The 2-color Rado Number of $x_1 + x_2 + \cdots + x_{m-1} = ax_m$, II

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Abstract

In the first installment of this series, we proved that, for every integer $a \geq 3$ and every $m \geq 2a^2 - a + 2$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = ax_m$$

is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$. Here we obtain the best possible improvement of the bound on m. We prove that if $3|a$ then the 2-color Rado number is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$ when $m \geq 2a+1$ but not when $m = 2a$, and that if $3 \nmid a$ then the 2-color Rado number is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$ when $m \geq 2a + 2$ but not when $m = 2a + 1$. We also determine the 2-color Rado number for all $a \geq 3$ and $m \geq \frac{a}{2} + 1$.

1. Introduction

A special case of the work of Richard Rado [5] is that for every integer $m \geq 3$ and all positive integers $a_1, \ldots, a_m$ there exists a smallest positive integer n with the following property: for every coloring of the elements of the set $[n] = \{1, \ldots, n\}$ with two colors, there exists a solution of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = a_mx_m$$

using elements of $[n]$ that are all colored the same. (Such a solution is called *monochromatic.*) The integer n is called the *2-color Rado number* of the equation.

In 1982, Beutelspacher and Brestovansky [1] proved that for every $m \geq 3$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = x_m$$

is $m^2 - m - 1$. In 2008 Guo and Sun [2] generalized this result by proving that, for all positive integers $a_1, \ldots, a_{m-1}$, the 2-color Rado number of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m$$

is $aw^2 + w - a$, where $a = \min\{a_1, \ldots, a_{m-1}\}$ and $w = a_1 + \cdots + a_{m-1}$. In the same year, Schaal and Vestal [7] dealt with the equation

$$x_1 + x_2 + \cdots + x_{m-1} = 2x_m.$$

They proved, in particular, that for every $m \geq 6$, the 2-color Rado number is $\lceil \frac{m-1}{2} \lceil \frac{m-1}{2} \rceil \rceil$. Building on the work of Schaal and Vestal, we proved in [6] that for every $a \geq 3$ and $m \geq 2a^2 - a + 2$, the 2-color Rado number of the equation $x_1 + \cdots + x_{m-1} = ax_m$ is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$. Our main purposes here are to obtain the best possible improvement of the bound on m, and to determine the Rado number in most cases where m falls below the improved bound.

We begin by using a sharpening of the arguments in [6] to prove (in Section 3) the following result.

Theorem 1. For every integer $a \geq 3$ and every $m \geq a^2 - a + 1$, the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_{m-1} = ax_m$$

is $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$.

Notation. We will denote $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$ by $C(m, a)$, and we will denote the equation indicated in the statement of Theorem 1 by $L(m, a)$. We will denote the 2-color Rado number of $L(m, a)$ by$R_2(m, a)$.

In order to present the rest of our results efficiently, we next prove (in Section 4) the following.

Theorem 2. Suppose $a+1 \leq m \leq 2a+1$. Then $R_2(m, a) = 1$ iff $m = a+1$. If $a + 2 \leq m \leq 2a + 1$, then $R_2(m, a) \in \{3, 4, 5\}$, and we have:

$R_2(m, a) = 3$ iff $m \leq \frac{3a}{2} + 1$ and $a \equiv m - 1 \pmod 2$.

$R_2(m, a) = 4$ iff either:

(i) $m \leq \frac{3a}{2} + 1$ and $a \not\equiv m - 1 \pmod 2$, or
(ii) $m > \frac{3a}{2} + 1$ and $a \equiv m - 1 \pmod 3$.

$R_2(m, a) = 5$ iff $m > \frac{3a}{2} + 1$ and $a \not\equiv m - 1 \pmod 3$.

Theorem 2 will be useful to us in Section 5, where we obtain our final lowering of the bound on m, which is as follows.

Theorem 3. Suppose $a \geq 3$. If $3|a$ then $R_2(m,a) = C(m,a)$ when $m \geq 2a+1$ but $R_2(2a,a) = 5$ and $C(2a,a) = 4$. If $3 \nmid a$ then $R_2(m,a) = C(m,a)$ when $m \geq 2a+2$ but $R_2(2a+1,a) = 5$ and $C(2a+1,a) = 4$.

By the results of [**7**], Theorem 3 also holds when $a = 2$.

Finally, in Section 6, we prove Theorems 4 and 5, which determine all values of $R_2(m,a)$ when $\frac{a}{2} + 1 \leq m \leq a$.

Theorem 4. If $\frac{2a}{3} + 1 \leq m \leq a$, then:

for $a = 3$ we have $R_2(a,a) = 9$, and

for $a \geq 4$ we have

$R_2(m,a) = 3$ if $a \equiv m-1 \pmod 2$ and
$R_2(m,a) = 4$ if $a \not\equiv m-1 \pmod 2$.

Theorem 5. If $\frac{a}{2} + 1 \leq m < \frac{2a}{3} + 1$ (so $a \geq 4$) then:

for $a \equiv m-1 \pmod 3$ we have $R_2(m,a) = 4$, and

for $a \not\equiv m-1 \pmod 3$ we have $R_2(m,a) = 5$ *except* that

$R_2(3,4) = 10$ and $R_2(4,5) = 9$, and
$R_2(m,a) = 6$ if $10 \leq a \leq 14$ and $m = a-4$.

Conventions and definitions. In working with a fixed 2-coloring of $[n]$, we will use the colors red and blue, and we will denote by R and B, respectively, the sets of elements colored red and blue. We will call a 2-coloring of $[n]$ *bad* if it yields no monochromatic solution of $L(m,a)$.

2. Preliminary lemmas

The results of [**6**] relied on the fact that if $m \geq 2a^2 - a + 2$ then $2m - 2 \leq C(m,a)$, and therefore numbers in $[2m-2]$ can be used in producing solutions of $L(m,a)$ in $[C(m,a)]$. The improvement presented in Theorem 1 rests on showing that we can obtain the same results using $[m-1]$ instead of $[2m-2]$, and that $[m-1] \subseteq [C(m,a)]$ if $m \geq a^2 - a + 2$. (The case $m = a^2 - a + 1$ will be handled separately, in Proposition 1 below.)

Lemma 1. Suppose $a \geq 3$ and $m \geq a^2 - a + 2$. Then $m - 1 \leq C(m, a)$.

Proof. If $m \geq a^2 + 1$, then $\frac{(m-1)^2}{a^2} \geq m - 1$ and the result follows. If $a^2 - a + 2 \leq m \leq a^2$ we can write $m = a^2 - a + b$, where $2 \leq b \leq a$. We have $\frac{m-1}{a} = a - 1 + \frac{b-1}{a}$, and therefore $\lceil \frac{m-1}{a} \rceil = a$ and $C(m, a) = a^2 - a + b - 1 = m - 1$. □

It is shown in Proposition 1 of [6] that, for $m \geq 3$, $C(m, a)$ is a lower bound for $R_2(m.a)$. So to prove Theorem 1 we must show that, for $m \geq a^2 - a + 1$, $C(m, a)$ is also an upper bound, i.e., every 2-coloring of $[C(m, a)]$ yields a monochromatic solution of $L(m, a)$.

To proceed, it will be convenient to recall the compact notation used in [6] to indicate solutions of $L(m, a)$.

Notation. If $n_1, \ldots, n_k$ are nonnegative integers whose sum is m, and $d_1, \ldots, d_k$ are elements of $[C(m, a)]$ such that we obtain a true equation from $L(m, a)$ by substituting d_1 for the variables $x_1, \ldots, x_{n_1}$, d_2 for the next n_2 variables, and so on, then we denote this true equation by

$$[n_1 \to d_1;\ n_2 \to d_2;\ \cdots;\ n_k \to d_k].$$

For example, the true instance

$$a + a + \cdots + a = a(m - 1)$$

of $L(m, a)$ will be denoted by

$$[m - 1 \to a;\ 1 \to m - 1].$$

Proposition 1. If $m = a^2 - a + 1$, then every 2-coloring of $[C(m, a)]$ yields a monochromatic solution of $L(m, a)$.

Proof. Note that if $m = a^2 - a + 1$, then $C(m, a) = (a - 1)^2$.

Suppose we have a bad 2-coloring of $C(m, a)$, and suppose, without loss of generality, that $1 \in R$. Then the solution $[a^2 - a \to 1;\ 1 \to a - 1]$ shows that $a - 1 \in B$, and multiplying the assigned values in this solution by $a - 1$ shows that $(a - 1)^2 \in R$. But the solution $[(a - 1)^2 \to 1;\ a \to (a - 1)^2]$ shows that $(a - 1)^2 \in B$, a contradiction. □

By Proposition 1, we can assume, in completing the proof of Theorem 1, that $m \geq a^2 - a + 2$, and therefore Lemma 1 applies.

Some of our arguments in Section 3 will require $a \geq 4$. When $a = 3$, Theorem 1 asserts that $R_2(m, 3) = C(m, 3)$ for $m \geq 7$, and this is proved in Section 6 of [6]. Accordingly, we need only consider $a \geq 4$ in what follows.

Conventions. In the remainder of Section 2, and in Section 3, we assume that $a \geq 4$ and $m \geq a^2 - a + 2$. We suppose that we have a bad 2-coloring of $[C(m,a)]$, and we seek a contradiction. We assume without loss of generality that $a - 2 \in R$.

As in [**6**], we proceed by considering two cases, depending on the coloring of the element $a-1$. If $a - 1 \in B$, then we can obtain our contradiction by using the same argument as in [**6**], since that argument uses only elements in $[m-1]$. (See [**6**], Section 3.) Accordingly, we adopt another convention.

Convention. We assume in the remainder of Section 2, and in Section 3, that $a - 1 \in R$.

Lemma 2. The elements 1 and a are in R.

The proof is as in Lemmas 4 and 5 of [**6**], which use only numbers in $[m-1]$.

Lemma 3. The numbers $m-a, \ldots, m-1$ are all in B.

Proof. We want to show that $m-a+j \in B$ for $0 \leq j \leq a-1$. Since 1, $a-1$, and a are all in R and we are assuming that there are no monochromatic solutions of $L(m,a)$ in $[C(m,a)]$, we need only consider the solution

$[m-2a+2j+1 \to a;\ a-1-j \to a-1;\ a-1-j \to 1;\ 1 \to m-a+j.]$ □

Lemma 4. The numbers $1, 2, \ldots, a$ are all in R.

Proof. For $0 \leq j \leq a-1$, consider the solution

$$[m-a+j \to j+1;\ a-j \to m-a+j]$$

and use the result of Lemma 3. □

The next result generalizes Lemma 9 from [**6**].

Lemma 5. If d is an integer such that $a|d$ and $m-1 \leq d \leq a(m-1)$, then $\frac{d}{a} \in B$.

Proof. Write $d = (m-1)j + k$, with $1 \leq j \leq a-1$ and $0 \leq k \leq m-1$. Then the solution

$$\left[m-1-k \to j;\ k \to j+1;\ 1 \to \frac{d}{a}\right]$$

shows that $\frac{d}{a} \in B$. □

3. The proof of Theorem 1

In this section we will use the results of Section 2, together with algebraic expressions for $C(m,a)$, to produce a red solution of $L(m,a)$ in $[C(m,a)]$. This will contradict our standing assumption that our 2-coloring of $[C(m,a)]$ is bad, and conclude the proof of Theorem 1.

The following Lemma is Lemma 10 from [6].

Lemma 6. Let $m = ua^2 + va + c$, with u as large as possible and $0 \leq v, c \leq a-1$.

(i) If $c = 1$ then $C(m,a) = \frac{(m-1)^2}{a^2}$.

(ii) If $c = 0$ then $C(m,a) = \frac{m^2-m+va}{a^2}$.

(iii) If $2 \leq c \leq a-1$ then $C(m,a) = \frac{m^2+(a-c-1)m+c-ac-vac+va+ta^2}{a^2}$, where $t = \left\lceil \frac{(c-1)(v+1)}{a} \right\rceil$.

When $c = 1$, the argument in [6] produces a red solution of $L(m,a)$ by using only elements of $C(m,a)$ that can be shown to be in R by using elements of $[m-1]$. So the same argument yields a red solution here. We turn to the remaining cases.

The Case $c = 0$

In this case we have $\frac{m}{a} \in B$ by Lemma 5. We choose an s such that $s \in R$, $s+1 \in B$, and $s+1 \leq \frac{m}{a}$. Using the expression for $C(m,a)$ in Lemma 6, we obtain

$$C(m,a) = \left(\frac{m-a}{a}\right)\frac{m}{a} + \frac{(a-1)m+va}{a^2}.$$

We let

$$\alpha = \frac{m-a}{a}(s+1) + \frac{(a-1)m+va}{a^2} \leq \left(\frac{m-a}{a}\right)\frac{m}{a} + \frac{(a-1)m+va}{a^2},$$

so $\alpha \leq C(m,a)$. As in [6], we see that $\alpha \in R$.

We now obtain a red solution of $L(m,a)$ by assigning the value α to x_{m-2}, x_{m-1} and x_m and the value s to $(a-2)(\frac{m-a}{a})$ other variables, and showing that we can assign values in R to the remaining $\frac{2m}{a}+a-5$ variables to complete the solution. In fact we will show that we can accomplish this by using only values in the set $[a]$. These values are all in R by Lemma 4.

The values assigned to the remaining variables must add up to

$$\frac{a-2}{a}(m-a)+\frac{a-2}{a^2}((a-1)m+va).$$

If we can show that using only the value a yields a sum that is at least this large, and using only the value 1 yields a sum that is at most this large, then there is a unique solution that uses values in one of the sets $\{j,\ j+1\}$, where $j \in [a-1]$.

Since $v \leq a-1$, we can achieve our first objective by showing that

$$a\left(\frac{2m}{a}+a-5\right) \geq \frac{a-2}{a}(m-a)+\frac{a-2}{a^2}((a-1)m+(a-1)a),$$

which simplifies to

$$a^2-5a+1-\frac{2}{a} \geq m\left(\frac{2-5a}{a^2}\right).$$

Since the right-hand side is negative, this is clearly true when $a \geq 5$. When $a=4$ it is true since $m \geq 14$ because $m \geq a^2-a+2$.

Since $v \geq 0$, we can achieve our second objective by showing that

$$\left(\frac{2m}{a}+a-5\right) \leq \frac{a-2}{a}(m-a)+\frac{a-2}{a^2}((a-1)m).$$

But this simplifies to $2a^3-7a^2 \leq m(2a^2-7a+2)$, which is true for all $a \geq 4$ and $m \geq a$. (It is *not* true when $a=3$ and $m>9$, and this is why we dealt with the case $a=3$ separately at the outset.)

The Case $2 \leq c \leq a-1$

In this case we have $\frac{m+a-c}{a} \in B$ by Lemma 5. We choose an s such that $s \in R$, $s+1 \in B$, and $s+1 \leq \frac{m+a-c}{a}$. Using the expression for $C(m,a)$ in Lemma 6, we obtain

$$C(m,a)=\left(\frac{m-c}{a}\right)\left(\frac{m+a-c}{a}\right)+\frac{(c-1)(m+a-c)+a\gamma}{a^2},$$

where $\gamma = ta-(c-1)(v+1)$, with t as in Lemma 6. Note that since

$$0 \leq t-\frac{(c-1)(v+1)}{a} \leq 1$$

by definition of t, we have $0 \leq \gamma \leq a$.

We consider the element

$$\beta = \left(\frac{m-c}{a}\right)(s+1) + \frac{(c-1)(m+a-c)+a\gamma}{a^2} \le C(m,a).$$

To see that $\beta \in R$, we consider the solution

$$\left[m-c \to s+1;\ c-2 \to \frac{m+a-c}{a};\ 1 \to \frac{m+a-c}{a} + \gamma;\ 1 \to \beta\right].$$

Note that $\frac{m+a-c}{a} + \gamma \in B$ by Lemma 5, since

$$\frac{m+a-c}{a} + \gamma \le \frac{m+a-c+a^2}{a} \le \frac{m+a-2+a^2}{a}$$

and it is easy to verify that $m + a - 2 + a^2 \le a(m-1)$ when $a \ge 4$ and $m \ge a^2 - a + 2$.

To obtain our red solution of $L(m,a)$, we assign the value β to x_m, x_{m-1} and x_{m-2}, and the value s to $(a-2)(\frac{m-c}{a})$ other variables, and show that we can assign values in R to the remaining $\frac{2(m-c)}{a} + c - 3$ variables to complete the solution. We again use values in the set $[a]$.

The values assigned to the remaining $\frac{2(m-c)}{a} + c - 3$ variables must add up to

$$\frac{a-2}{a}(m-c) + \frac{a-2}{a^2}((c-1)(m+a-c)+a\gamma). \tag{1}$$

If we can show that using only the value a (respectively, 1) yields a sum that is at least (respectively, at most) this large, then, as before, there must be a solution that uses values in one of the sets $\{j,\ j+1\}$, where $j \in [a-1]$.

Using the fact that $\gamma \le a$, we can achieve our first objective by showing that

$$a\left(\frac{2(m-c)}{a} + c - 3\right) \ge \frac{a-2}{a}(m-c) + \frac{a-2}{a^2}((c-1)(m+a-c)+a^2),$$

which simplifies to

$$c^2(a-2)+c(a^3-2a^2-a+2)+(-4a^3+3a^2-2a) \ge m(-a^2-3a+2+c(a-2)).$$

If we regard a as a constant and denote the quantity on the left-hand side of this inequality by $f(c)$, then the derivative

$$f'(c) = 2c(a-2) + (a^3 - 2a^2 - a + 2)$$

is easily seen to be positive for $c \ge 0$ and $a \ge 4$, so the minimum value of $f(c)$ for $2 \le c \le a-1$ occurs at $c = 2$. Since

$$m(-a^2-3a+2+c(a-2)) \le m(-a^2-3a+2+(a-1)(a-2)) = m(4-6a),$$

we only need to verify that $f(2) \geq m(4-6a)$, i.e.,

$$2a^3 + a^2 + 4 \leq m(6a-4),$$

and this is true for $a \geq 4$ and $m \geq a^2 - a + 2$.

To achieve our second objective, it will suffice, by using expression (1) and the fact that $0 \leq \gamma$, to show that

$$\left(\frac{2(m-c)}{a} + c - 3\right) \leq \frac{a-2}{a}(m-c) + \frac{a-2}{a^2}((c-1)(m+a-c)).$$

This inequality simplifies to

$$c^2(a-2) + c(a^2-3a+2) - 2a^2 - 2a \leq m(a^2 - 5a + 2 + c(a-2)).$$

Denoting the quantity on the left-hand side by $g(c)$, we have

$$g'(c) = 2c(a-2) + (a^2 - 3a + 2),$$

so $g'(c) > 0$ for $c \geq 0$ and $a \geq 4$. Therefore the maximum value of $g(c)$ for $2 \leq c \leq a-1$ occurs at $c = a-1$. Since

$$m(a^2 - 5a + 2 + c(a-2)) \geq m(a^2 - 5a + 2 + 2(a-2)) = m(a^2 - 3a - 2),$$

we need only verify that $g(a-1) \leq m(a^2 - 3a - 2)$, i.e., that

$$2a^3 - 10a^2 + 8a - 4 \leq m(a^2 - 3a - 2).$$

This is easily verified for $a \geq 4$ and $m \geq a^2 - a + 2$. (But it fails when $a = 3$ and $m > 8$, again indicating why we dealt separately with the case $a = 3$.)
□

4. The proof of Theorem 2

The following lemma will be useful in proving Theorems 2, 4, and 5.

Lemma 7. Suppose $\frac{a}{2} + 1 \leq m \leq 2a + 1$. Then:

(1) there exists a solution of $L(m, a)$ using only values in $\{1, 2\}$,

(2) there exists a solution of $L(m, a)$ using only values in $\{1, 3\}$ iff $a \equiv m - 1 \pmod 2$,

(3) there exists a solution of $L(m, a)$ using only values in $\{2, 3\}$ iff $\frac{2a}{3} + 1 \leq m \leq \frac{3a}{2} + 1$, and

(4) there exists a solution of $L(m,a)$ using only values in $\{1,4\}$ iff $a \equiv m-1 \pmod 3$.

Proof. There exists a solution using values in $\{1,2\}$ iff either $m-1 \le a \le 2(m-1)$ or $m-1 \le 2a \le 2(m-1)$. If $\frac{a}{2}+1 \le m \le a+1$ then the first alternative holds, and if $a+1 \le m \le 2a+1$ then the second holds. This proves statement (1). There exists a solution of $L(m,a)$ using values in $\{2,3\}$ iff either $2(m-1) \le 2a \le 3(m-1)$ or $2(m-1) \le 3a \le 3(m-1)$, i.e., iff $\frac{2a}{3} \le m-1 \le \frac{3a}{2}$. This proves statement (3).

As we assign values in $\{1,3\}$ to $x_1,\ldots,x_{m-1}$, the total values achieved by the left side of $L(m,a)$ are exactly those integers that have the same parity as $m-1$ and are between $m-1$ and $3(m-1)$, inclusive. So there exists a solution using values in $\{1,3\}$ iff $a \equiv m-1 \pmod 3$ and either $m-1 \le a \le 3(m-1)$ or $m-1 \le 3a \le 3(m-1)$. As in the proof of (1), one of these pairs of inequalities must hold, and this proves (2). The proof of (4) is similar. □

The proof of Theorem 2. Assume that $a+1 \le m \le 2a+1$.

It is clear that $R_2(m,a)=1$ iff we can obtain a solution of $L(m,a)$ by assigning all the variables the same value, and this is so iff $m-1=a$. Note that $R_2(m,a)$ can never be 2, for if we color 1 and 2 differently then we can only obtain a monochromatic solution of $L(m,a)$ in [2] by coloring all the variables the same, but then $R_2(m,a)=1$.

For the remainder of the proof we assume $a+2 \le m \le 2a+1$, so that $R_2(m,a) \ge 3$, and there is no solution of $L(m,a)$ that assigns all the variables the same color.

We next establish the conditions under which $R_2(m,a)=3$. Suppose first that we have a bad 2-coloring of [3], with, say, $1 \in R$. By statement (1) of Lemma 7, there is a solution of $L(m,a)$ using values in $\{1,2\}$, and both values must be used in the solution. So $2 \in B$. It then follows from statements (2) and (3) of Lemma 7 that if $a \equiv m-1 \pmod 2$ and $m \le \frac{3a}{2}+1$ then we must color 3 both blue and red in order to avoid a monochromatic solution of $L(m,a)$ in [3], so $R_2(m,a)=3$. If $a \not\equiv m-1 \pmod 2$ (or, respectively, if $m > \frac{3a}{2}+1$) then we can color 3 red (or, respectively, blue) and obtain a bad 2-coloring of [3], so $R_2(m,a) > 3$.

By what we have just shown, either of conditions (i) or (ii) in the statement of Theorem 2 implies that $R_2(m,a) \ge 4$. To prove that each implies $R_2(m,a)=4$, we suppose we have a bad 2-coloring of [4], with $1 \in R$, and we seek a contradiction, assuming that (i) or (ii) holds. By using statement (1) of Lemma 7, and then doubling all the values in its proof, we get $2 \in B$ and $4 \in R$.

If (i) holds then by statement (3) of Lemma 7 we have $3 \in R$. To get a contradiction, we obtain a red solution of $L(m,a)$. We start by assigning the value 4 to x_m and the value 1 to each of the other variables. We must show that we can increase the total value of the left side by $4a-(m-1)$ by increasing some of the values on the left side by 2 or 3. So we need to write $4a-(m-1)$ as a sum of 2's and 3's, using at most $m-1$ terms. Our bounds on m imply that $2 \leq 4a-(m-1) \leq 3(m-1)$, so this is possible (using at most two 2's).

If (ii) holds then by statement (4) of Lemma 7 we can obtain a red solution of $L(m,a)$ using values in $\{1,4\}$.

We have shown that each of (i) and (ii) implies $R_2(m,a)=4$. Conversely, if $R_2(m,a)=4$ then since $R_2(m,a)\neq 3$, either (i) holds or $m > \frac{3a}{2}+1$. In the latter case we must have $a \equiv (m-1) \pmod 3$, for otherwise by statements (3) and (4) of Lemma 7, the coloring $R=\{1,4\}, B=\{2,3\}$ is bad, contradicting $R_2(m,a)=4$.

Finally, suppose $m>\frac{3a}{2}+1$ and $a \not\equiv m-1 \pmod 3$. Then $R_2(m,a)\geq 5$. To prove equality, suppose for a contradiction that we have a bad 2-coloring of $[5]$, with $1 \in R$. As above, we see that $2 \in B$ and $4 \in R$, and as in our proof that condition (i) implies $R_2(m,a)=4$ we get a contradiction if $3 \in R$. So suppose $3 \in B$.

We claim that $5 \in R$. To see this we construct a solution of $L(m,a)$ in which we assign the value 5 to x_m and values in $\{2,3,5\}$ to all the other variables. If we start by assigning the value 2 to all the other variables, then we must increase the value of the left side by $5a-2(m-1)$ by increasing some of the 2's by 1 or 3 each. Note that $5a-2(m-1)\geq 0$ since $m \leq 2a+1$, and, since $a \leq m-2$, we have $5a-2(m-1) \leq 5(m-2)-2(m-1)=3m-8$. Any nonnegative integer less than or equal to $3m-5$ can be written in the form $3q+r$, with $0\leq q \leq m-2$ and $0 \leq r \leq 2$, with $r \leq 1$ if $q=m-2$. So we can achieve the desired solution (using at most two 1's), and $5 \in R$.

We now obtain a red solution of $L(m,a)$ (and therefore a contradiction) by assigning the value 4 to x_m and values in $\{1,4,5\}$ to all the other variables. If we start by assigning the value 1 to each of $x_1,\ldots,x_{m-1}$, then to finish we must write $4a-(m-1)$ as a sum of 3's and 4's, using at most $m-1$ terms. Our bounds on m yield $4a-(m-1)\geq 4a-2a \geq 6$ and $4a-(m-1)\leq 4(m-1)$. Therefore it is easy to show that the desired expression for $4a-(m-1)$ exists.

We have shown that if $m>\frac{3a}{2}+1$ and $a\not\equiv m-1 \pmod 3$ then $R_2(m,a)=5$. Conversely, if $R_2(m,a)=5$ then $R_2(m,a)$ is neither 3 nor 4, so by what we have already shown, we must have $m>\frac{3a}{2}+1$ and $a \not\equiv m-1$ (mod 3). □

5. The proof of Theorem 3

We have $C(2a+1,a)=4$, and, since $a\geq 3$, $C(2a,a)=4$ as well. If $3|a$, then by Theorem 2 we have $R_2(2a+1,a)=4$ and $R_2(2a,a)=5$, while if $3\nmid a$ then $R_2(2a+1,a)=5$. Therefore, to prove Theorem 3 it will suffice to prove the following.

Proposition 2. For $a\geq 3$ and $m\geq 2a+2$, $R_2(m,a)=C(m,a)$.

By Theorem 2 of [**6**], we know that Proposition 2 holds when $a=3$. By Theorem 1, we also know that it holds when $m>a^2-a$. So we adopt the following conventions for the remainder of this section.

Conventions. We have $a\geq 4$, and $2a+2\leq m\leq a^2-a$. We write $m=av+c$, with $2\leq v\leq a-1$ and $0\leq c\leq a-1$, so that when $v=2$ we have $c\geq 2$ and when $v=a-1$ we have $c=0$. We suppose that we have a bad 2-coloring of $[C(m,a)]$, with $1\in R$, and we seek a contradiction.

We consider three cases.

Case 1: $1,2\in R$

Lemma 8. When $1,2\in R$, we have $C(m,a)\in R$.

Proof. Since $m\geq 2a+2$, we have $\frac{m-1}{a}\geq 2+\frac{1}{a}$ and $\lceil\frac{m-1}{a}\rceil\geq 3$. So if we let $n=\lceil\frac{m-1}{a}\rceil$ then $n+1\leq C(m,a)$ and, for $k\in\{n,\ n+1\}$,

$$m-1\leq ak\leq 2(m-1).$$

So we may assign either of the values $n,\ n+1$ to x_m and obtain a solution of $L(m,a)$ by assigning a value of 1 or 2 to each of $x_1,\ldots,x_{m-1}$. Since $1,2\in R$, we have $n,\ n+1\in B$.

To show that $C(m,a)\in R$, it therefore suffices to show that

$$n(m-1)\leq aC(m,a)\leq(n+1)(m-1).$$

The first inequality holds because $n=\lceil\frac{m-1}{a}\rceil$. The second inequality asserts that $C(m,a)\leq n(\frac{m-1}{a})+\frac{m-1}{a}$, and this is true because $C(m,a)$ exceeds $n(\frac{m-1}{a})$ by less than 1. □

We now obtain a contradiction by showing that, for some positive integer $j\leq a$, we obtain a red solution of $L(m,a)$ by assigning the value $C(m,a)$ to x_m and to $a-j$ of the variables $x_1,\ldots,x_{m-1}$, and assigning the value 1 or 2 to each of the remaining $a(v-1)+j+c-1$ variables. To show this, we must show that for some positive $j\leq a$,

$$a(v-1)+j+c-1\leq jC(m,a)\leq 2(a(v-1)+j+c-1). \qquad (2)$$

Subcase 1: $c = 0$

In this case, Lemma 6 yields $C(m,a) = v^2$, so we must show that

$$a(v-1) + j - 1 \le jv^2 \le 2(a(v-1) + j - 1)$$

for some positive $j \le a$. The first inequality clearly holds when $j = a$. We now choose j to be the smallest positive integer such that the first inequality holds. We claim that, for this j, the second inequality holds.

If $j = 1$ the second inequality says that $v^2 \le 2a(v-1)$, and since $a \ge v+1$ it suffices to show that $v^2 \le 2(v^2 - 1)$. But this is clearly true, since $v \ge 2$. If $j > 1$ then by the minimality of j we have $a(v-1) + j - 2 > (j-1)v^2$, so $jv^2 < v^2 + a(v-1) + j - 2$ and it suffices to show that

$$v^2 + a(v-1) + j - 2 \le 2(a(v-1) + j - 1).$$

This inequality reduces to $v^2 \le a(v-1) + j$, and since $a \ge v + 1$ it suffices to show that $v^2 \le v^2 - 1 + j$, which is clearly true.

Subcase 2: $c = 1$

The argument for this case is nearly identical to that for $c = 0$. We omit the details.

Subcase 3: $2 \le c \le a - 1$

In this case, Lemma 6 yields $C(m,a) = v^2 + v + t$, where $t = \lceil \frac{(c-1)(v+1)}{a} \rceil$. So when $j = a$ the first inequality in statement (2) says that $av + c - 1 \le a(v^2 + v + t)$, which is clear. We choose the smallest positive j such that the first inequality of statement (2) holds.

If $j = 1$, the second inequality in (2) says that $v^2 + v + t \le 2(a(v-1) + c)$. Since $c \ne 0$, our conditions on m imply that $a \ge v + 2$, so since $t \le v + 1$ it will suffice to show that $v^2 + 2v + 1 \le 2(v^2 + v - 2 + c)$. This reduces to $5 \le v^2 + 2c$, which is clearly true.

If $j > 1$ then by the minimality of j we have

$$a(v-1) + j + c - 2 > (j-1)(v^2 + v + t),$$

so $j(v^2 + v + t) < a(v-1) + j + c - 2 + v^2 + v + t$, and to verify the second inequality in (2) we want to show that

$$a(v-1) + j + c - 2 + v^2 + v + t \le 2(a(v-1) + j + c - 1),$$

i.e., $v^2 + v + t \le a(v-1) + j + c$. Since $c \ne 0$ implies $a \ge v + 2$, it suffices to show that $v^2 + v + t \le v^2 + v - 2 + j + c$, i.e., $t \le j + c - 2$. But $t \le c - 1$, so this is clear.

Case 2: $1 \in R$, $2 \in B$, $3 \in B$

Subcase 1: $c - 1 < \frac{a}{3}$

In this subcase we will produce a red solution of $L(m, a)$ by using values that are at most $3v$. Note that if $c = 0$ or 1 then $C(m, a) = v^2$ and $v \geq 3$, so it is clear that $3v \leq C(m, a)$. If $c \geq 2$ then $C(m, a) = v^2 + v + t$ and it is again clear that $3v \leq C(m, a)$.

By assigning a value of 2 or 3 to each of the variables $x_1, \ldots, x_{m-1}$, we can achieve for the left side of $L(m, a)$ any total value between $2(av + c - 1)$ and $3(av + c - 1)$, inclusive. By the assumption of the current subcase, this implies that when $c > 0$ we can achieve a solution of $L(m, a)$ using 2's and 3's on the left side and any of $2v + 1, \ldots, 3v$ on the right. When $c = 0$ we can use any of $\{2v, \ldots, 3v - 1\}$ on the right. So $2v + 1, \ldots, 3v \in R$ when $c > 0$ and $2v, \ldots, 3v - 1 \in R$ when $c = 0$.

When $c = 0$,

$$[m - a - 1 \to 1;\ a - 1 \to 2v;\ 1 \to 2v + 1;\ 1 \to 3v - 1]$$

is a red solution of $L(m, a)$.

To obtain a red solution when $c > 0$, start by assigning the value $3v$ to x_m and the value 1 to each of the other variables. We must then increase the total value of the left side by $2av - (c - 1)$, which is easily seen to be at least $3(2v)$ for any $a \geq 4$, since $v \geq 2$ and $c \leq a - 1$. Write

$$2av - (c - 1) = q(2v) + r,$$

with $3 \leq q < a$ and $0 \leq r \leq 2v - 1$. If we increase the values of each of $x_1, \ldots, x_q$ to $2v + 1$, then we must still increase the total value of the left side by r. Since $r \leq 2v - 1$ and $q \geq 3$, we can accomplish this by again increasing the values of some of $x_1, \ldots, x_q$ without increasing any value $2v + 1$ by more than $v - 1$ (even if $v - 1 = 1$).

Subcase 2: $c - 1 \geq \frac{a}{3}$

In this subcase we will use values no larger than $3v + 1$. Note that $3v + 1 \leq C(m, a)$ since $c \geq 2$, so $C(m, a) = v^2 + v + t$ and $t \geq 1$.

By the assumption of the current subcase, we see as in the preceding subcase that we now have $2v + 2, \ldots, 3v + 1 \in R$.

To obtain a red solution of $L(m, a)$ we start by assigning the value $3v + 1$ to x_m and the value 1 to each of the other variables. We must then increase the total value of the left side by $a(2v + 1) - (c - 1)$, which is easily shown to be least $4(2v + 1)$ for any $a \geq 5$. Assuming for the moment that $a \geq 5$, write

$$a(2v + 1) - (c - 1) = q(2v + 1) + r,$$

with $4 \leq q < a$ and $0 \leq r \leq 2v$. Increase the values of $x_1, \ldots, x_q$ to $2v+2$. Since $r \leq 2v$ and $q \geq 4$, we can then increase the total value of the left side by r by increasing some of the values $2v+2$ by no more than $v-1$ each.

If $a = 4$, then by our bounds on m we have $10 \leq m \leq 12$. In the current subcase we also have $c-1 \geq \frac{4}{3}$, so $c \geq 3$. Thus $m = 11$, and $[10 \to 2;\ 1 \to 5]$ and $[6 \to 2;\ 4 \to 3;\ 1 \to 6]$ are solutions of $L(m,a)$. Since $2, 3 \in B$, we have $5, 6 \in R$, so $[8 \to 1;\ 2 \to 6;\ 1 \to 5]$ is a red solution of $L(m,a)$.

Case 3: $1 \in R,\ 2 \in B,\ 3 \in R$

In this case we will use numbers no larger than $3v$. As in Case 2, all these numbers are in $[C(m,a)]$.

Subcase 1: m is odd

First suppose v is even. Then c must be odd, so $c > 0$. It follows that for any $k \in \{v+2, v+4, \ldots, 3v\}$, ak is an even number such that $m-1 \leq ak \leq 3(m-1)$, and therefore we can achieve the value ak by assigning each variable on the left side of $L(m,a)$ a value of 1 or 3. So $v+2, v+4, \ldots, 3v$ are all in B. To obtain a blue solution of $L(m,a)$, we start by assigning the value $3v$ to x_m and the value 2 to each of the remaining variables. To achieve a solution, we must then increase the total value on the left side of $L(m,a)$ by $av+2-2c$, which is easily seen to be at least v. So we write

$$av + 2 - 2c = qv + r,$$

where $1 \leq q \leq a$, $0 \leq r < v$, and r is even. If we increase the values of $x_1, \ldots, x_q$ to $v+2$, we can then increase the value of x_1 to $v+2+r$ and obtain a blue solution of $L(m,a)$. Note that $v+2+r$ is even and at most $3v$.

Now suppose v is odd. Then $v+1, v+3, \ldots, 3v-1$ are all even, and as in the preceding paragraph we see that they are all in B. To obtain a blue solution of $L(m,a)$, we start by assigning the value $3v-1$ to x_m and the value 2 to each of the other variables. We must then increase the total value of the left side by $av+2-2c-a$. Since v is odd, $v \geq 3$, and using this it is easy to show that $av+2-2c-a \geq v+1$. So we write

$$av + 2 - 2c - a = q(v+1) + r,$$

with $1 \leq q < a$, r even, and $0 \leq r \leq v-1$ since v is odd. If we increase the value of x_1 to $v+3+r$ and the values of $x_2, \ldots, x_q$ to $v+3$, we obtain a blue solution of $L(m,a)$, since $v+3+r \leq 2v+2 \leq 3v-1$ because $v \geq 3$. (We could have done this argument by increasing values to $v+1$ instead of $v+3$, but doing it as we have will be useful in dealing with the next subcase.)

Subcase 2: m is even and a is even

If m and a are even and z is an even integer such that $m-2 \le (a-1)z \le 3(m-2)$, then we can obtain a solution of $L(m,a)$ by assigning the value z to x_m and x_{m-1} and assigning a value of 1 or 3 to each of the remaining variables. It is straightforward to verify that $m-2 \le (a-1)z \le 3(m-2)$ whenever $v+2 \le z \le 3v$. So if v is even then $v+2, v+4, \ldots, 3v$ are all in B, and if v is odd then $v+3, v+5, \ldots, 3v-1$ are all in B.

We can now obtain a blue solution of $L(m,a)$ by repeating the arguments given for Subcase 1, because we didn't use the value $v+1$ in the argument given there when v was odd.

Subcase 3: m is even and a is odd.

In this subcase, $a-1$ is even, so we can now do the argument of the first paragraph of Subcase 2 without the restriction that z be even, and conclude that $\{v+2, v+3, \ldots, 3v\} \subseteq B$. We can then obtain a blue solution of $L(m,a)$ by using the argument given for even v in Subcase 1, regardless of the parity of v. In the present situation we will not know that the remainder r is even, but that doesn't matter now. □

6. The proofs of Theorems 4 and 5

The proof of Theorem 4. Suppose that $\frac{2a}{3}+1 \le m \le a$. As in the proof of Theorem 2, we have $R_2(m,a) \ge 3$, since $m \ne a+1$.

It is shown in Theorem 2 of [6] that $R_2(3,3) = 9$, so we can assume that $a \ge 4$. If we take a bad 2-coloring of [3] with $1 \in R$, then by statement (1) of Lemma 7 and the fact that $R_2(m,a) \ne 1$, we have $2 \in B$. So by statement (3) of Lemma 7 we must have $3 \in R$. If $a \equiv m-1 \pmod 2$, then by statement (2) of Lemma 7 we have a red solution of $L(m,a)$, so $R_2(m,a) = 3$.

If $a \not\equiv m-1 \pmod 2$ then the coloring $R = \{1,3\}, B = \{2\}$ is bad, so $R_2(m,a) \ge 4$. To prove equality, suppose for a contradiction that we have a bad 2-coloring of [4], with $1 \in R$. Then, as above, we have $2 \in B$ and $4 \in R$. We again have $3 \in R$ by statement (3) of Lemma 7. To obtain a red solution of $L(m,a)$ we assign the value 1 to all the variables, and show that we can increase the total value of the left side by $a-(m-1)$ by increasing some of the 1's on the left side by 2 or 3 each. This is possible if $2 \le a-(m-1) \le 3(m-1)$. The second inequality holds since $m \ge \frac{a}{4}+1$. The first inequality holds if $m \le a-1$. So we have a red solution unless

$m = a$. But if $m = a$ then, since $a \geq 4$, $[a-4 \to 3;\ 3 \to 4;\ 1 \to 3]$ is a red solution. □

Lemma 9. $R_2(4,5) = 9$.

Proof. We first determine the unique bad 2-coloring of [8] that has $1 \in R$. As in the proof of Theorem 4 we must have $2 \in B$ and $4 \in R$, and it then follows from statement (2) of Lemma 7 that $3 \in B$. The solutions $[3 \to 5;\ 1 \to 3]$ and $[2 \to 6;\ 2 \to 3]$ then yield $5, 6 \in R$, and the solutions $[2 \to 7;\ 1 \to 6;\ 1 \to 4]$ and $[2 \to 8;\ 2 \to 4]$ show that $7, 8 \in B$. With the coloring $R = \{1,4,5,6\}, B = \{2,3,7,8\}$, the left side of $L(m,a)$ would have total value at most 18 in any red solution, so x_4 would have to be assigned the value 1. But the left side couldn't have total value 5, so there is no red solution. In a blue solution, the left side of $L(m,a)$ would have total value at most 24, so x_4 would have to be 2 or 3. But the left side couldn't have total value 10 or 15, so there is no blue solution.

Suppose now that we have a bad 2-coloring of [9] with $1 \in R$. By what we have just shown, we must have $3 \in B$ and $4,5,6 \in R$. Then the solution $[1 \to 9;\ 3 \to 3]$ shows that $9 \in R$, so $[1 \to 5;\ 1 \to 6;\ 1 \to 9;\ 1 \to 4]$ is a red solution. We conclude that $R_2(4,5) = 9$. □

The proof of Theorem 5. Suppose that $\frac{a}{2} + 1 \leq m < \frac{2a}{3} + 1$ (so $a \geq 4$). Then the coloring $R = \{1\}, B = \{2,3\}$ of [3] is bad by statement (3) of Lemma 7, so $R_2(m,a) \geq 4$. For any bad 2-coloring of [4] with $1 \in R$, we have $2 \in B$ and $4 \in R$ as above, so if $a \equiv m-1 \pmod 3$ then by statement (4) of Lemma 7 we have a red solution of $L(m,a)$. So if $a \equiv m-1 \pmod 3$ then $R_2(m,a) = 4$.

Now suppose that $a \not\equiv m-1 \pmod 3$. Then, by statements (3) and (4) of Lemma 7, the coloring $R = \{1,4\}, B = \{2,3\}$ of [4] is bad, so $R_2(m,a) \geq 5$. Suppose we have a bad 2-coloring of [5], with $1 \in R$. Then as above we have $2 \in B$ and $4 \in R$. If $3 \in R$ then, as in the second half of the last paragraph of the proof of Theorem 4, we have a red solution of $L(m,a)$ using values in $\{1,3,4\}$. (The requirement $m \leq a-1$ at the end of the argument is satisfied since $m < \frac{2a}{3} + 1$ and m is an integer.) So $3 \in B$.

We claim that $5 \in R$. To see this we obtain a solution of $L(m,a)$ using values in $\{2,3,5\}$, and note that any such solution must involve the value 5 by statement (3) of Lemma 7. To obtain our solution, we start by assigning the value 2 to all the variables. We must then increase the total value of the left side by $2a - 2(m-1)$ by increasing the values of some of the variables on the left side by 1 or 3 each. As in the third-to-last paragraph of the proof of Theorem 2, to show that this is possible we need only verify that $0 \leq 2a - 2(m-1) \leq 3m - 5$. The first inequality holds since $m \leq a+1$, and the second states that $2a \leq 5m - 7$, which is true since $m \geq \frac{a}{2} + 1$ and

$a \geq 4$.

We now try to obtain a red solution of $L(m,a)$ by using values in $\{1,4,5\}$. If we start by assigning the value 1 to all the variables, then to achieve a solution we must write $a-(m-1)$ as a sum of 3's and 4's, using at most $m-1$ terms. This is possible if $a-(m-1) \leq 4(m-1)$ (which clearly holds) and $a-(m-1)$ is either 3 or 4 or at least 6. Since $m-1 < \frac{2a}{3}$ we have $a-(m-1) > \frac{a}{3}$, so $a-(m-1) \geq 2$. So we have a red solution, and thus $R_2(m,a) = 5$, unless $a-(m-1) = 2$ or 5.

If $a-(m-1) = 2$ then $m = a-1$, so $\frac{a}{2}+1 \leq a-1 < \frac{2a}{3}+1$ and therefore $a = 4$ or 5. For $a = 5$ we have $R_2(4,5) = 9$ by Lemma 9, and for $a = 4$ it is shown in [3] that $R_2(3,4) = 10$.

If $a-(m-1) = 5$ then $m = a-4$, so $\frac{a}{2}+1 \leq a-4 < \frac{2a}{3}+1$ and therefore $10 \leq a \leq 14$. It is easy to verify that in this case the coloring $R = \{1,4,5\}, B = \{2,3\}$ of [5] is bad, so $R_2(m,a) \geq 6$. For any bad 2-coloring of [6] with $1 \in R$ we have $3 \in B$, as above, and the solution $[5 \to 6;\ a-9 \to 3]$ shows that $6 \in R$. But then $[1 \to 6;\ a-5 \to 1]$ is a red solution of $L(m,a)$. So $R_2(m,a) = 6$. □

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