THE t–PEBBLING NUMBER AND $2t$–PEBBLING PROPERTY ON THE GRAPH $D_{n,C_{2m}}$

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Abstract. Given a distribution D of pebbles on the vertices of a graph G, a pebbling move on G consists of taking two pebbles off from a given vertex and placing one of them onto an adjacent vertex (the other one is discarded). The pebbling number of a graph, denoted by $f(G)$, is the minimal integer k such that any distribution of k pebbles on G allows one pebble to be moved to any specified vertex by a sequence of pebbling moves. In this paper, we calculate the t–pebbling number of the graph $D_{n,C_{2m}}$. Moreover, we verify the q-t-pebbling number in order to show that the graph $D_{n,C_{2m}}$ has $2t$–pebbling property.

Keywords: pebbling, t-pebbling number, 2t-pebbling property.

1. Introduction

Graph pebbling is a mathematical game and area of interest played on a graph with pebbles on the vertices. The game of pebbling was first suggested by Lagarias and Saks, as a tool for solving a particular problem in number theory. The pebbling number of a graph was first introduced into the literature by Chung [1]. A pebbling move consists of removing two pebbles from one vertex, throwing one away, and putting the other pebble on an adjacent vertex. The pebbling number of a specified vertex v in a graph G is the smallest number $f(G,v)$ with the property that from any distribution of $f(G,v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum of $f(G,v)$ over all the vertices of graph G. If one pebble is placed at each vertex other than the root vertex r, then no pebble can be moved to r. Also, if w is at distance d from r, and 2^d-1 pebbles are placed at w, then no pebble can be moved to r. We record this as $f(G) \geqslant max\{|V(G)|, 2^{d(G)}\}$, where $d(G)$ is the diameter of G.

Furthermore, $f(K_n) = n$ and $f(P_n) = 2^{n-1}$ (see [1]), where K_n denotes a complete graph with n vertices and P_n denotes a path with n vertices. Similarly, the $t-$pebbling number of v in G is the smallest number $f_t(G, v)$ such that from every placement of $f_t(G, v)$ pebbles, it is possible to move t pebbles to v. The $t-$pebbling number of G is the smallest number $f_t(G)$ such that no matter how $f_t(G)$ pebbles are placed on the vertices of G, t pebbles can be moved to any vertex by a sequence of pebbling moves. Obviously, $f(G) = f_1(G)$, $f_t(G) = max\{f_t(G, v)|v \in V(G)\}$.

Theorem 1.1. [2] *Let C_n denote a simple cycle with n vertices, where $n \geq 3$, then*
(i)$f(C_{2m}) = 2^m$. (ii)$f(C_{2m+1}) = 2\lfloor\frac{2^{m+1}}{3}\rfloor + 1 = \frac{2^{m+2}-(-1)^m}{3}$.

Theorem 1.2. [3] *Let C_n denote a simple cycle with n vertices, where $n \geq 3$, then*
(i)$f_t(C_{2m}) = t \cdot 2^m$.
(ii)$f_t(C_{2m+1}) = 2\lfloor\frac{2^{m+1}}{3}\rfloor + 1 + 2^m(t-1) = \frac{2^{m+2}-(-1)^m}{3} + 2^m(t-1)$.

Theorem 1.3. [7] *Let P_n be a path on n vertices, then $f_t(P_n) = t(2^{n-1})$.*

Theorem 1.4. [9] *The pebbling number of $D_{n,C_{2m}}$ is $[f(C_{2m}) - 1](n-2) + f(P_{2m+1})$.*

Furthermore, a graph G has the 2−pebbling property if for any distribution with more than $2f(G) - q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using pebbling moves, to get two pebbles to any vertex. Lourdusamy[4] extended the definitions of the 2−pebbling property to the $2t-$pebbling property. Given a distribution on G, let q be the number of vertices with at least one pebble. We say that a graph G has the $2t-$pebbling property if, for any distribution with more than $2f_t(G) - q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using pebbling moves, to get $2t$ pebbles to any vertex.

Theorem 1.5. [5] *Let K_n be the complete graph on n vertices. Then*
(i)K_n has the $2t-$pebbling property for all positive integer t;
(ii)if G satisfies the $2t-$pebbling property, then for all positive integer t

$$f_t(K_n \times G) \leqslant f(K_n)f_t(G)$$

In this paper, G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For u, $v \in V(G)$ the distance between u and v in G denoted by $d(u, v)$. Moreover, denote by $\tilde{D}(G)$ and $\tilde{D}(v)$ the number of pebbles on G and the number of pebbles on v after a specified sequence of pebbling moves. Let D be a distribution of pebbles on the vertices of G. For any vertex v of G, $D(v)$ denotes the number of pebbles on v in D and denotes the size of D as $|D|$, i.e. $|D| = \sum_{v \in V} D(v)$. If D is a distribution

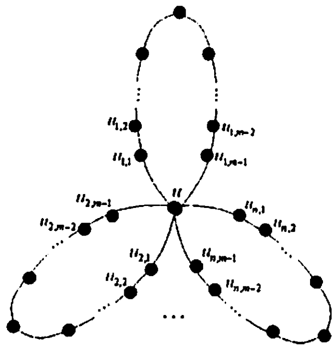

FIGURE 1. The graph D_{n,C_m}.

of pebbles on the vertices of G and there is some choice of a root r such that it is impossible to move a pebble to r, then we say that D is a *bad* distribution.

There are many existing results regarding $f(G)$. In [2], Pachter et al. gave the pebbling number and $t-$pebbling number of C_n, i.e. (see Theorem 1.1 and 1.2). In particular, Lourdusamy et al.[4-7] showed that the star graph, the $n-$cube, the complete graph, the fan graph, the wheel graph, and the even cycle have the $2t-$pebbling property. Moreover, Zetu Gao et al.[8] gave the t-pebbling number and 2t-pebbling property of generalized friendship graphs, and Han et al.[9] gave the pebbling number of the graph D_{n,C_m}(see Theorem 1.4. In exploring these results, we are naturally led to consider the relevant parameter of the graph $D_{n,C_{2m}}$. As shown in Fig.1, the graph D_{n,C_m} consists of n cycles with one common vertex, which denoted by u, and each cycle has m vertices besides the center point u.

This paper is organized as follows. In Section 2, we start with showing some preliminary lemmas and theorems based on the pigeonhole principle, and then, we calculate the $t-$pebbling number of the graph $D_{n,C_{2m}}$ by considering the number of occupied vertices of the graph. Finally, we prove that the graph$D_{n,C_{2m}}$ has the $2t-$pebbling property in Section 3.

2. $t-$PEBBLING NUMBER

This section studies the $t-$pebbling number of $D_{n,C_{2m}}$. First, we introduce the following lemmas, which is necessary for the proof of the main theorems.

Lemma 2.1. [9] *Let f be the pebbling number of C_m and place $(f-1)n+k$ pebbles on n cycles of D_{n,C_m} arbitrarily, then at least $\left\lfloor \frac{f-1+k}{f} \right\rfloor$ pebbles can be moved to the center point u, where k and n are positive integers.*

Based on Lemma 2.1, we have

Lemma 2.2. *Let $(2^m-1)n+k$ objects be in n boxes and let x_i be the number of objects in the ith box, then*

$$\sum_{i=1}^{n} \left\lfloor \frac{x_i}{2^m} \right\rfloor \geqslant \left\lfloor \frac{t \cdot 2^m - 1 + k}{2^m} \right\rfloor$$

where k, n, t and m are positive integers.

Proof. For $t=1$, the Lemma 2.2 is true by Lemma 2.1. We use induction on t to show the cases when $t \geqslant 1$. Note y_i and x_i be the number of objects in the ith box when $t=l$ and $t=l+1$.

First, suppose Lemma 2.2 is true when $t=l$, certainly, $\sum_{i=1}^{n} \left\lfloor \frac{y_i}{2^m} \right\rfloor \geqslant \left\lfloor \frac{l \cdot 2^m - 1 + k}{2^m} \right\rfloor = l + \left\lfloor \frac{k-1}{2^m} \right\rfloor$.

Next, we show that $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{2^m} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{y_i}{2^m} \right\rfloor \geqslant 1$. Let $\frac{x_i}{2^m} - \left\lfloor \frac{x_i}{2^m} \right\rfloor = a^i$, $\left\lfloor \frac{x_i}{2^m} \right\rfloor = a_0^i$, and $\frac{y_i}{2^m} - \left\lfloor \frac{y_i}{2^m} \right\rfloor = b^i$, $\left\lfloor \frac{y_i}{2^m} \right\rfloor = b_0^i$, then $\frac{x_i}{2^m} = a^i + a_0^i$ and $\frac{y_i}{2^m} = b^i + b_0^i$. We have

$$\sum_{i=1}^{n} \frac{x_i}{2^m} - \sum_{i=1}^{n} \frac{y_i}{2^m} = n = \sum_{i=1}^{n} (a_0^i - b_0^i) + \sum_{i=1}^{n} (a^i - b^i),$$

since $0 \le a^i - b^i \le 1$ ($\sum_{i=1}^{n} (a^i - b^i) \le n$), certainly, $\sum_{i=1}^{n} (a_0^i - b_0^i) \ge 0$. As a_0^i and b_0^i are positive integers, $\sum_{i=1}^{n} (a_0^i - b_0^i) \ge 1$. Thus, $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{2^m} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{y_i}{2^m} \right\rfloor \geqslant 1$, which means $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{2^m} \right\rfloor \ge l + \left\lfloor \frac{k-1}{2^m} \right\rfloor + 1$. Therefore, by the induction hypothesis, it suffices to show that the result holds for all t. □

Motivated by prior work, we consider the $t-$pebbling number of the graph $D_{n,C_{2m}}$.

Theorem 2.3. *The $t-$pebbling number of the graph $D_{n,C_{2m}}$ is $(n-2)(f(C_{2m})-1) + f_t(P_{2m+1})$.*

Proof. Note u as the center vertex of all the cycles in $D_{n,C_{2m}}$. Let $C^{(i)}$ be the cycle with the vertex $u_{i,m}$ and let $C^{(i)}/u$ be the cycle without the center point u. without losing generality, we may assume that $u_{1,m}$ is the target vertex.

First, suppose that there are $f_t(D_{n,C_{2m}}) - 1$ pebbles on the vertices of $D_{n,C_{2m}}$. Let D be the pebbling distribution $D(u_{n,m}) = t \cdot 2^{2m} - 1$, $D(u_{i,m}) = 2^m - 1$ for $i \in \{2, 3, \cdots n-1\}$, $D(u) = D(u_{i,j}) = 0$ ($i =$

$2, 3, \cdots n, j \neq m)$ and $D(u_{1,m}) = 0$. Then D is a bad distribution for $D_{n,C_{2m}}$ when size $|D| = f_t(D_{n,C_{2m}}) - 1$.

Next, we consider the distribution with $f_t(D_{n,C_{2m}})$ pebbles on the vertices of $D_{n,C_{2m}}$. The graph $D_{n,C_{2m}}$ has three kinds of target vertices, i.e.,(1) the center vertex u. (2) $u_{i,j}$, where $j \neq m$. (3) $u_{i,m}$, where $d(u_{i,m}, u) = m$. The proof of (1) and (2) are easy to be checked, so we consider the third situation in two cases.

Case 1: To prove the target vertex $u_{1,m}$ is occupied by pebbles, namely $= D(u_{1,m}) = l$, where $1 \leqslant l \leqslant t-1$. Since there are l pebbles on $u_{1,m}$, we have $D(D_{n,C_{2m}}/u_{1,m}) = f_t(D_{n,C_{2m}}) - l$, which we can rewritten as

$$\begin{aligned} D(D_{n,C_{2m}}/u_{1,m}) &= (n-2)(2^m-1) + t \cdot 2^{2m} - l \\ &\geqslant (n-2)(2^m-1) + (t-l) \cdot 2^{2m} = f_{t-l}(D_{n,C_{2m}}) \end{aligned}$$

This implies that we could use the remaining $f_t(D_{n,C_{2m}}) - l$ pebbles to put $(t-l)$ additional pebbles on $u_{1,m}$. Thus, the total number of pebbles on $u_{1,m}$ is $l + t - l = t$.

Case2: To prove the case $D(u_{1,m}) = 0$. For convenience, the cycle $C^{(1)}$ is divided into two parts $\mathscr{P}_a = \langle u_{1,1}, u_{1,2}, \cdots u_{1,m}\rangle$ and $\mathscr{P}_b = \langle u_{1,2m-1}, u_{1,2m-2}, \cdots, u_{1,m}\rangle$.

Subcase 2.1: To prove the case when $\mathscr{P}_a$ and $\mathscr{P}_b$ are not occupied by pebbles. This implies that there are $f_t(D_{n,C_{2m}})$ pebbles on the other $(n-1)$ cycles. $|D| = (n-1)(2^m-1) + t \cdot 2^{2m} - (2^m-1)$. According to Lemma 2.2, $\tilde{D}(u) = \left\lfloor \frac{2^m - 1 + t \cdot 2^{2m} - (2^m-1)}{2^m} \right\rfloor = t \cdot 2^m$.

Subcase 2.2: For the cycle $C^{(1)}$, to prove the cases when $\mathscr{P}_a$ and $\mathscr{P}_b$ are occupied by pebbles.

Subcase 2.2.1: $D(\mathscr{P}_a)$ or $D(\mathscr{P}_b)$ is more than $t \cdot 2^{m-1}$. Obviously, at least t pebbles can be moved to $u_{1,m}$.

Subcase 2.2.2: Both $D(\mathscr{P}_a)$ and $D(\mathscr{P}_b)$ are less than $t \cdot 2^{m-1}$.

First, Let H be the subgraph of $D_{n,C_{2m}}$, denoted by $H = D_{n,C_{2m}} - \{u_{1,1}, \cdots, u_{1,2m-1}\}$. Let D' be a distribution H, it implies that $|D'|$ pebbles are artificially redistributed on the other $(n-1)$ cycles, then we have

$$|D'| = \sum_{v \in V(H)} D(v) = f(D_{n,C_{2m}}) - (D(\mathscr{P}_a) + D(\mathscr{P}_b))$$

which, we can rewrite as

$$|D'| = (n-1)(2^m-1) + t \cdot 2^{2m} - (2^m-1) - (D(\mathscr{P}_a) + D(\mathscr{P}_b)),$$

by Lemma 2.2, we have

$$\tilde{D}(u) = \left\lfloor t \cdot 2^m - \frac{D(\mathscr{P}_a) + D(\mathscr{P}_b)}{2^m} \right\rfloor \geqslant t \cdot 2^m - \frac{D(\mathscr{P}_a) + D(\mathscr{P}_b)}{2^m} - 1.$$

Second, for the cycle $C^{(1)}$, $\tilde{p}(u) + D(\mathscr{P}_a) + D(\mathscr{P}_b) \geqslant t \cdot 2^m - \frac{D(\mathscr{P}_a)+D(\mathscr{P}_b)}{2^m} - 1 + D(\mathscr{P}_a) + D(\mathscr{P}_b) \geqslant t \cdot 2^m + \frac{2(2^m-1)}{2^m} - 1 \geqslant t \cdot 2^m = f_t(C_{2m})$.

Thus, t pebbles can be moved to $u_{1,m}$.

Therefore, $f_t(D_{n,C_{2m}}) = (n-2)(f(C_{2m}) - 1) + f_t(P_{2m+1})$. □

3. $2t$–Pebbling Property

The q-t-pebbling number $f_t^q(G)$ features prominently in the following proof, so we define it at first.

Definition 3.1. *The q-t-pebbling number $f_t^q(G)$ is the minimal positive integer such that, for every distribution of $f_t^q(G)$ pebbles, t pebble can be moved to any specified vertex by a sequence of pebbling moves, where q is the number of occupied vertices.*

Before we proceed with the next lemma we need to introduce the notation used in its proof. Let $C_{2m} = [u, a_1, \cdots, a_{m-1}, v, b_{2m-1}, \cdots, b_1]$. For C_{2m}, define $\mathscr{P}_A = [a_1, a_2 \cdots, a_{m-1}, v]$ and $\mathscr{P}_B = [b_{m+1}, b_{m+2} \cdots, b_{2m-1}, v]$. Without loss of generality, assume that $D(\mathscr{P}_A) \leqslant D(\mathscr{P}_B)$. Let s be the number of vertices of $\mathscr{P}_A$ with at least one pebble.

First, we introduce Lemma 3.2, which is necessary for the proof of the main Theorem.

Lemma 3.2. $f_t^q(C_{2m}) \leqslant t \cdot 2^m - q + 1$. *($t \geqslant 2$)*

Proof. The argument depends on the number of occupied vertices on C_{2m}.
Case 1: $q = 2s + 1$ $(0 \leqslant s \leqslant m-1)$. Lemma 3.2 turns to $f_t^{2s+1}(C_{2m}) \leqslant t \cdot 2^m - 2s$, we let D be a distribution of $(t \cdot 2^m - 2s)$ pebbles on C_{2m} for all $t \geqslant 2$. Suppose that $D(a_i) = D(b_{2m-j}) = 1$ $(i, j \in \{1, 2, \cdots, s\})$, then $D(u) = t \cdot 2^m - 4s$. The pebbles on the vertex u can be partitioned into 2 groups, $D_A(u)$ and $D_B(u)$. We attempt to transfer $D_A(u)$ pebbles to v through the path $\mathscr{P}_A$, while moving $D_b(u)$ pebbles to the target vertex through the path $\mathscr{P}_B$. Let $D_A(u) = 2^s(2^{m-s} - 1) - 2^s + 2 = 2^m - 2^{s+1} + 2$ and let $D_B(u) = D(u) - D_A(u) = (t-1) \cdot 2^m + 2^{s+1} - 4s - 2$. For the path $\mathscr{P}_A$, $D(a_i) = \frac{2^m - 2^{s+1} + 2^{i+1}}{2^i}$, with $1 \leqslant i \leqslant s$, in particular, $D(a_i) = D(a_s) = 2^{m-s}$ when $i = s$, then one pebble can be moved to v. For the path $\mathscr{P}_B$, $D(b_{2m-j}) = (t-1) \cdot 2^{m-j} + \frac{2^s - 2s + 2^j - 2}{2^j - 1}$,in particular, $D(b_{2m-j}) = D(b_{2m-s}) \geqslant 2^{m-j}$ when $j = s$, then at least $(t-1)$ pebbles can be moved

to v. As the target vertex can obtain pebbles from both $\mathscr{P}_A$ and $\mathscr{P}_B$, we can move at least t pebbles to v.

Case 2: $q = 2s+2$. To prove Lemma 3.2, we let D be a distribution of $(t\cdot 2^m - 2s - 1)$ pebbles on C_{2m} for all $t \geqslant 2$. Suppose that $D(a_i) = D(b_{2m-(j+1)}) = 1$ $(i,j \in \{1,2,\cdots,s\})$, then $D(u) = t\cdot 2^m - 4s - 2$. Similarly, $D_A(u) = 2^m - 2^{s+1} + 2$ and $D_B(u) = D(u) - D_A(u) = (t-1)\cdot 2^m + 2^{s+1} - 4s - 4$. As in Case 1, we can move at least one pebble to v along the path $\mathscr{P}_A$. For the path $\mathscr{P}_B$, $D(b_{2m-j}) = (t-1)\cdot 2^{m-j} + \frac{2^s-2s+2^j-3}{2^{j-1}}$, in particular, when $j = s+1$, $D(b_{2m-j}) = (t-1)\cdot 2^{m-(s+1)} + \frac{2^s-2s+2^{s+1}-3}{2^s}$. Thus, at least t pebbles can be moved to v along the path $\mathscr{P}_A$ and $\mathscr{P}_B$.□

Based on Theorem 2.3 and Lemma 3.2, we will prove the graph $D_{n,C_{2m}}$ has the 2t-pebbling property in the following theorem.

Theorem 3.3. *Graph $D_{n,C_{2m}}$ has the 2t-pebbling property.*

Proof. Let x_i be the number of pebbles on the cycle $C^{(i)}$ and let y_i be the number of pebbles on u, which are moved from the cycle $C^{(i)}$. Since the distances between u and $u_{i,j}$ $(i \in \{1,\cdots,n\}, j \neq m)$, which are less than m, that is to say, more pebbles are required during the pebbling move, when the target vertex is $u_{i,m}$. Without loss of generality, assume that the target vertex is $u_{1,m}$ and denote that $C^{(1)}$ is the target cycle. As the conditions $D(C^{(1)}) \neq 0$ are easy to verify, we consider the cases for $D(C^{(1)}) = 0$, based on the number of occupied vertices on $C^{(i)}$ for $i \neq 1$. Let D be a distribution on the graph $D_{n,C_{2m}}$ with $2f_t(D_{n,C_{2m}}) - \sum_{i=1}^{n} q_i + 1$ pebbles, where q_i is the occupied vertices on $C^{(i)}$ for $1 \leqslant i \leqslant n$, and denote $p = |D|$. Obviously,

$$\begin{aligned} p &= 2f_t(D_{n,C_{2m}}) - \sum_{i=1}^{n} q_i + 1 \\ &= 2(n-2)(2^m-1) + t\cdot 2^{2m+1} - \sum_{i=1}^{n} q_i + 1 = \sum_{i=2}^{n} x_i. \qquad (3.1) \end{aligned}$$

In [3], Herscovici gave the t-pebbling number of C_{2m}, i.e., $f_t(C_{2m}) = t\cdot 2^m$, then $y_i = \lfloor \frac{x_i}{2^m} \rfloor \geqslant \frac{x_i}{2^m} - 1$. Naturally,

$$(3.2)\quad \sum_{i=2}^{n} y_i = \sum_{i=2}^{n} \left\lfloor \frac{x_i}{2^m} \right\rfloor \geqslant \sum_{i=2}^{n} \left(\frac{x_i}{2^m} - 1 \right) = \frac{\sum_{i=2}^{n} x_i}{2^m} - (n-1)$$

Since $(n-1) \leqslant \sum_{i=1}^{n} q_i \leqslant (2m-1)(n-1)$, we consider the following cases based on the occupied vertices on $C^{(i)}$ for $i \in \{2,3,\cdots,n\}$.

Case 1: Only $u_{i,m}$ $(i \neq 1)$ are occupied, that is, $D(u_{i,m}) \neq 0$ $(i = 2,3,\cdots,n)$, $D(u) = D(u_{1,m}) = D(u_{i,j}) = 0$ $(i \neq 1, j \neq m)$. Obviously,

$\sum_{i=2}^n q_i = n-1$, then we can rewrite (3.2) as $\sum_{i=2}^n y_i = t \cdot 2^{m+1} + \frac{(2^m-3)(n-1)-2^{m+1}+3}{2^m}$.

If $n-1 \geqslant \left\lceil \frac{2^{m+1}-3}{2^m-3} \right\rceil$, then $\sum_{i=2}^n y_i \geqslant t \cdot 2^{m+1}$. Notice that $\lim_{n\to\infty} \left\lceil \frac{2^{m+1}-3}{2^m-3} \right\rceil = 3$ when $m \geqslant 3$. We consider two subcases: $n-1 \leqslant 2$ and $m = 2$.

subcase 1.1: $n-1 \leqslant 2$. If $n-1=1$, there is only one vertex occupied, then we have $p \geqslant 2f_t(D_{2,C_{2m}}) - q + 1 = t \cdot 2^{2m+1}$, that is $D(u_{n,m}) \geqslant t \cdot 2^{2m+1}$. As Lourdusamy et.al. mentioned in [7] that $f_t(P_n) = t \cdot 2^{n-1}$, then $2t$ pebbles are able to be moved to $u_{1,m}$ from $u_{n,m}$, for the reason that $d(u_{1,m}, u_{n,m}) = 2m$. If $n-1=2$, it implies that $\sum_{i=2}^n q_i = 2$, we have

$$p \geqslant 2f_t(D_{3,C_{2m}}) - q + 1 = 2(2^m - 1) + t \cdot 2^{2m+1} - 1.$$

If x_2 is a multiple of 2^m, denoted by $x_2 = k \cdot 2^m$, then $x_3 = p - x_2 = t \cdot 2^{2m+1} + 2^{m+1} - 3 - k \cdot 2^m$. By Theorem 1.2, $y_2 = \lfloor \frac{x_2}{2^m} \rfloor = k$ and $y_3 \geqslant \lfloor \frac{x_3}{2^m} \rfloor \geqslant t \cdot 2^{m+1} + 2 - k - 1$, we have $y_2 + y_3 \geqslant t \cdot 2^{m+1}$. Otherwise, if x_2 is not a multiple of 2^m, denoted by $x_2 = k \cdot 2^m + w$, where $1 \leqslant w \leqslant 2^m - 1$ and $x_3 = p - x_2$. Similarly, $y_2 = k$ and $y_3 \geqslant t \cdot 2^{m+1} + 2 - k - \lfloor \frac{3+w}{2^m} \rfloor$, we have $y_2 + y_3 \geqslant t \cdot 2^{m+1}$, which means $2t$ pebbles can be moved to $u_{1,m}$ for $n-1 \geqslant 1$.

subcase 1.2: $m=2$. For the graph D_{n,C_4}, $\sum_{i=2}^n q_i = n-1 \leqslant 3n-5$ $(n \geqslant 2)$. It implies that, $p \geqslant 2f_t(D_{n,C_4}) - (n-1) + 1 \geqslant 2f(D_{n,C_4}) - (3n-5) + 1 = 3(n-1) + 32t - 3$. According to Lemma 2.1, at least $\lfloor \frac{3+32t-3}{2^2} \rfloor = 8t$ pebbles can be moved to u, in other words, we can move $2t$ pebbles to $u_{1,m}$.

Case 2: $2 \leqslant q_i \leqslant 2m-2$. It implies that, (3.2) can be rewritten as $\sum_{i=2}^n y_i \geqslant t \cdot 2^{m+1} + \frac{(n-1)(2^m-2m)-2^{m+1}+3}{2^m}$.

If $n-1 \geqslant \left\lceil \frac{2^{m+1}-3}{2^m-2m} \right\rceil$, then $\sum_{i=2}^n y_i \geqslant t \cdot 2^{m+1}$. Notice that $\lim_{m\to\infty} \left\lceil \frac{2^{m+1}-3}{2^m-2m} \right\rceil = 3$ when $m \geqslant 3$. We consider subcases when $m \leqslant 4$ and $n-1 \leqslant 2$.

Subcase 2.1: $m \leqslant 4$. It is easy to verify the graph D_{n,C_4} has the $2t$-pebbling property. We use the similar algorithm to show the cases $m=3$ and $m=4$. If $m=3$, for the graph D_{n,C_6}, $\sum_{i=2}^n q_i = n-1 \leqslant 7n-13$ $(n-1 \geqslant 3)$. It implies that, $p \geqslant 7(n-1) + 2^7 t - 7$. By Lemma 2.1, at least $\lfloor \frac{7+2^7 t-7}{2^3} \rfloor = t \cdot 2^4$ pebbles can be moved to u. Similarly, for the graph D_{n,C_8}, $\sum_{i=2}^n q_i = n-1 \leqslant 15n-29$ $(n-1 \geqslant 3)$. It implies that, $p \geqslant 15(n-1) + 2^9 t - 15$. According to Lemma 2.1, at least $\lfloor \frac{15+2^9 t-15}{2^4} \rfloor = t \cdot 2^5$ pebbles can be moved to u, in other words, we can move $2t$ pebbles to $u_{1,m}$.

Subcase 2.2: $n-1 \leqslant 2$. If $n-1=1$, we have $p \geqslant t \cdot 2^{2m+1} - q + 1$. According to Lemma 3.2, 2^{m+1} pebbles can be moved to u, and then at least 2 pebbles are able to be moved to the target vertex. If $n-1=2$, it implies that $4 \leqslant \sum_{i=2}^{n} q_i \leqslant 2(2m-2)$, we have

$$p \geqslant 2f_t(D_{3,C_{2m}}) - \sum_{i=2}^{n} q_i + 1 \geqslant t \cdot 2^{2m+1} + 2^{m+1} - 4m + 3.$$

If x_2 is a multiple of 2^m, denoted by $x_2 = k \cdot 2^m$, then $x_3 = p - x_2 = t \cdot 2^{2m+1} + 2^{m+1} - 4m + 3 - k \cdot 2^m$, similarly, we have $y_2 = k$ and $y_3 = \lfloor \frac{x_3}{2^m} \rfloor \geqslant t \cdot 2^{m+1} - k$. Otherwise, if x_2 is not divisible by 2^m, denoted by $x_2 = k \cdot 2^m + w$, where $1 \leqslant w \leqslant 2^m - 1$ and $x_3 = p - x_2$. We have $y_3 \geqslant t \cdot 2^{m+1} - k$, then $y_2 + y_3 \geqslant t \cdot 2^{m+1}$, which means $2t$ pebbles can be moved to $u_{1,m}$ for $n - 1 \geqslant 1$.

Case 3: $q_i = 2m - 1$. It implies that, we can rewrite (3.2) as

$\sum_{i=2}^{n} y_i \geqslant t \cdot 2^{m+1} + \frac{(n-1)(2^m-2m-1)-2^{m+1}+3}{2^m}$. If $n - 1 \geqslant \left\lceil \frac{2^{m+1}-3}{2^m-2m-1} \right\rceil$, then $\sum_{i=2}^{n} y_i \geqslant t \cdot 2^{m+1}$. Notice that $\lim_{n\to\infty} \left\lceil \frac{2^{m+1}-3}{2^m-2m-1} \right\rceil = 3$ when $m \geqslant 5$. We consider two subcases: $m \leqslant 4$ and $n - 1 \leqslant 2$.

Subcase 3.1: $m \leqslant 4$. Without loss of generality, let every $x_i = 2m - 1$ with $i = \{3, \cdots, n\}$, then $x_2 = p - \sum_{i=3}^{n} x_i = p - (2m-1)(n-1)$. According to Lemma 3.2 $f_y^{2m-1}(C_{2m}) \leqslant y \cdot 2^m - 2m + 2$, then $y_2 = \lfloor \frac{x_2+2m-2}{2^m} \rfloor \geqslant t \cdot 2^{m+1}$.

Subcase 3.2: $n - 1 \leqslant 2$, as the case $n - 1 = 1$ is easy to verify, we show the cases $n - 1 = 2$ as follows: we have $x_2 + x_3 = p = t \cdot 2^{2m+1} + 2^{m+1} - 4m + 1$, If $x_2 = 2m - 1$, then $x_3 = p - x_2$. Obviously, there is no pebble can be moved to u through the cycle $C^{(2)}$. We use the Lemma3.2 to obtain that $y_3 = \lfloor \frac{x_3+2m-2}{2^m} \rfloor = t \cdot 2^{m+1} + \frac{2^{m+1}-4m}{2^m} \geqslant t \cdot 2^{m+1}$. If $2m - 1 < x_2 \leqslant 2m + 1$, then $x_3 \geqslant p - (2m+1)$. By Lemma3.2, $y_3 = \lfloor \frac{x_3+2m-2}{2^m} \rfloor \geqslant t \cdot 2^{m+1} - 1$. Since $y_2 = 1$, $y_2 + y_3 \geqslant t \cdot 2^{m+1}$. If $x_2 > 2m + 1$, according to Lemma 3.2, we have $\sum_{i=2}^{n} y_i = \left\lfloor \frac{\sum_{i=2}^{n} x_i + 2m - 2}{2^m} \right\rfloor \geqslant t \cdot 2^{m+1}$.

Finally, each distribution with more than $2f_t(D_{n,C_{2m}}) - \sum_{i=1}^{n} q_i$ pebbles is able to move at least $t \cdot 2^{m+1}$ pebbles to the center vertex$u$, and then $2t$ pebbles can be moved to the target vertex $u_{1,m}$.

Therefore, over the course of the algorithm, the graph $D_{n,C_{2m}}$ satisfied the $2t-$pebbling property. □

4. Further Problems

There are many types of graphs such as trees, product graphs and hypercubes. It would be interesting to study whether a clique block graphs combine together through a common vertex or vertex set have similar conclusions.

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