# Note on a Turán-type problem on distances*

Xueliang Li, Jing Ma, Yongtang Shi, Jun Yue
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
lxl@nankai.edu.cn, majingnk@gmail.com;
shi@nankai.edu.cn, yuejun06@126.com

**Abstract**

A new Turán-type problem on distances of graphs was introduced by Tyomkyn and Uzzell. In this paper, we focus on the case of distance two. We show that for any positive integer $n$, a graph $G$ on $n$ vertices without three vertices pairwise at distance 2 has at most $(n-1)^2/4+1$ pairs of vertices at distance 2, if $G$ has a vertex $v \in V(G)$ whose neighbors are covered by at most two cliques. This partially answers a guess of Tyomkyn and Uzzell in [Tyomkyn, M., Uzzell, A.J.: A new Turán-type problem on distances of graphs. Graphs Combin. **29**(6), 1927–1942 (2012)].

**Keywords**: distance, Turán-type problem, forbidden subgraph

# 1 Introduction

In [10], Tyomkyn and Uzzell introduced a new Turán-type problem on distances of graphs, which is an extension of the problem studied by Bollobás and Tyomkyn in [6], namely, determining the maximum number of paths with length $k$ in a tree $T$ on $n$ vertices.

The problem on counting paths of a given length in a graph $G$ has been studied since 1971, see, e.g., [1, 2, 3, 4, 5, 8, 9] and the references therein. On the other hand, counting paths of length $k$ in trees can be interpreted as counting pairs of vertices at distance $k$. Tyomkyn and Uzzell asked a natural question as follows.

**Question.** For a graph $G$ on $n$ vertices, what is the maximum possible number of pairs of vertices at distance $k$ ?

Let $G = (V, E)$ be a connected simple graph. The distance between two vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest

*Supported by NSFC No. 11371205 and PCSIRT.

path between $u$ and $v$ in $G$. Let $N_G(v)$ be the neighborhood of $v$, and $d_G(v) = |N_G(v)|$ denote the degree of a vertex $v$. The greatest distance between any two vertices in $G$ is the diameter of $G$, denoted by $diam(G)$. The set of neighbors of a vertex $v$ in $G$ is denoted by $N(v)$ or $N^1(v)$, and the set of vertices, whose distance is $i$ from $v$, is denoted by $N^i(v)$, where $i \in \{1, 2, 3, \cdots, diam(G)\}$. Suppose that $V'$ is a nonempty subset of $V$. The subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ induced by $V'$ and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of $G$. A clique in a graph $G$ is a subset of vertices such that every two vertices in the subset are connected by an edge.

If $H$ is a graph, then we say that $G$ is *$H$-free* if $G$ does not contain a copy of $H$ as an induced subgraph. A claw, denoted by $C$, is the complete bipartite graph $K_{1,3}$. Thus, $G$ is said to be *claw-free* if it does not contain any induced subgraph that is isomorphic to $C$.

A graph $G$ is a *quasi-line graph* if for every vertex $v \in V(G)$, the neighborhood of $v$ can be partitioned into two sets $A$, $B$ in such a way that both $A$ and $B$ are cliques. Note that there may be edges between $A$ and $B$. Thus all line graphs are quasi-line graphs, and all quasi-line graphs are claw-free, but if we converse either of the statements, it is not true. For other notation and terminology not defined here, we refer to [7].

For a graph $G$, a new graph $G_k$ is defined to be the graph with vertex set $V(G)$ and $\{x, y\} \in E(G_k)$ if and only if $x$ and $y$ are at distance $k$ in $G$. We call $G_k$ the *distance-$k$ graph*. Such vertices $x$ and $y$ are called *$k$-neighbors*. We call $d_{G_k}(x)$ the *$k$-degree* of $x$. We say that a graph $G$ is $k$-isomorphic to a graph $H$ if $G_k$ is isomorphic to $H$. Let $\omega(G)$ denote the clique number of a graph $G$, which is the number of vertices in a maximum clique of $G$.

It is interesting to maximize the number of edges in $G_k$ over all graphs $G$ on $n$ vertices. In [6], Bollobás and Tyomkyn proved that if $G$ is a tree, then $e(G_k)$ is maximal when $G$ is a $t$-broom for some $t$.

**Theorem 1.1.** *Let $n \geq k$. If $G$ is a tree on $n$ vertices, then $e(G_k)$ is maximal when $G$ is a $t$-broom. If $k$ is odd, then $t = 2$. If $k$ is even, then $t$ is within 1 of*

$$\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{n-1}{k-2}}.$$

For a general graph $G$, Tyomkyn and Uzzell [10] gave a conjecture and proved a part of it in the following theorem.

**Conjecture 1.1.** [10] Let $k \geq 3$ and $t \geq 2$. There is a function $h_2 : N \times N \to N$ such that if $n \geq h_2(k, t)$, then $e(G_k)$ is maximized over all $G$ with $|G| = n$ and $\omega(G_k) \leq t$ when $G$ is $k$-isomorphic to a $t$-broom for some $t$.

**Theorem 1.2.** *[10] There is a constant $k_0$ and a function $n_0 : N \to N$ such that for all $k \geq k_0$, all $n \geq n_0(k)$ and all graphs $G$ of order $n$ with no three vertices pairwise at distance $k$,*

$$e(G_k) \leq (n - k + 1)^2/4.$$

*Moreover, if the equality holds, then $G$ is $k$-isomorphic to the double broom.*

For the detailed proof and some terminology, we refer to [10]. Actually, Tyomkyn and Uzzell tried to do better about the bound, but there is no good way. For the case of $k = 2$, they believe that for $n \geq 5$, a triangle-free $G_2$ can have no more than $(n-1)^2/4 + 1$ edges. They mentioned that this is clearly true for $n = 5$ and a computer search verifies that it also holds for $6 \leq n \leq 11$. However, they could not prove it in general. In this paper, we will give an partial answer to their guess. Our main result is as follows:

**Theorem 1.3.** *Let $G$ be a graph on $n$ vertices, which has no three vertices pairwise at distance 2. If there exists an vertex $v \in V(G)$, whose neighbors are covered by at most two cliques, then $G$ has at most $(n-1)^2/4 + 1$ pairs of vertices at distance 2.*

From Theorem 1.3, we can get the following corollary.

**Corollary 1.1.** *Let $G$ be a quasi-line graph on $n$ vertices, which has no three vertices pairwise at distance 2. Then $G$ has at most $(n-1)^2/4 + 1$ pairs of vertices at distance 2.*

# 2 Preliminaries

In this section, we will figure out the structure of a graph $G$ with the property that $G_2$ is triangle-free.

Let $C_k$ be the cycle with $k$ vertices $v_1, v_2, \cdots, v_k$ such that $v_1v_k, v_iv_{i+1} \in E(C_k)$ for $i = 1, 2, \cdots, k-1$. First of all, we define two graphs $C_6^{'}$ and $C_6^{''}$ which can be obtained from $C_6$ as follows: $C_6^{'} = C_6 + v_1v_3$, $C_6^{''} = C_6 + v_1v_3 + v_3v_5$.

**Lemma 2.1.** *If $G_2$ is triangle-free, then $G$ is claw-free, $C_6$-free, $C_6^{'}$-free and $C_6^{''}$-free.*

*Proof.* By contradiction. Suppose that $G$ has a claw $C$ as an induced subgraph. Let $V(C) = \{v, u_1, u_2, u_3\}$ and $v$ is adjacent to $u_i$, $i = 1, 2, 3$. Then the three vertices $u_1$, $u_2$ and $u_3$ form a triangle in $G_2$, a contradiction.

Suppose that $G$ is not $C_6$-free. Let $C_6 = v_1v_2 \ldots v_6v_1$. Then $v_1v_3$, $v_3v_5$, $v_1v_5 \in e(G_2)$, which implies that the three vertices $v_1$, $v_3$ and $v_5$ form a triangle in $G_2$, a contradiction.

Similarly, $G$ is $C_6^{'}$-free and $C_6^{''}$-free. □

The *independence number* $\alpha(G)$ of a graph $G$ is the cardinality of a maximum independent set of $G$. Recall that an independent set of $G$ is a subset of vertices in $G$ such that no two of them are connected by an edge of $G$. For a claw-free graph, there is a well-known result [9] as follows.

**Lemma 2.2.** *Let $G$ be a claw-free graph with independence number at least three, then every vertex $v$ satisfies exactly one of the following:*
*(1) $N_G(v)$ is covered by two cliques,*
*(2) $N_G(v)$ contains an induced $C_5$.*

From Lemma 2.2, we know that for a claw-free graph $G$, the subgraph induced by the neighborhood of a vertex $v \in G$ is covered by at most two cliques, or contains an included $C_5$. In the following, we give an observation about the subgraph induced by $N_G^2(v)$.

**Observation 1.** Let $G$ be a graph with diameter two and $v \in V(G)$. If $G_2$ is triangle-free, then $G[N_G^2(v)]$ is a clique.

Let $G^{(d)}$ denote a graph $G$ with diameter $d$. Let $v_0v_1\ldots v_d$ be a spindle of $G^{(d)}$ and $V_d = N_{G^{(d)}}(v_{d-1})\backslash v_{d-2}$. We define an operation as follows: Delete all the edges between $v_{d-1}$ and $V_d$, and then join $v_{d-2}$ with all the vertices of $V_d$. We call this operation *"move $V_d$ to $v_{d-2}$"*.

**Lemma 2.3.** *If $G$ be a graph with diameter $d \geq 2$, then $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$, where $G^{(d-1)}$ is obtained by applying the above operation on $G^{(d)}$.*

*Proof.* Let $v_0v_1\ldots v_d$ be a spindle of $G^{(d)}$. After applying the above operation on $G^{(d)}$, the only change on the number of vertex pairs $\{u,v\}$ such that $d_{G^{(d)}}(u,v) = 2$ is brought by the movement of $V_d$. Thus, to prove $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$, it suffices to show that for every spindle of $G^{(d)}$, the number of vertex pairs such that the distance between them is two is not decreasing after using the operation "move $V_d$ to $v_{d-2}$". So we only need to show that for some spindle, after using the above operation, $|\{u|d_{G^{(d)}}(u,v_d) = 2\}| \leq |\{u'|d_{G'}(u',v_d) = 2\}|$, where $G'$ is obtained by moving $V_d$ to $v_{d-2}$ in $G^{(d)}$. Since $|\{u|d_{G^{(d)}}(u,v_d) = 2\}| = |N_{G^{(d)}}(v_{d-1})\backslash v_d|$ and $|\{u'|d_{G'}(u',v_d) = 2\}| = |N_{G^{(d)}}(v_{d-2})\backslash v_{d-2} \cup N_{G^{(d)}}(v_{d-1})\backslash v_d \backslash v_{d-2}|$, we can get that $|\{u|d_{G^{(d)}}(u,v_d) = 2\}| \leq |\{u'|d_{G'}(u',v_d) = 2\}|$. Therefore, we have $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$. □

**Corollary 2.1.** *If $G$ is a graph with diameter $d \geq 2$, then $e(G_2^{(d)}) \leq e(G_2^{(2)})$.*

*Proof.* By Lemma 2.3, we know that $e(G_2^{(d)}) \leq e(G_2^{(d-1)}) \leq \cdots \leq e(G_2^{(2)})$. □

By Corollary 2.1, we know that to maximize $|e(G_2^{(d)})|$ ($d \geq 2$), it suffices to get the maximum value of $|e(G_2^{(2)})|$. Thus, in the following part of the paper, we focus on the graph $G^{(2)}$. For convenience, we write $G$ instead of $G^{(2)}$.

# 3 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. For convenience, we use $\{x, y\}$ or vertex-pair to stand for the vertex-pair such that $d_G(x, y) = 2$. Actually, Theorem 1.3 can be stated as the following theorem.

**Theorem 3.1.** *Let $G$ be a graph with $|V(G)| \geq 5$. If there is a vertex $v \in V(G)$ whose neighborhood is covered by at most two cliques, then a triangle-free $G_2$ can have no more than $(n-1)^2/4 + 1$ edges.*

*Proof.* Since $G_2$ is triangle-free, Lemma 2.1 implies that $G$ is claw-free. Let $v \in V(G)$, whose neighbors is covered by at most two cliques. Then the proof will be given by the following cases.

**Case 1.** $N_G(v)$ is covered by only one clique.

Let $V_1 = N_G(v)$ and $V_2 = N_G^2(v)$. By Observation 1, $G[N_G^2(v)]$ is a clique. Suppose $V_1 = V_{11} \cup V_{12}$ and $V_2 = V_{21} \cup B \cup V_{22}$, where a vertex of $V_{21}$ is only adjacent to vertices of $V_{11}$ but not any vertex of $V_{12}$, a vertex of $V_{22}$ is only adjacent to vertices of $V_{12}$ but not any vertex of $V_{11}$, and a vertex of $B$ is adjacent to vertices of both $V_{11}$ and $V_{12}$. Without loss of generality, we suppose that $|V_{11}| \geq |V_{12}|$.

By considering whether $V_{12} = \emptyset$ or not, we discuss as follows.

**Subcase 1.1.** $V_{12} = \emptyset$.

Then, $V_{22} = \emptyset$ and $G[V_1, V_2]$ is a complete bipartite graph. Hence, $e(G_2) = |V_2| \leq n - 2 \leq (n-1)^2/4 + 1$.

**Subcase 1.2.** $V_{12} \neq \emptyset$.

In this subcase, we will give the proof in detail as follows.

**Subsubcase 1.2.1.** $V_{22} \neq \emptyset$.

Let $d = |V_{22}|$. We can define a new graph $G'$ as follows: $V(G') = V(G)$, $V_1' = N_{G'}(v) = V_{11}' \cup V_{12}'$ and $V_2' = N_{G'}^2(v) = V_{21}' \cup V_{22}'$, where both $G'[V_1']$ and $G'[V_2']$ are cliques, both $G'[V_{11}', V_{21}']$ and $G'[V_{12}', V_{22}']$ are complete bipartite graphs, and $|V_{11}'| = 1$, $|V_{22}'| = 1$.

Suppose $d = 1$. In the following, we show that $G'$ can be obtained from $G$ by applying the corresponding operations mentioned in the following paper. Moreover, we show that such operations ensure that the number of vertex-pairs remains the same or increases.

Firstly, delete the edges between $V_{12}$ and $B$, and it is obvious that the number of $\{u_1, u_2\}$'s is not decreasing, where $u_1, u_2 \in V(G)$. Then $V_{21}' = V_{21} \cup B$ and $V_{22}' = V_{22}$.

Secondly, suppose $u \in V_{11}$ such that $u$ is adjacent to all the vertices in $V'_{21}$. Delete the edges between $V_{11} \setminus \{u\}$ and $V'_{21}$, meanwhile, connect all the vertices in $V_{11} \setminus u$ and all the vertices in $V'_{12}$. Then $V'_{11} = \{u\}$ and $V'_{12} = \{V_{11} \setminus \{u\}\} \cup V_{12}$. Therefore, we get the graph $G'$ (see Figure 1). Let $a = |V_{11}|$, $b = |V_{12}|$, $c = |V_{21}| + |B|$. Since $e(G_2) = a + bc + c + 1$ and $e(G'_2) = (a + c - 1)b + (a + c + 1)$, then $e(G'_2) - e(G_2) = (a - 1)b \geq 0$, that is, after applying the above operation, we ensure that the number of $\{u, v\}$'s is not decreasing.

For the graph $G'$, we have $e(G'_2) = xy + x + 2$ where $x = |V'_{12}|$, $y = |V'_{21}|$, such that $x + y = n - 3$ and $x \geq 1$, $y \geq 1$. Thus, $e(G'_2) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

Now suppose $d \geq 2$. Let $u_1 \in V_{21}$ and $u_2 \in V_{12}$. Delete the edges between $V_{21} \setminus \{u_1\} \cup B$ and $V_{22}$ and the edges between $V_{12} \setminus \{u_2\}$ and $V_{11}$; meanwhile, move the vertices of $V_{21} \setminus \{u_1\} \cup B$ to $V_{11}$ (the new vertex set obtained is denoted by $V'_{12}$), and the vertices in $V_{12} \setminus \{u_2\}$ with $V_{22}$ (the new vertex set obtained is denoted by $V'_{21}$), therefore, we get the graph $G'$ (see Figure 1), where $V'_{11} = \{u_1\}$ and $V'_{22} = \{u_2\}$. Let $a = |V_{11}|$, $b = |V_{12}|$, $c = |V_{21}| + |B|$. Since $e(G_2) = ad + bc + c + d$ and $e(G'_2) = (a + c - 1)(b + d - 1) + (a + c - 1 + 1) + 1$, then $e(G'_2) - e(G_2) = a(b-1) + (c-1)(d-2) \geq 0$, that is, using the above operation we ensure that the number of $\{w_1, w_2\}$'s is not decreasing, where $w_1, w_2 \in V(G')$.

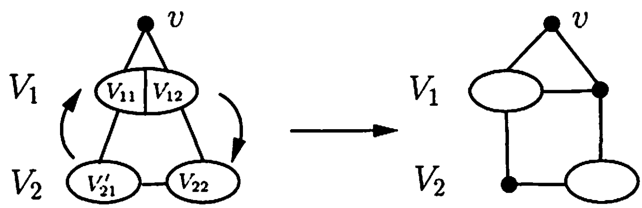


Figure 1: A new graph $G'$ for $d \geq 2$

For the graph $G'$, we have $e(G'_2) = xy + x + 2$, where $x = |V'_{12}|$, $y = |V'_{21}|$ such that $x + y = n - 3$ and $x \geq 1$, $y \geq 1$. Thus, $e(G'_2) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

**Subsubcase 1.2.2.** $V_{22} = \emptyset$.

For this case, delete the edges between $V_{11}$ and $B$, then it returns to **Subsubcase 1.2.1**.

By combining all the situations in **Subcase 1.1**, we get that $e(G) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

**Case 2.** $G[N_G(v)]$ is covered by two cliques.

In this case, $G[N_G(v)]$ is covered by two cliques, denoted by $V_1$ and $U_1$. According to the condition whether $e(U_1, V_1) = 0$, we prove it by the following subcases.

**Subcase 2.1.** $e(U_1, V_1) = 0$.

Let $V_2 = N_G^2(v)$. $V_2$ can be divided into three parts $A$, $B$ and $C$, where a vertex of $A$ is only adjacent to vertices of $V_1$, a vertex of $B$ is adjacent to vertices of both $V_1$ and $U_1$, and a vertex of $C$ is only adjacent to vertices of $U_1$. Now we give the following claim.

**Claim 1.** $G[V_1, A]$ and $G[U_1, C]$ are both complete bipartite graphs.

*Proof.* If $G[V_1, A]$ is not a complete bipartite graph, then there are vertices $u_1 \in A$, $u_2 \in V_1$, $w \in U_1$ such that $d_G(u_1, u_2) = d_G(u_1, w) = d_G(u_2, w) = 2$. Thus, the three vertices $u_1, u_2, w$ form a triangle in $G_2$, a contradiction.

Similarly, $G[U_1, C]$ is also a complete bipartite graph. □

Now we divide vertex set $B$ into three parts $B_1$, $B_2$ and $B_3$, where a vertex of $B_1$ is only adjacent to vertices of $V_1$, a vertex of $B_2$ is adjacent to vertices of both $V_1$ and $U_1$, and a vertex of $B_3$ is only adjacent to vertices of $U_1$.

**Claim 2.** Both $G[V_1, B_1 \cup B_2]$ and $G[U_1, B_2 \cup B_3]$ are complete bipartite graphs.

*Proof.* To prove Claim 2, it suffices to show that every vertex $u \in B$ is adjacent to all the vertices of $V_1$ or $U_1$ , that is, at least one of $G[V_1, u]$ and $G[U_1, u]$ is a complete bipartite graph. Suppose that there is a vertex $u \in B$ such that neither $G[V_1, u]$ nor $G[U_1, u]$ is a complete bipartite graph. Then there are vertices $w_1 \in V_1$, $w_2 \in U_1$ such that $d_G(u, w_1) = d_G(u, w_2) = d_G(w_1, w_2) = 2$. Thus, $uw_1, w_1w_2, w_2u \in e(G_2)$, contradicting to the fact that $G_2$ is triangle-free. □

In the following, we apply some operations on $G$ to maximize $|e(G_2)|$.

We define a new graph $G'$ as follows. $G'$ with $V_1' = V_1$, $U_1' = U_1$, $V_2' = A' \cup C'$ ($A' = A \cup B_1 \cup B_2$ and $C' = B_3 \cup C$), where $G[V_1', A']$, $G[V_2', C']$ and $G[A', C']$ are complete bipartite graphs, $G[V_1']$, $G[V_2']$, $G[A']$ and $G[B']$ are complete graphs.

Form $G$ to $G'$, we perform the following operations: Delete the edges between $B_1$ and $U_1$, remove the edges between $B_2$ and $U_1$, and the edges between $B_3$ and $V_1$. It is obvious that the number of vertex-pairs at distance two is not decreasing.

Now we construct a new graph $G''$ (see Figure 2) which is obtained from $G'$, that is, $V(G'') = V(G')$, by moving the vertex in $A' \setminus \{u\}$ to $V_1'$, and the vertices in $C' \setminus \{w\}$ to $U_1'$. The new vertex sets are denoted by $V_1''$ and $U_1''$, where $u \in A'$ and $w \in B'$.

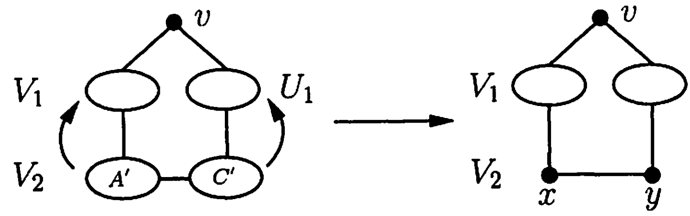


Figure 2: A new graph $G''$

Let $a = |V_1'|$, $b = |V_2'|$, $c = |A'|$, $d = |B'|$. Since $e(G_2') = ab+ad+bc+c+d$ and $e(G_2'') = (a+c-1)(b+d-1)+(a+c-1)+(b+d-1)+2$, then $e(G_2'') - e(G_2') = (c-1)(d-1) \geq 0$, that is, after using the move operation the number of vertex-pairs whose distance is two is not decreasing.

Now, for the graph $G''$, let $x = |V_1''|$ and $y = |U_1''|$. Then $e(G_2'') = xy + x + y + 2$, where $x + y = n - 3$ and $x, y \geq 1$. By some calculations, we get that $e(G_2'') \leq (n-1)^2/4 + 1$. And the equality holds if and only if $n$ is odd and $x = y = (n-1)/2$, where $n \geq 5$.

Combining all the situations in **Case 1**, $e(G_2) \leq (n-1)^2/4 + 1$ follows from the condition that $N_G(v)$ is covered by at least two cliques.

**Subcase 2.2.** $e(U_1, V_1) \neq 0$.

In this subcase, if there is not a subset $B$ of $V_2$ such that the vertices in $B$ can form the vertex-pairs with some vertices of $V_{11} \subseteq V_1$ and meanwhile with some vertices of $U_{11} \subseteq U_1$, then we delete the edges between $U_1$ and $V_1$. Now it returns to **Subcase 2.1.**

If there is a subset $B$ of $V_2$ such that the vertices in $B$ can form the vertex-pairs both with some vertices of $V_{11} \subseteq V_1$ and with some vertices of $U_{11} \subseteq U_1$, then divide $B$ into two parts $B_1$ and $B_2$, delete the edges between $V_1$ and $U_1$, move $B_1$ to $V_1$, and move $B_2$ to $U_1$. Now we want to prove that there always exist such $B_1$ and $B_2$ to ensure that the number of the vertex-pairs is not decreasing. Let $a = |V_{11}|$, $b = |U_{11}|$, $c = |B_1|$, $d = |B_2|$. By only considering the vertex-pairs of those sets, the change of the number is at least $(a+c)(b+d)-(c+d)(a+b) = (a-d)(b-c)$, which is no less than 0 (by some knowledge of the inequality, no matter how much $(a+c)$ and $(c+d)$ are, we can find some number to make it right). Now this subcase returns to **Subcase 2.1.**

Combining all the above cases, we complete the proof of Theorem 3.1. □

From the proof of Theorem 3.1, we can easily get Corollary 1.1. By

Lemmas 2.1 and 2.2, if we can prove the following statement: All the claw-free graphs with diameter two, which has no three vertices pairwise at distance 2, has at most $(n-1)^2/4+1$ pairs of vertices at distance 2, then we can confirm the guess of Tyomkyn and Uzzell.

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