = G - (V(B_0) \cup V(B_1) \cup V(B) \cup (V(B_2) - \{v_3\}))$ and $D' = D - \{v_1, v_2\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. Then $|D_0 - V(G')| = 2$ or 3. $|D_0 - V(G')| = 2$ implies that $D_0 - V(G') = \{v_1, v_2\}$. $D_0' = D_0 - \{v_1, v_2\}$ and it is a γ_p-set of G'. Thus $D_0' = D'$, which implies $D_0 = D$. If $|D_0 - V(G')| = 3$, without loss of generality, we assume $D_0 - V(G') = \{v_1, v_2, u\}$ and $v_3 \in D_0$. Also, there is a neighbor w' of v_3 such that $w' \in V(G') - D_0$. Let $D_0' = (D_0 - \{v_1, v_2, u\}) \cup \{w'\}$ and it is a γ_p-set of G'. Hence, $D_0' = D'$. However, $P(w', D_0') = \emptyset$. It is a contradiction. Therefore, $D_0 = D$.

Case 3.2: There is no block containing v_2 except B_1 and B_2.
If there is a cut-vertex w in B_2 except v_2 and v_3, let B be a block containing

w except B_2. Since $\{v_1, v_2\} \subseteq D$ and $P(v_2, D) \neq \emptyset$, we know that $w \notin D$. Thus there are at least two cut-vertices in B except w. We look B as B_1 and discuss it same as Case 1 or Case 2. So we may assume there is no cut-vertex in B_2 except v_2 and v_3. As $P(v_2, D) \neq \emptyset$, it follows that $v_3 \notin D$.

If there is no block containing v_3 except B_2 and B_3, then let $G' = G - (V(B_0) \cup V(B_1) \cup V(B_2))$ and $D' = D - \{v_1, v_2\}$. Then G' is a block graph with less than $b - 1$ blocks and D' is a paired-dominating set of G' such that $P(x, D') \neq \emptyset$ for every vertex in D'. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. If $v_3 \notin D_0$, then $D_0 - V(G') = \{v_1, v_2\}$. $D'_0 = D_0 - \{v_1, v_2\}$ and it is also a γ_p-set of G'. Hence, $D'_0 = D_0$, this implies $D_0 = D$. If $v_3 \in D_0$, then $|D_0 - V(G')| = 3$ or 4. With the similar argument to Case 3.1, it is easy to prove $D_0 = D$.

If there is a block B containing v_3 except B_2 and B_3, then B is not an end block by $v_3 \notin D$. Similar to Case 3.1, we can prove that v_3 is dominated by $D - \{v_1, v_2\}$. Let $G' = G - (V(B_0) \cup V(B_1) \cup V(B_2) - \{v_3\})$ and $D' = D - \{v_1, v_2\}$. Then G' is a block graph with less than b blocks and D' is a paired-dominating set of G' such that $P(x, D') \neq \emptyset$ for every vertex in D'. Applying inductive hypothesis to G', D' is a unique γ_p-set of G'. Let D_0 be a γ_p-set of G. Then $|D_0 - V(G')| = 2$ or 3. With the similar argument to Case 3.1, it is easy to prove $D_0 = D$. □

3 Constructive characterization of trees with unique γ_p-set

Let $T = (V, E)$ be a tree with vertex set V and edge set E. A vertex of T is said to be a *support vertex* if it is adjacent to at least one leaf (i.e., a vertex with degree one).

To provide a constructive characterization of trees with unique γ_p-set, we describe a procedure to build a family $\mathscr{F}$ of labeled trees as follows. The label of a vertex v is also called its status denoted by $sta(v)$. There are two kinds of status , say A and B, used to label the tree. We call a vertex u is *strong* if and only if $sta(u) = A$ and it is the only private neighbor of a vertex labeled B with regard to the sets consisting of all vertices labeled

B. Let $\mathscr{F}$ be the family of labeled trees such that:

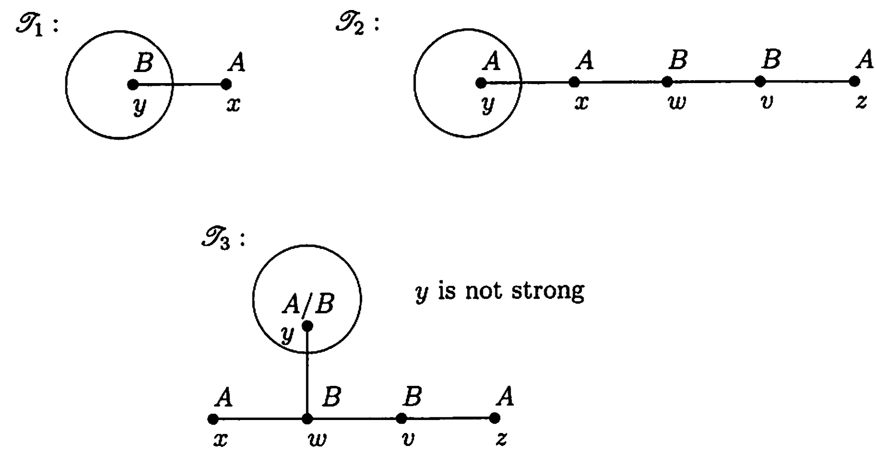
(i) it contains P_4 in which two leaves have status A, the two support vertices have status B.

(ii) it is closed under the three operations $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$, which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the attacher.

Operation $\mathscr{T}_1$: Assume $sta(y) = B$. Then add a vertex x and edge xy. Let $sta(x) = A$.

Operation $\mathscr{T}_2$: Assume $sta(y) = A$. Then add a path x, w, v, z and edge xy. Let $sta(x) = sta(z) = A$ and $sta(w) = sta(v) = B$.

Operation $\mathscr{T}_3$: Assume y is not strong. Then add a path x, w, v, z and edge yw. Let $sta(x) = sta(z) = A$, $sta(w) = sta(v) = B$.



The three operations are illustrated in the above figure. Let $\mathscr{T} = \{T \mid T$ is a tree with a unique γ_p-set of $T\}$. Suppose that $T \in \mathscr{F}$. Let $B(T) = \{v \in V(T) \mid sta(v) = B\}$ and $A(T) = \{v \in V(T) \mid sta(v) = A\}$.

Observation 4 *Let $T \in \mathscr{F}$ and $v \in V(T)$.*

(a) If $sta(v) = A$, then v is adjacent to at least one vertex in $B(T)$.

(b) If $sta(v) = B$, then v has at least one private neighbor in $A(T)$ with regard to $B(T)$. Moreover, v has at least one neighbor in $B(T)$.

(c) *If v is a leaf, then $sta(v) = A$.*

(d) *If v is a support vertex, then $sta(v) = B$.*

(e) *$G[B(T)]$ has exactly one perfect matching.*

Proof We only give the proof of (e), others are obvious. Let $s(T)$ be the number of operations required to construct T. We use induction on $s(T)$. If $s(T) = 0$, it is obviously true. For all trees $T' \in \mathscr{F}$ with $s(T') < k$ ($k \geq 1$ is an integer), we assume $G[B(T')]$ has exactly one perfect matching. Let $T \in \mathscr{F}$ with $s(T) = k$. Then T is obtained from T' by one of Operation $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$. By inductive hypothesis, $G[B(T')]$ has exactly one perfect matching, say M'. If T is obtained from $T'\mathscr{F}$ by Operation $\mathscr{T}_1$, then $B(T) = B(T')$. So $G[B(T)]$ has exactly one perfect matching, i.e., M'. If T is obtained from a tree $T' \in$ by Operation $\mathscr{T}_2$ or $\mathscr{T}_3$, then $B(T) = B(T') \cup \{w, v\}$. Let M be a perfect matching in $G[B(T)]$. Since v have to paired with w. So $wv \in M$. $M - \{vw\}$ is a perfect matching in $G[B(T')]$. Hence, $M' = M - \{vw\}$, this implies $M = M' \cup \{vw\}$. So M is the only perfect matching in $G[B(T)]$. □

Lemma 5 $\mathscr{F} \subset \mathscr{T}$.

Proof Let $T \in \mathscr{F}$, we want to show $T \in \mathscr{T}$. By Observation 4 (a) and (e), $B(T)$ is a paired-dominating set of T. By Observation 4 (b) and Theorem 3, $B(T)$ is a unique γ_p-set of T. So $T \in \mathscr{T}$. □

Lemma 6 *Let T be a tree with order at least 3, if $T \in \mathscr{T}$, then $T \in \mathscr{F}$.*

Proof We use induction on n, the order of T. If $n \leq 4$, then $T \cong P_4$ as $T \in \mathscr{T}$. Let $T \in \mathscr{T}$ be a tree with order n and D be a unique γ_p-set of T. Assume now that $T' \in \mathscr{F}$ for all tree $T' \in \mathscr{T}$ with order $4 \leq n' \leq n$. By Theorem 3, D is a paired-dominating set such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$. Let $P: v_0, v_1, \cdots, v_k$ be a longest path in T. Thus v_0, v_k are leaves. T can be looked as a rooted tree at v_k. By Proposition 2, $v_0, v_k \notin D$ and $v_1 \in D$. Since $T \in \mathscr{T}$, then $k \geq 3$. If $k = 3$, then T is a double star (a tree obtained by adding an edge between centers of two stars) and T can be obtained from P_4 by a series Operations $\mathscr{T}_1$. So in the following proof, we assume $k \geq 4$.

If there is a support vertex w such that there are at least two leaves, say u and v, in its neighborhood. Let $T' = T - u$, then D is a paired-dominating set of T' such that $P(x, D) \neq \emptyset$ for every vertex $x \in D$. By Theorem 3, D is a unique γ_p-set of T', that is $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. Since w is a support vertex in T', by Observation 4 (d), $sta(w) = B$. Then T can be obtained from T' by one Operation $\mathscr{T}_1$. Therefore $T \in \mathscr{F}$. In the following proof, we may assume that every support vertex is adjacent to exactly one leaf. In particular, we may assume $d(v_1) = 2$.

Case 1: $d(v_2) \geq 3$

Let w be any neighbor of v_2 except v_1 and v_3. If w is a support vertex, then $w \in D$ by Proposition 2. In this case, at least one vertex in $\{v_1, w, v_1', w'\}$, where v_1' (w', respectively) is the paired vertices of v_1 (w, respectively), have no private neighbors with regard to D. So any neighbor of v_2 except v_1 and v_3 is leaf. Since v_2 is a support vertex, we may assume that it is adjacent to exactly one leaf, say w. In this case, $v_1, v_2 \in D$ and they are paired in D.

Case 1.1: $d(v_3) \geq 3$ and v_3 is a support vertex.

Let $T' = T - \{v_0, v_1, v_2, w\}$ and $D' = D - \{v_1, v_2\}$. Then D' is a paired-dominating set of T' and $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. By Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. Since v_3 is a support vertex in T', so $sta(v_3) = B$ by Observation 4 (d). Thus v_3 is not strong. T can be obtained from T' by one Operation $\mathscr{T}_3$. Therefore $T \in \mathscr{F}$.

Case 1.2: $d(v_3) \geq 3$ and v_3 is not a support vertex.

Let $u_1, u_2, \cdots, u_a$ be neighbors of v_3 except v_2 and v_4. If there are two vertices u_i, u_j ($1 \leq i \neq j \leq a$) such that its neighbors except v_3 are all leaves. Then $u_i, u_j \in D$ and at least one vertex in $\{u_i, u_j, u_i', u_j'\}$, where u_i' (u_j', respectively) is the paired vertex of u_i (u_j, respectively), have no private neighbors with regard to D, a contradiction. If there is exactly one vertex, say u_1, such that its neighbors except v_3 are leaves. Let $T' = T - \{v_0, v_1, v_2, w\}$ and $D' = D - \{v_1, v_2\}$. Then, Similarly, D' is a paired-dominating set of T' and $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. So by Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$.

Since u_1 is a support vertex in T', $sta(u_1) = B$. And $sta(v) = A$ for all $v \in N(u_1) - \{v_3\}$ as v is a leaf in T' and Observation 4 (c). By Observation 4 (b), $sta(v_3) = B$. Thus v_3 is not strong. T can be obtained from T' by one Operation $\mathscr{T}_3$. Therefore $T \in \mathscr{F}$. If there is no vertex in $\{u_1, u_2 \cdots, u_a\}$ such that its neighbors except v_3 are all leaves. Let x_i $(1 \le i \le a)$ be a neighbor of u_i except v_3 which is not a leaf. Since P is a longest path in T, x_i is a support vertex. For any u_i, if there exists another neighbor x_i' $(\neq v_3)$ which is also a support vertex. Then at least one vertex in $\{x_i, x_i^1, x_i', x_i''\}$, where x_i^1 (x_i'', respectively) is the paired vertex of x_i (x_i', respectively), have no private neighbors with regard to D. So for any u_i $(1 \le i \le a)$, there exists exactly one neighbor x_i $(\neq v_3)$ such that x_i is a support vertex. In this case, $\{u_1, x_1, u_2, x_2 \cdots, u_a, x_a\} \subseteq D$ and there must exist one leaf adjacent to u_i for $i = 1, 2, \cdots, a$. Let $T' = T - \{v_0, v_1, v_2, w\}$ and $D' = D - \{v_1, v_2\}$. Similarly, D' is a paired-dominating set of T' and $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. So by Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. Since x_i is a support vertex, $sta(x_i) = B$. Also, for any $v \in N(x_i) - \{u_i\}$, $sta(v) = A$. So $sta(u_i) = B$. Since u_i $(1 \le i \le a)$ is adjacent to at least one leaf, so v_3 is not strong in T'. Thus T can be obtained from T' by one Operation $\mathscr{T}_3$. Therefore $T \in \mathscr{F}$.

Case 1.3: $d(v_3) = 2$.
Let $T' = T - \{v_0, v_1, v_2, v_3, w\}$ and $D' = D - \{v_1, v_2\}$. Then D' is a paired-dominating set of T' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. By Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. If $sta(v_4) = A$, then T can be obtained from T' by one Operation $\mathscr{T}_2$ and one Operation $\mathscr{T}_1$. Therefore $T \in \mathscr{F}$. If $sta(v_4) = B$, we use one Operation $\mathscr{T}_1$ to T', the attacher is v_4, we get T''. Though $sta(v_3) = A$, v_3 is not strong. Then T can be obtained from T' by one Operation $\mathscr{T}_3$. Therefore $T \in \mathscr{F}$.

Case 2: $d(v_2) = 2$.
In this case, $v_1, v_2 \in D$ and they are paired in D. v_3 is the only private neighbor of v_2. If $d(v_3) \ge 3$, then v_3 must be dominated by one of its neighbors other than v_2. It is a contradiction. So $d(v_3) = 2$. Since $T \in \mathscr{T}$, $d(v_4) \ge 2$.

Case 2.1: $d(v_4) = 2$.
Let $T' = T - \{v_0, v_1, v_2, v_3\}$ and $D' = D - \{v_1, v_2\}$. Then D' is a paired-dominating set of T' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. By Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. As v_4 is a leaf in T', by Observation 4 (c), $sta(v_4) = A$. Then T can be obtained from T' by one Operation $\mathscr{T}_2$. Therefore $T \in \mathscr{F}$.

Case 2.2: $d(v_4) \geq 3$.
If v_4 is a support vertex or one of its neighbors is a support vertex, then $v_4 \in D$. This contradicts that v_3 is the only private neighbor of v_2. Let $u_1, u_2, \cdots, u_a$ be neighbors of v_4 other than v_3 and v_5 and P_i $(1 \leq i \leq a)$ be the longest path start at v_4 through edge v_4u_i. Then the length of P_i is three or four. Let $P_1, P_2, \cdots, P_b$ be the path of length 3 and $P_{b+1}, \cdots, P_a$ be the path of length 4. Let T_{u_i} be a rooted subtree at u_i. If there is a rooted subtree T_{u_i} $(b + 1 \leq i \leq a)$ which is not a path, then we discuss it same as Case 1. So $T_{u_i} \cong P_4$ for $b + 1 \leq i \leq a$. If $b = 0$, then let $T' = T - (V(T_{v_4}) - \{v_4\})$ and $D' = D \cap V(T')$. Then D' is a paired-dominating set of T' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. By Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. Since v_4 is a leaf in T', so $sta(v_4) = A$. T can be obtained from T' by $a+1$ Operations $\mathscr{T}_2$. Therefore $T \in \mathscr{F}$. If $b > 0$, Let $T' = T - (V(T_{v_3}) \cup \bigcup_{i=b+1}^{a} V(T_{u_i}))$ and $D' = D \cap V(T')$. Then D' is a paired-dominating set of T' such that $P(x, D') \neq \emptyset$ for every vertex $x \in D'$. By Theorem 3, $T' \in \mathscr{T}$. Applying inductive hypothesis to T', $T' \in \mathscr{F}$. For any vertex u_i $(1 \leq i \leq b)$, there is exactly one support vertex in $N(u_i) - \{v_4\}$, as $T' \in \mathscr{T}$. Let x_i be such a neighbor. Then $sta(x_i) = B$. By Observation 4 (b) and (c), $sta(u_i) = B$ for $1 \leq i \leq b$. If $sta(v_4) = B$, then By Observation 4 (e), $sta(v_5) = B$. In this case, v_4 has no private neighbor in $A(T')$, and hence $sta(v_4) = A$. Then T can be obtained from T' by $a - b + 1$ Operations $\mathscr{T}_2$. Therefore $T \in \mathscr{F}$. □

By Lemmas 5, 6, we get the following Theorem.

Theorem 7 $\mathscr{T} = \mathscr{F} \cup \{P_2\}$.

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