A Note on the Density of M-sets in Geometric Sequence

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Abstract

For a given set M of positive integers, a well known problem of Motzkin asks for determining the maximal density $\mu(M)$ among sets of nonnegative integers in which no two elements differ by an element of M. The problem is completely settled when $|M| \leq 2$, and some partial results are known for several families of M for $|M| \geq 3$, including the case where the elements of M are in arithmetic progression. We resolve the problem in case of geometric progressions and geometric sequences.

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1 Introduction

For $x \in \mathbb{R}$ and a set S of nonnegative integers, let $\#(S,x)$ denote the number of elements $n \in S$ such that $n \leq x$. The upper density of S,

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denoted by $\overline{\delta}(S)$, is defined as

$$\overline{\delta}(S) := \limsup_{x\to\infty} \frac{\#(S,x)}{x}.$$

Given a set of positive integers M, S is said to be an M-set if $a \in S$, $b \in S$ imply $a - b \notin M$. Motzkin in [4] asked to determine $\mu(M)$ defined as

$$\mu(M) := \sup_{S} \overline{\delta}(S)$$

where S varies over all M-sets. Cantor & Gordon in [1] determined $\mu(M)$ when $|M| \leq 2$, and gave the the following lower bound for $\mu(M)$:

$$\mu(M) \geq \sup_{\gcd(c,m)=1} \frac{1}{m} \min_{i} |cm_i|_m, \tag{1}$$

where m_i are the elements of M, m and c are any two relatively prime positive integers, and $|x|_m$ denotes the absolute value of the absolutely least remainder of x mod m. A useful upper bound for $\mu(M)$ is due to Haralambis in [3]:

$$\mu(M) \leq \alpha \tag{2}$$

provided there exists a positive integer k such that $S(k) \leq (k+1)\alpha$ for every M-set S with $0 \in S$.

The problem of determining the density of M-sets is completely resolved in several special cases, including when M consists of numbers in arithmetic progression [2]. Due to the fact that $\mu(cM) = \mu(M)$ for any positive integer c, the problem in case of a geometric progression amounts to determining $\mu(\mathscr{G}_{r,k})$ with $\mathscr{G}_{r,k} = \{1, r, r^2, \ldots, r^k\}$ for $r > 1$ and $k \geq 1$. For positive and relatively prime integers a, b, the geometric progression with first term 1 and common ratio a/b yields the geometric sequence $\mathscr{G}_{a,b;k} = \{a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k\}$. For positive integers a, b, k, with $\gcd(a,b) = 1$ and $k \geq 2$, we show that $\mu(\mathscr{G}_{a,b;k}) = \mu(\mathscr{G}_{a,b;2})$. A similar argument proves that $\mu(\mathscr{G}_{r,k}) = \mu(\mathscr{G}_{r,1})$ for $r > 1$ and $k \geq 1$, and this extends to the case of the infinite geometric progression $\mathscr{G}_r = \{1, r, r^2, \ldots\}$ for $r > 1$.

2 Results

Throughout this section, let a, b, k be positive integers with $\gcd(a,b) = 1$ and $k \geq 2$.

Theorem 1. *Let a, b, k be positive integers, with* $\gcd(a,b) = 1$ *and* $k \geq 2$, *and let* $\mathscr{G}_{a,b;k} = \{a^k, a^{k-1}b, \ldots, ab^{k-1}, b^k\}$. *Then*

$$\mu(\mathscr{G}_{a,b;k}) = \mu(\{a,b\}) = \frac{\lfloor \frac{1}{2}(a+b) \rfloor}{a+b}.$$

Proof. Note that $\mu(\mathscr{G}_{a,b;k}) \leq \mu(\{a^k, a^{k-1}b\}) = \mu(\{a,b\})$. If a, b are odd, all elements of $M = \mathscr{G}_{a,b;k}$ are odd, and the assertion is obvious since $\{1, 3, 5, \ldots\}$ is an M-set with density $\frac{1}{2}$.

Suppose $a+b$ is odd. We use (1) to show that $\mu(\{a,b\}) = \frac{a+b-1}{2(a+b)}$ is a lower bound for $\mu(\mathscr{G}_{a,b;k})$. Let $m = a+b$, and choose c such that $a^k c \equiv \frac{a+b-1}{2} \pmod{a+b}$. Since $b \equiv -a \pmod{a+b}$, it easily follows that $a^i b^{k-i} c \equiv a^i(-a)^{k-i} c = (-1)^{k-i} a^k c \equiv \pm \frac{a+b-1}{2} \pmod{a+b}$ for $0 \leq i \leq k-1$. This provides the desired lower bound, and the proof of the result. ■

The special case of the geometric progression may be obtained from Theorem 1 by choosing $a = 1$ and $b = r$. For $r > 1$ and $k \geq 1$, let $\mathscr{G}_{r,k} := \mathscr{G}_{1,r;k} = \{1, r, r^2, \ldots, r^k\}$. By Theorem 1, we have

$$\mu(\mathscr{G}_{r,k}) = \mu(\{1,r\}) = \frac{\lfloor \frac{1}{2}(r+1) \rfloor}{r+1}.$$

This result extends to the case of the infinite geometric progression.

Theorem 2. *For $r > 1$, let $\mathscr{G}_r = \{1, r, r^2, \ldots\}$. Then*

$$\mu(\mathscr{G}_r) = \mu(\{1,r\}) = \frac{\lfloor \frac{1}{2}(r+1) \rfloor}{r+1}.$$

Proof. Note that $\mu(\mathscr{G}_r) \leq \mu(\{1,r\})$. If r is odd, all elements of $\mathscr{G}_r$ are odd and so $\mathscr{G}_r$ has density $\frac{1}{2}$. For even r, it suffices to show that $\mu(\{1,r\}) = \frac{r}{2(r+1)}$ is a lower bound for $\mu(\mathscr{G}_r)$. Let $c \equiv \frac{r}{2} \pmod{r+1}$. Then $r^i c \equiv (-1)^i \frac{r}{2} \pmod{r+1}$ for each $i \geq 0$. This provides the desired lower bound and the claim. ■

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References

[1] D. G. Cantor and B. Gordon, Sequences of integers with missing differences, *J. Combin. Theory Ser. A* **14** (1973), 281–287.

[2] S. Gupta and A. Tripathi, Density of M-sets in Arithmetic Progression, *Acta Arith.* **89.3** (1999), 255–257.

[3] N. M. Haralambis, Sets of Integers with Missing Differences, *J. Combin. Theory Ser. A* **23** (1977), 22–33.

[4] T. S. Motzkin, Unpublished problem collection.