# $k$-Domination stable graphs upon edge removal

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**Abstract**

Let $k$ be a positive integer and $G = (V(G), E(G))$ a graph. A subset $S$ of $V(G)$ is a $k$-dominating set if every vertex of $V(G) - S$ is adjacent to at least $k$ vertices of $S$. The $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. A graph $G$ is called $\gamma_k^-$-stable if $\gamma_k(G - e) = \gamma_k(G)$ for every edge $e$ of $E(G)$. We first give a necessary and sufficient condition for $\gamma_k^-$-stable graphs. Then for $k \geq 2$ we provide a constructive characterization of $\gamma_k^-$-stable trees.

**Keywords:** $k$-domination stable graphs, $k$-domination.

**AMS Subject Classification:** 05C69

## 1 Introduction

We consider finite, undirected, and simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *degree* of $v$, denoted by $d_G(v)$, is the size of its open neighborhood. We denote by $K_{1,t}$ a star of order $t+1$ and by $K_{1,0}$ the graph of order one. Specifically, for a vertex $v$ in a rooted tree $T$, we denote by $C(v)$

and $D(v)$ the set of children and descendants, respectively, of $v$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D(v) \cup \{v\}$, and is denoted by $T_v$.

In [1] Fink and Jacobson generalized the concept of dominating sets. Let $k$ be a positive integer. A subset $S$ of $V(G)$ is *$k$-dominating* if every vertex of $V(G) - S$ is adjacent to at least $k$ vertices of $S$. The *$k$-domination number* $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. Thus the 1-dominating set is a dominating set and so $\gamma_1(G) = \gamma(G)$. If $S$ is a $k$-dominating set of $G$ of size $\gamma_k(G)$, then we call $S$ a $\gamma_k(G)$-set. A graph $G$ is called *$\gamma_k^-$-stable* if $\gamma_k(G - e) = \gamma_k(G)$ for every edge $e$ of $E(G)$. An edge $e$ whose deletion from $G$ does not affect the $k$-domination number is called a *stable edge.*

In [2] Hartnell and Rall characterized the trees $T$ whose domination numbers are unaffected by deletion of any edge, that is $\gamma(T - e) = \gamma(T)$ for every edge $e$ of $E(T)$.

In this paper, we first give a necessary and sufficient condition for $\gamma_k^-$-stable graphs. Then we provide for $k \geq 2$ a constructive characterization of $\gamma_k^-$-stable trees.

## 2 $\gamma_k^-$-stable graphs

The following observation is straightforward.

**Observation 1** *Every $k$-dominating set of a graph $G$ contains any vertex of degree at most $k - 1$.*

Since every $k$-dominating set of a spanning graph of $G$ is also a $k$-dominating set of $G$ we have the following observation.

**Observation 2** *For any graph $G$ and edge $e \in E(G)$, $\gamma_k(G - e) \geq \gamma_k(G)$.*

Next we give a necessary and sufficient condition for $\gamma_k^-$-stable graphs.

**Theorem 3** *Let $k$ be a positive integer. A graph $G$ is $\gamma_k^-$-stable if and only if for each pair of adjacent vertices $u, v \in V(G)$, there exists a $\gamma_k(G)$-set $D$ such that one of the following conditions holds:*

*i) $u, v$ are both in $D$ or both in $V(G) - D$,*

*ii) if $u \in D$ and $v \notin D$, then $v$ is $(k+1)$-dominated by $D$.*

**Proof.** Let $u, v$ be any pair of adjacent vertices for which there is a $\gamma_k(G)$-set $D$ such that one of Conditions (i) or (ii) is verified. Then by removing $uv$, the set $D$ remains a $k$-dominating set of $G - uv$ and so $\gamma_k(G - uv) \leq |D|$. Equality is obtained from Observation 2.

Now assume that $G$ is a $\gamma_k^-$-stable graph. Let $uv$ be any stable edge and $D$ a $\gamma_k(G-uv)$-set. Then $|D| = \gamma_k(G)$ and $D$ is a $\gamma_k(G)$-set. Clearly if $u, v \in D$ or $u, v \notin D$, then Condition (i) holds. Without loss of generality, assume that $u \in D$ and $v \notin D$. Then $v$ is $k$-dominated by $D$ in $G - uv$ and so it is $(k+1)$-dominated by $D$ in $G$. Hence Condition (ii) follows. □

We note that for $k = 1$ Theorem 3 has been obtained by Walikar and Acharya [3].

For the purpose of characterizing $\gamma_k^-$-stable trees for every integer $k \geq 2$, we define the family $\mathcal{H}_k$ of all trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ $(p \geq 1)$ of trees, where $T_1 = K_{1,m}$ $(m \neq k$ and $m \geq 1)$, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the following operations.

- Operation $\mathcal{O}_1$: Add a star $K_{1,t}$ $(0 \leq t \leq k-2)$ of center vertex $x$ by adding an edge from $x$ to a vertex $y$ in $T_i$ with degree at most $k-2$ in $T_i$.

- Operation $\mathcal{O}_2$: Add a star $K_{1,t}$ where $t \geq k+1$ for a given integer $k \geq 2$ by adding an edge from the center vertex $x$ of the star to any vertex $y$ of $T_i$.
- Operation $\mathcal{O}_3$: Add a star $K_{1,k}$ of center vertex $x$ by adding an edge from $x$ to a vertex $y$ of $T_i$ that belongs to a $\gamma_k(T_i)$-set.
- Operation $\mathcal{O}_4$: Add a star $K_{1,k-1}$ of center vertex $x_1$ and $p \geq k-1$ new stars of center vertices $x_2, ..., x_{p+1}$ each one of order at most $k-1$, by adding edges from each $x_i$ to a leaf $y$ of $T_i$ such that $\gamma_k(T_i - y) < \gamma_k(T_i)$, with a further condition if $p = k-1$, then the support vertex $z$ of $y$ belongs to some $\gamma_k(T_i)$-set.

The following observation will be useful for the next.

**Observation 4** *Let $k \geq 2$ be an integer and $T_w$ a tree rooted on a vertex $w$ of degree at least $k-1$ and such that all descendants of $w$ are of degree less than $k$. If $T$ is a tree obtained from $T_w$ by adding an edge between $w$ and a vertex $v$ of a tree $T'$, then $\gamma_k(T') \leq \gamma_k(T) - |V(T_w)| + 1$, with equality if either $d_{T_w}(w) \geq k$ or $v$ belongs to a $\gamma_k(T')$-set.*

**Proof.** Let $S$ be a $\gamma_k(T)$-set. Then by Observation 1, $S$ contains $V(T_w) - \{w\}$ and without loss of generality $w \notin S$ else replace $w$ in $S$ by $v$. Thus $S \cap V(T')$ is a $k$-dominating set of $T'$, so $\gamma_k(T') \leq \gamma_k(T) - |V(T_w)| + 1$. Now let $S'$ be a $\gamma_k(T')$-set. If $d_{T_w}(w) \geq k$ or $v \in S'$, then $S' \cup (V(T_w) - \{w\})$ is a $k$-dominating set of $T$. Hence $\gamma_k(T) \leq \gamma_k(T') + |V(T_w)| - 1$ and the equality follows. □

**Lemma 5** *For every integer $k \geq 2$, if $T \in \mathcal{H}_k$, then $T$ is $\gamma_k^-$-stable.*

**Proof.** Let $T$ be a tree of $\mathcal{H}_k$ for some integer $k \geq 2$. Then $T$ is obtained from a sequence $T_1, T_2, \ldots, T_p$ $(p \geq 1)$ of trees, where $T_1 = K_{1,m}$ $(m \neq k)$, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the four operations defined above. We use an induction on the number of operations performed to construct

$T$. Clearly the property is true if $p = 1$. This establishes the basis case.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in \mathcal{H}_k$ that can be constructed from a sequence of length at most $p - 1$, and let $T' = T_{p-1}$. By the induction hypothesis, $T'$ is a $\gamma_k^-$-stable tree and hence every edge of $E(T')$ is stable. For any edge $uv \in E(T')$, let $D_{uv}$ denote a $\gamma_k(T')$-set for which $u, v$ satisfy Condition (i) or (ii) of Theorem 3. Let $T$ be a tree obtained from $T'$ and consider the following four cases.

- $T$ is obtained from $T'$ by using Operation $\mathcal{O}_1$. Clearly by Observation 1 and since $d_{T'}(y) \leq k-2$, $\gamma_k(T) = \gamma_k(T') + |V(K_{1,t})|$. Let $uv$ be any edge of $E(T')$. Since $d_{T'}(y) \leq k-2$, $y \in D_{uv}$ and $D_{uv} \cup V(K_{1,t})$ is a $\gamma_k(T)$-set for which $u, v$ and every two adjacent vertices of $V(K_{1,t}) \cup \{y\}$ satisfy Condition (i) of Theorem 3, $T$ is a $\gamma_k^-$-stable tree.

- $T$ is obtained from $T'$ by using Operation $\mathcal{O}_2$. Then by Observation 4 $\gamma_k(T) = \gamma_k(T') + |V(K_{1,t})| - 1$. Let $uv$ be any edge of $E(T')$. Clearly $D'' = D_{uv} \cup (V(K_{1,t}) - \{x\})$ is a $\gamma_k(T)$-set, where $x$ is $(k+1)$-dominated by $D''$. Thus the pair $u, v$ and every two adjacent vertices in $V(K_{1,t}) \cup \{y\}$ satisfy Condition (i) or (ii) of Theorem 3. It follows that $T$ is a $\gamma_k^-$-stable tree.

- $T$ is obtained from $T'$ by using Operation $\mathcal{O}_3$. Then by Observation 4 $\gamma_k(T) = \gamma_k(T') + |V(K_{1,k})| - 1$. Let $uv$ be any edge of $E(T')$. It follows that $D_{uv} \cup (V(K_{1,k}) - \{x\})$ is a $\gamma_k(T)$-set for which $u, v$ satisfy one of the two conditions of Theorem 3. For the remaining edges incident with $x$, let $S'$ be a $\gamma_k(T')$-set containing $y$. Then $S' \cup (V(K_{1,k}) - \{x\})$ is a $\gamma_k(T)$-set that $(k+1)$-dominates $x$. Hence Condition (ii) is satisfied for every pair $x, b$, where $b \in N_T(x)$. Therefore $T$ is a $\gamma_k^-$-stable tree.

- $T$ is obtained from $T'$ by using Operation $\mathcal{O}_4$. Let $H_i$ be the added star of center $x_i$ with $1 \leq i \leq p+1$. Then $\gamma_k(T) = \gamma_k(T') + \sum_{i=1}^{p+1} |V(H_i)| - 1$. Let $uv$ be any edge of $E(T')$. Since

$d_{T'}(y) = 1$, $y \in D_{uv}$ and $D_{uv} \cup \left( \bigcup_{i=1}^{p+1} V(H_i) - \{x_1\} \right)$ is a $\gamma_k(T)$-set for which $u, v$ and every two adjacent vertices of $\{y\} \cup \left( \bigcup_{i=2}^{p+1} V(H_i) \right)$ satisfy one of the two conditions of Theorem 3, it remains to see all edges incident with $x_1$ in $T$. If $p = k - 1$, then by the construction there is a $\gamma_k(T')$-set containing $z$. Let $D_z$ be such a set. Then $y \in D_z$ and $(D_z - \{y\}) \cup \left( \bigcup_{i=1}^{p+1} V(H_i) \right)$ is a $\gamma_k(T)$-set that contains $x_1$ and all leaves neighbored to $x_1$ and that $(k+1)$-dominates $y$. If $p \geq k$, let $D_y$ be a $\gamma_k(T' - y)$-set. Since by construction $y$ satisfies $\gamma_k(T' - y) < \gamma_k(T')$ and $D_y \cup \left( \bigcup_{i=1}^{p+1} V(H_i) \right)$ is a $\gamma_k(T)$-set that contains $x_1$ and all leaves neighbored to $x_1$ and that $(k+1)$-dominates $y$. In both cases Condition (ii) is satisfied for the pair $x_1, y$ and Condition (i) is satisfied for every pair $x_1, b$ where $b$ is any leaf-neighbor of $x_1$. Therefore $T$ is a $\gamma_k^-$-stable tree. □

**Lemma 6** *Let $k \geq 2$ be an integer. If $T$ is a nontrivial $\gamma_k^-$-stable tree, then $T \in \mathcal{H}_k$.*

**Proof.** Let $k \geq 2$ be an integer and assume that $T$ is a $\gamma_k^-$-stable tree of order at least two. We use an induction on the order $n$ of $T$. Clearly if $T$ is a star $K_{1,m}$, then $m \neq k$, and hence $T$ belongs to $\mathcal{H}_k$. Assume that every $\gamma_k^-$-stable tree $T'$ of order $2 \leq n' < n$ is in $\mathcal{H}_k$. Let $T$ be a $\gamma_k^-$-stable tree of order $n$ and $D$ any $\gamma_k(T)$-set. If the maximum degree $\Delta(T) \leq k - 1$, then $T \in \mathcal{H}_k$ and is obtained from $T_1 = K_{1,m}$ with $m \leq k - 1$ by applying Operation $\mathcal{O}_1$. Suppose that $\Delta(T) = k$. Since $T$ is $\gamma_k^-$-stable, all pairs of adjacent vertices of $T$ satisfy only Condition (i) of Theorem 3. Hence every vertex of maximum degree $z$ has to be adjacent to another vertex of maximum degree for otherwise every $\gamma_k(T)$-set would contain all $N_T(z)$ and not $z$. Now let $w$ be a vertex of maximum degree and assume that $u$ is the unique vertex of maximum degree adjacent to $w$. Since $wu$ is a stable edge, let $S_{wu}$ be a $\gamma_k(T)$-set that contains $w, u$. Hence $S_{wu}$ also contains all the remaining vertices adjacent to $w$ but then $S_{wu} - \{w\}$

is a $k$-dominating set of $T$ smaller than $S_{wu}$, a contradiction. It follows that every vertex of maximum degree has to be neighbored to two other vertices of maximum degree. Then, since the graph is finite, there has to be a cycle, which is a contradiction. Thus from now on we can assume that $\Delta(T) \geq k+1$. Since stars $K_{1,m}$ with $m \neq k$ belong to $\mathcal{H}_k$ we assume that $T$ has diameter at least three.

We now root $T$ at a leaf $r$. Let $w$ be a vertex of degree at least $k$ at maximum distance from $r$. Let $u$ be the parent of $w$ in the rooted tree. Thus every descendant of $w$ has degree at most $k-1$ and hence $D$ contains all vertices of $D(w)$. If $r = u$, then $u$ is a leaf, $d_T(w) = \Delta(T)$ and $T \in \mathcal{H}_k$ since it is obtained from a star $K_{1,t}$ ($t \geq k+1$) of center $w$ by applying Operation $\mathcal{O}_1$ at least once. Thus suppose that $r \neq u$ and let $v$ be the parent of $u$. We distinguish between three cases.

**Case 1.** $d_T(w) \geq k+2$. If $w \in D$, then $u \notin D$ and so we can replace $w$ by $u$ in $D$. Thus we may assume that $w \notin D$. Let $T' = T - T_w$. Then by Observation 4 $\gamma_k(T') = \gamma_k(T) - |V(T_w)| + 1 = \gamma_k(T) - \gamma_k(T_w)$. Suppose now that $T'$ is not $\gamma_k^-$-stable. Thus there is an edge $xy \in E(T')$ such that $\gamma_k(T' - xy) > \gamma_k(T')$. Note that the removing of $xy$ from $T'$ provides two subtrees $T'(x)$ and $T'(y)$ containing $x$ and $y$, respectively. Also the removing of $xy$ from $T$ provides two subtrees $T(x)$ and $T(y)$. Without loss of generality, we can assume that $T'(y) = T(y)$, and so $T'(x)$ is a subtree of $T(x)$. Clearly $\gamma_k(T' - xy) = \gamma_k(T'(x)) + \gamma_k(T'(y))$ and $\gamma_k(T - xy) = \gamma_k(T(x)) + \gamma_k(T(y))$. It follows by Observation 4 that

$$\begin{aligned}
\gamma_k(T - xy) &= \gamma_k(T(x)) + \gamma_k(T(y)) \\
&= \gamma_k(T_w) + \gamma_k(T'(x)) + \gamma_k(T'(y)) \\
&= \gamma_k(T_w) + \gamma_k(T' - xy) \\
&> \gamma_k(T_w) + \gamma_k(T') = \gamma_k(T),
\end{aligned}$$

contradicting the fact that $T$ is $\gamma_k^-$-stable. Therefore $T'$ is $\gamma_k^-$-stable and so by induction on $T'$, we have $T' \in \mathcal{H}_k$. Consequently $T \in \mathcal{H}_k$ and is obtained from $T'$ by using Operation $\mathcal{O}_2$ followed repetitively by Operation $\mathcal{O}_1$ if $T_w$ is not a star.

**Case 2.** $d_T(w) = k+1$. Then since $D(w) \subset D$ no $\gamma_k(T)$-set contains both $w, u$. Let $T' = T - T_w$. Then by Observation 4 $\gamma_k(T') = \gamma_k(T) - \gamma_k(T_w) = \gamma_k(T) - |V(T_w)| + 1$. Now let $w'$ be any vertex of $C(w)$. Then $w'$ is in every $\gamma_k(T)$-set. Since $T$ is $\gamma_k^-$-stable, the edge $ww'$ is stable. By Theorem 3 there is a $\gamma_k(T)$-set $S$ that contains $w$, too, and so $u \notin S$, or that $(k+1)$-dominates $w$, that is $w \notin S$ and $u \in S$. In the first case we can replace $w$ by $u$ in $S$. In any case we may assume that $u \in S$, implying that $u$ belongs to at least the $\gamma_k(T')$-set $S \cap V(T')$. Now applying the same argument to that used in Case 1, we can see that $T'$ is a $\gamma_k^-$-stable tree. By induction on $T'$, we have $T' \in \mathcal{H}_k$. Therefore $T \in \mathcal{H}_k$ and is obtained from $T'$ by using Operation $\mathcal{O}_3$ followed repetitively by Operation $\mathcal{O}_1$ if $T_w$ is not a star.

**Case 3.** $d_T(w) = k$. Clearly to $k$-dominate $w$ every $\gamma_k(T)$-set contains either $u$ or $w$ but not both since such a set minus $w$ is a $k$-dominating set of $T$. Also since $T$ is $\gamma_k^-$-stable, $wu$ is a stable edge and hence $w, u$ have to satisfy Condition (ii) Theorem 3. It follows that there is a $\gamma_k(T)$-set, say $S$, such that $w \in S$, $u \notin S$ and $u$ is $(k+1)$-dominated by $S$. Therefore $d_T(u) \geq k+1$, that is $|C(u)| \geq k \geq 2$. Seeing the previous cases we can assume that every vertex in $C(u)$ has degree at most $k$. If there is a vertex $w' \in C(u)$ such that $w' \neq w$ and $d_T(w') = k$, then $w' \in S$ and hence $\{u\} \cup S - \{w, w'\}$ is a $k$-dominating set of $T$ smaller than $S$, a contradiction. Thus every vertex in $C(u) - \{w\}$ has degree at most $k-1$, that is for every $b \in C(u) - \{w\}$ the subtree induced by $b$ and its children is a star of order at most $k-1$. Note that $C(u) - \{w\}$ is in every $\gamma_k(T)$-set. Now let $T'$ be the tree obtained from $T$ by removing all vertices in $D(u)$. Then $u$ is a leaf in $T'$ and belongs to every $\gamma_k(T')$-set. It is easy to see that $\gamma_k(T) = \gamma_k(T') + |D(u)| - 1$. We observe that for the previous $\gamma_k(T)$-set $S$ containing $w$ and $(k+1)$-dominating $u$, $S' = S \cap V(T')$ is a $\gamma_k(T' - u)$-set, where $v$ may belong or not to $S'$ and $S' \cup \{u\}$ is a $\gamma_k(T')$-set. It follows that $\gamma_k(T' - u) < \gamma_k(T')$. Also if $d_T(u) = k+1$, then $N_T(u) \subset S$, implying that $v \in S'$ and so $v$ belongs to the $\gamma_k(T')$-set $S' \cup \{u\}$. Assume now that $T'$ is not a $\gamma_k^-$-stable tree. Then there is an edge $e = xy$ such that $\gamma_k(T' - xy) > \gamma_k(T')$. Such an edge $xy$ is different from $uv$ since $u, v$ are either both in the $\gamma_k(T')$-set $S' \cup \{u\}$, that is $u, v$ satisfy (i) of Theorem 3 or $v \notin S'$ and so $v$ is $(k+1)$-dominated by $S' \cup \{u\}$, that is $u, v$ satisfy Condition (ii). Now let $T'(x)$, $T'(y), T(x)$ and $T(y)$ as defined in Case 1. Then $\gamma_k(T' - xy) = \gamma_k(T'(x)) + \gamma_k(T'(y))$ and

$$\begin{aligned}
\gamma_k(T - xy) &= \gamma_k(T(x)) + \gamma_k(T(y)) \\
&= \big(\gamma_k(T'(x)) + |D(u)| - 1\big) + \gamma_k(T'(y)) \\
&= \gamma_k(T' - xy) + |D(u)| - 1 \\
&> \gamma_k(T') + |D(u)| - 1 = \gamma_k(T),
\end{aligned}$$

a contradiction to the fact that $T$ is $\gamma_k^-$-stable. Therefore $T'$ is a $\gamma_k^-$-stable tree and hence by induction on $T'$, $T' \in \mathcal{H}_k$. Consequently $T \in \mathcal{H}_k$ and is obtained from $T'$ by using Operation $\mathcal{O}_4$ followed repetitively by Operation $\mathcal{O}_1$ if $T_a$ is not a star for some $a \in C(u)$. $\square$

According to Lemmas 5 and 6 we have the following result.

**Theorem 7** *Let $k \geq 2$ be an integer. A nontrivial tree $T$ is $\gamma_k^-$-stable if and only if $T \in \mathcal{H}_k$.*

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