On the Ramsey Numbers $R(S_{2,m}, K_{2,q})$ and $R(sK_2, K_s + C_n)$

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Abstract

We determine the Ramsey numbers $R(S_{2,m}, K_{2,q})$ for $m \in \{3,4,5\}$ and $q \geq 2$. In addition, we obtain $R(tS_{2,3}, K_{2,2})$ and $R(S_{2,3}, sK_{2,2})$ for $s \geq 2$, $t \geq 1$. We also obtain $R(sK_2, \mathcal{H})$, where $\mathcal{H}$ is the union graphs which each component is isomorphic to the connected spanning subgraph of $K_s + C_n$ for $n \geq 3$ and $s \geq 1$.

Key words and phrases: *bipartite, double stars, (G, H)-good graph, pancyclic, Ramsey number, union graph.*

1 Introduction

We consider finite undirected graphs without loops and multiple edges. Let $G(V, E)$ be a graph, the notation $V(G)$ and $E(G)$ (in short V and E) stand for the vertex set and the edge set of the graph G, respectively. A graph $H(V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For $A \subseteq V$, $G[A]$ represents the *subgraph induced* by A in G. Let H be a subgraph of G. A graph $G - H$ is a subgraph of G obtained by deleting the vertices of H and all the edges incident to them. A subgraph H is called a *spanning subgraph* of G if $V(H) = V(G)$. A graph G on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$. Moreover, order of graph G is denoted by $|G|$.

For given graphs G and H, a graph F is called a *(G, H)-good graph* if F contains no G and $\overline{F}$ contains no H. Any (G, H)-good graph on n vertices is denoted by (G, H, n)-good graph. The *Ramsey number* $R(G, H)$ is defined as the smallest natural number n such that no (G, H, n)-good graph exists. A nice survey for an attractive applications of various branches of Ramsey theory in harmonic analysis, metric spaces, ergodic theory, computational geometry, probabilistic, and information theory on dual source codes can be seen in Rosta [13].

2 Some Preliminary Results and Lemmas

The Ramsey numbers $R(G, H)$ for connected graphs G and H have been intensively studied since Chvátal and Harary [7] established a general lower bound of $R(G, H)$, i. e. $R(G, H) \geq (c(G) - 1)(\chi(H) - 1) + 1$, where $\chi(H)$ is the chromatic number of H and $c(G)$ is the number of vertices of the largest component of G. Chvátal [8] showed that $R(T_n, K_m) = (n - 1)(m - 1) + 1$, with $n, m \geq 1$. In [2], Baskoro et al. have proved that $R(T_n, W_4) = 2n - 1$ for $n \geq 4$ and $T_n \neq S_n$; $R(T_n, W_5) = 3n - 2$ for $n \geq 3$ and $T_n \neq S_n$; $R(S_n, W_4) = 2n - 1$ for odd $n \geq 5$; $R(S_n, W_4) = 2n + 1$ for even $n \geq 4$. Furthermore, Hasmawati et al. [9] showed that $R(S_n, W_m) = 3n - 2$ for odd $m \leq 2n - 1$ and $n \geq 3$. In [15], Surahmat et al. gave $R(C_n, W_m) = 3n - 2$ for odd $m \geq 5$ and $n \geq \frac{5m-9}{2}$. In [16], they have obtained $R(C_n, W_m) = 2n - 1$ for even m and $n \geq \frac{5m}{2} - 1$.

The Ramsey numbers of the graphs as stated above attain the Chvátal-Harary bound. However, for some combination of graphs G and H the Ramsey numbers $R(G, H)$ do not satisfy the Chvátal-Harary bound. Namely, $R(S_n, W_6) = 2n + 1$ for $n \geq 3$ is proved by Chen et al. [6]. $R(S_n, W_8) = 2n + 2$ for even $n \geq 6$ is established by Zhang et al. [17]. $R(S_n, W_4) = 2n + 1$ for even $n \geq 4$ is given by Baskoro et al. [2]. $R(S_n(1, 1), W_6) = 2n$ for $n \geq 4$ and $R(S_n(1, 2), W_6) = 2n$ for $n \equiv 0 (\mathrm{mod}\, 3) \geq 6$ is showed by Chen et al. in [5]. $R(K_{2,3}, K_{2,3}) = 10$ is proved by Burr [4]. $R(K_{2,n}, K_{2,n}) \leq 4n - 2$ for $n \geq 2$ and $R(K_{2,n}, K_{2,n}) = \{6, 10, 14, 18, 21, 26, 30, 33, 38, 42, 46, 50, 54, 57, 62\}$ for $2 \leq n \leq 16$ are showed by Exoo et al. [10]. This makes the problems on the Ramsey number of graphs become quite interesting, especially for combination of disconnected graphs G and H.

Let $k \geq 1$ be an integer. Let G_i be a connected graph with the vertex set V_i and the edge set E_i for $i = 1, 2, ..., k$. The *union* $G = \bigcup_{i=1}^{k} G_i$ has the vertex set $V = \bigcup_{i=1}^{k} V_i$ and the edge set $E = \bigcup_{i=1}^{k} E_i$. Let F be a connected graph. If $G_1 = G_2 = ... = G_k = F$ then we denote the union by kF. Some recent results on the Ramsey number for a combination of disconnected (union) and connected graphs are presented as follows. Baskoro et al. [1] showed that $R(kS_n, W_4) = (k+1)n$, for even $n \geq 4$, $k \geq 2$; and $R(kS_n, W_4) = (k+1)n - 1$ for odd $n \geq 5$, $k \geq 1$. In more general, Hasmawati et al. [11] have proved that if G_i and H are connected graphs satisfying $|G_i| \geq |G_{i+1}|$, $|G_i| > (|G_i| - |G_{i+1}|)(\chi(H) - 1)$ for every $i = 1, 2, ..., k-1$ and $R(G_i, H) = (|G_i| - 1)(\chi(H) - 1) + 1$ for any i, then $R(\bigcup_{i=1}^{k} G_i, H) = (|G_k| - 1)(\chi(H) - 1) + \sum_{i=1}^{k-1} |G_i| + 1$, where $\chi(H)$ is the chromatic number of H. Furthermore, Sudarsana et al. [14] have improved the Hasmawati's result by showing that if the condition $R(G_i, H)$ attaining the Chavátal-Harary bound is only satisfied by $i = k$ then the conclusion is still true.

In this note, we discuss the Ramsey numbers $R(S_{n,m}, K_{p,q})$. We also obtain the Ramsey numbers $R(sK_2, \mathcal{H})$ if $\mathcal{H}$ is the union graphs which each component is isomorphic to the connected spanning subgraphs of $K_s + C_n$.

Before end this section, we shall have the following lemmas which will be useful in proving our results.

Lemma 1 *(Bondy [3]). Let G be a graphs of order n. If $\delta(G) \geq \frac{n}{2}$ then either G is pancyclic or n is even and $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 2 *(Hasmawati et al. [11]). If G and H are connected graphs and $k \geq 1$ then $R(kG, H) \leq R(G, H) + (k-1)|G|$.*

3 On the $(S_{2,m}, K_{2,q})$-Ramsey Numbers

The notation S_n and $K_{p,q}$ denote a star with n vertices and a complete bipartite graph on $p+q$ vertices, respectively. A *double stars* $S_{n,m}$ is a graph constructed from two stars S_n and S_m by joining the two centers by an edge. In this section, we determine the Ramsey numbers $R(S_{2,m}, K_{2,q})$ for $m \in \{3, 4, 5\}$ and $q \geq 2$. We also obtain $R(tS_{2,3}, K_{2,2})$ and $R(S_{2,3}, sK_{2,2})$ for $s \geq 2, t \geq 1$.

First of all, let us give some notations used in proving the theorems. Let G be a graph. For $x \in V(G)$ and $B \subseteq V(G)$, $N_B(x) = \{y \in B : xy \in E(G)\}$ is the vertex set consisting of all the neighbors of x in B. Moreover, $N_B[x] = \{x\} \cup N_B(x)$. $N_B(x) = N(x)$ if $B = V(G)$. The degree of a vertex x in G is denoted by $d(x)$, obviously $d(x) = |N(x)|$. The minimum (maximum) degree of G is denoted by $\delta(G)$ $(\Delta(G))$.

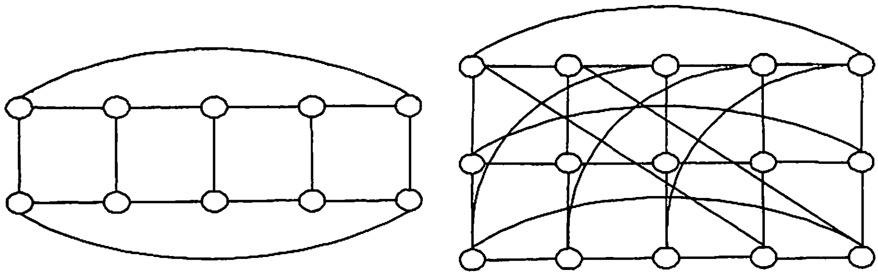

Figure 1: The illustration of the construction of the graphs $L_5(2)$ and $L_5(3)$.

Let $C'_n = (c'_1c'_2...c'_{n-1}c'_nc'_1)$, $C''_n = (c''_1c''_2...c''_{n-1}c''_nc''_1)$ and $C'''_n = (c'''_1c'''_2...c'''_{n-1}c'''_nc'''_1)$ be the three cycles of order n. Construct the graphs $L_n(2) = C'_n \cup C''_n \cup \{c'_ic''_i : c'_i \in V(C'_n), c''_i \in V(C''_n), 1 \leq i \leq n\}$ and $L_n(3) = C'_n \cup C''_n \cup C'''_n \cup \{c'_ic''_i, c''_ic'''_i; 1 \leq i \leq n, c'_1c'''_{n-1}, c'_2c'''_n, c'''_ic'_{i+2}; 1 \leq i \leq n-2 : c'_i \in V(C'_n), c''_i \in V(C''_n), c'''_i \in V(C'''_n)\}$. Clearly, the graph $L_n(2)$ is 3-regular connected of order $2n$ and $L_n(3)$ is 4-regular connected of order $3n$, see Figure 1.

Lemma 3 $R(S_{2,3}, K_{2,2}) = 6$.

Proof. The cycle C_5 is a $(S_{2,3}, K_{2,2}, 5)$-good graph because C_5 and its complement contains no $S_{2,3}$ and $K_{2,2}$, respectively. So, $R(S_{2,3}, K_{2,2}) \geq 6$. The inequality $R(S_{2,3}, K_{2,2}) \leq 6$ is showed by the following reasons. Let F be a graph on six vertices that contains no $S_{2,3}$. We will show that $\overline{F} \supseteq K_{2,2}$. Take a vertex $x \in V(F)$ with $|N(x)| = \Delta(F)$. Write $B = V(F) \backslash N[x]$. If $\Delta(F) \geq 3$ then $zy \notin E(F)$ for every $y \in N(x)$ and $z \in B$. Therefore, we obtain that the vertex set $N(x) \cup B$ induces a $K_{2,2}$ in $\overline{F}$. Next, consider that $\Delta(F) \leq 2$. If $\Delta(F) \leq 1$ then obviously the vertex set $N[x] \cup B$ induces a $K_{2,2}$ in $\overline{F}$. Now, we let $\Delta(F) = 2$. Take a vertex y in $N(x)$. Clearly, $|N(y)| \leq 2$ and hence $\overline{F}[\{x, y\} \cup B] \supseteq K_{2,2}$. This concludes the proof. □

Theorem 1 *If* $q \geq 2$ *is an integer then* $R(S_{2,3}, K_{2,q}) = q + 4$.

Proof. Since C_{q+3} does not contain $S_{2,3}$ and $\overline{C}_{q+3}$ does not contain $K_{2,q}$ then C_{q+3} is a $(S_{2,3}, K_{2,q})$-good graph on $q+3$ vertices. Therefore, $R(S_{2,3}, K_{2,q}) \geq q + 4$. Furthermore, we will show the inequality

$R(S_{2,3}, K_{2,q}) \leq q+4$ by induction on q. For $q = 2$, Lemma 3 gives $R(S_{2,3}, K_{2,2}) = 6$. Let F be a graph on $q+4$ vertices and contains no $S_{2,3}$. By induction hypothesis, $\overline{F}$ contains $K_{2,q-1}$. Let B and Y be the independent sets of $K_{2,q-1}$ in $\overline{F}$ with $|B| = 2$ and $|Y| = q-1$. Write $D = V(F) \setminus V(K_{2,q-1})$ and clearly $|D| = 3$. Next, we observe the relation of the vertex set D and B in F. Define $E(D,B) = \{db : d \in D, b \in B\}$. Since F does not contain $S_{2,3}$ then $|E(D,B)| \leq 4$. Let $D = \{d_1, d_2, d_3\}$ and $B = \{a, b\}$. If $|E(D,B)| = 4$ then the vertex set $D \cup B$ forms a P_5 or a C_4 in F. Without loss of generality, let $P_5 = (d_1 a d_2 b d_3)$ and $C_4 = (d_1 a d_2 b d_1)$. Since F contains no $S_{2,3}$ then it is easy to verify that the set $\{b, d_2\} \cup Y \cup \{d_1\}$ and $\{a, b\} \cup Y \cup \{d_3\}$ form a $K_{2,q}$ in $\overline{F}$ when $P_5 = (d_1 a d_2 b d_3)$ and $C_4 = (d_1 a d_2 b d_1)$, respectively. Meanwhile, if $|E(D,B)| \leq 2$ then there exists a vertex $d \in D$ with $ad, bd \notin E(F)$, which implies that the set $\{a, b, d\} \cup Y$ forms a $K_{2,q}$ in $\overline{F}$. Now, we observe $|E(D,B)| = 3$. Without loss of generality, we distinguish the following two cases:

Case 1. $ad_1, ad_2 \in E(F)$ and $bd_3 \in E(F)$.

Since $F \supseteq S_{2,3}$ then $ad_3, ab \notin E(F)$ and one of the following conditions holds:

(i). $d_1 z \in E(F)$ for exactly one $z \in Y$,

(ii). $d_1 z \notin E(F)$ for every $z \in Y$.

If we obtain (i) then $d_1 b, d_1 d_3 \notin E(F)$. This implies that the vertex set $\{a, d_1\} \cup H$ forms a $K_{2,q}$ in $\overline{F}$ with $H = \{b, d_3\} \cup Y \backslash z$. If the condition (ii) holds then $d_1 b \notin E(F)$ or $d_1 d_3 \notin E(F)$. So the set $\{a, d_1\} \cup C$ or $\{a, d_1\} \cup L$ forms $K_{2,q}$ in $\overline{F}$, where $C = \{b\} \cup Y$ and $L = \{d_3\} \cup Y$.

Case 2. $ad_1, ad_2, ad_3 \in E(F)$.

Since F contains no $S_{2,3}$ then $d_1 b, d_2 b \notin E(F)$ and $d_1 z, d_2 z \notin E(F)$ for all z in Y. This implies that the set $\{d_1, d_2\} \cup M$ shape a $K_{2,q}$ in $\overline{F}$, where $M = \{b\} \cup Y$. This completes the proof. □

Lemma 4 *Let $q \geq 2$, $m \geq 3$ be integers. Then, $R(S_{2,m}, K_{2,q}) \leq 2m+q-2$.*

Proof. We prove this by induction on m. For $m = 3$, the lemma holds by Theorem 1. Let F be an arbitrary graph on $2m+q-2$ vertices and suppose that $\overline{F}$ does not contains $K_{2,q}$. We will show that F contains $S_{2,m}$. By induction hypothesis, F contains $S_{2,m-1}$. Let $D = V(S_{2,m-1})$ and $H = V(F) \backslash D$. Clearly, $|H| = m+q-3$. Let x, y be two vertices in D with $d(x) = m-1$ and $d(y) = 2$. By a contrary, suppose that F contains no $S_{2,m}$. Then, $xh \notin E(F)$ for all h in H since otherwise the vertex set $\{h\} \cup D$ forms a $S_{2,m}$ in F. Now, consider the vertex y. Since F does not contain $S_{2,m}$ then $|N_H(y)| \leq m-3$. This implies $|B| \geq q$ and hence the set $\{x, y\} \cup B$ forms a $K_{2,q}$ in $\overline{F}$, where $B = H \backslash N_H(y)$, a contradiction. So $F \supseteq S_{2,m}$. This concludes that $R(S_{2,m}, K_{2,q}) \leq 2m+q-2$. □

Theorem 2 *Let $q \geq 2$ be an integer. Then,*

$$R(S_{2,4}, K_{2,q}) = \begin{cases} q+6, & \text{for } q \text{ odd,} \\ q+5, & \text{for } q \text{ even.} \end{cases}$$

Proof. If q is odd then $q+5$ is even. Consider a graph $L_{\frac{q+5}{2}}(2)$. Clearly, $L_{\frac{q+5}{2}}(2)$ contains no $S_{2,4}$ and $\overline{L}_{\frac{q+5}{2}}(2)$ contains no $K_{2,q}$. This implies that $L_{\frac{q+5}{2}}(2)$ is a $(S_{2,4}, K_{2,q}, q+5)$-good graph and hence $R(S_{2,4}, K_{2,q}) \geq q+6$. The inequality $R(S_{2,4}, K_{2,q}) \leq q+6$ is derived from Lemma 4. So, $R(S_{2,4}, K_{2,q}) = q+6$.

Next, if q is even then $q+4$ is even. It is not difficult to verify that the graph $L_{\frac{q+4}{2}}(2)$ is a $(S_{2,4}, K_{2,q})$-good graph on $q+4$ vertices. Therefore, $R(S_{2,4}, K_{2,q}) \geq q+5$. Now, we will show that $R(S_{2,4}, K_{2,q}) \leq q+5$. Let F be a graph on $q+5$ vertices that contains no $S_{2,4}$. We shall show that $\overline{F}$ contains $K_{2,q}$. Let x be a vertex in $V(F)$ with $|N(x)| = \Delta(F)$ and write $B = V(F) \backslash N[x]$. Then, $\Delta(F) \geq 3$ since otherwise we obtain $|B| \geq q+2$ and hence $\overline{F}[N[x] \cup B] \supseteq K_{2,q}$. Now, consider $\Delta(F) \geq 4$. Since F does not contain $S_{2,4}$ then $zy \notin E(F)$ for all $z \in N(x)$, $y \in B$. Then, the vertex set $N(x) \cup B$ induces a $K_{2,q}$ in $\overline{F}$. This implies that $d(x) = 3$ for all $x \in V(F)$, which is impossible since $q+5$ is odd. Therefore, there exists at least one vertex, say y, in $V(F)$ such that $d(y) \leq 2$. For $d(y) = 2$, let w be a vertex in $N(y)$. Since $F \not\supseteq S_{2,4}$ then $|N_C(w)| \leq 2$, where $C = V(F) \backslash N(y)$. Consequently, $\overline{F}[\{y, w\} \cup P] \supseteq K_{2,q}$, where $P = V(F) \backslash N[y] \cup N_C(w)$. Now, if $d(y) = 1$ then it can be verified that the vertex sets $\{y, w\} \cup P$ and $\{y\} \cup P \cup N_C(w)$ induce a $K_{2,q}$ in $\overline{F}$ when $|N_C(w)| \leq 3$ and $|N_C(w)| \geq 4$, respectively. This completes the proof. □

Theorem 3 *If $q \equiv 2 \pmod 3 \geq 5$ then $R(S_{2,5}, K_{2,q}) = q+8$.*

Proof. Since $q \equiv 2 \pmod 3$ then $q+7$ can be divided by 3. Consider the graph $L_{\frac{q+7}{3}}(3)$. Since the graph $L_{\frac{q+7}{3}}(3)$ is 4-regular connected of order $q+7$ then $L_{\frac{q+7}{3}}(3)$ does not contain $S_{2,5}$. Now, let x and y be any two vertices in $L_{\frac{q+7}{3}}(3)$. Write $A = V(L_{\frac{q+7}{3}}) \backslash N[x] \cup N[y]$. Since $|N[x]| = |N[y]| = 5$ and $|N[x] \cap N[y]| \leq 2$ then $|A| = q+7 - |N[x]| - |N[y]| + |N[x] \cap N[y]| \leq q-1$. This implies that the complement of graph $L_{\frac{q+7}{3}}(3)$ does not contain $K_{2,q}$. Therefore, $L_{\frac{q+7}{3}}(3)$ is a $(S_{2,5}, K_{2,q})$-good graph on $q+7$ vertices for $q \equiv 2 \pmod 3$. So $R(S_{2,5}, K_{2,q}) \geq q+8$. The inequality $R(S_{2,5}, K_{2,q}) \leq q+8$ is obtained by applying Lemma 4. □

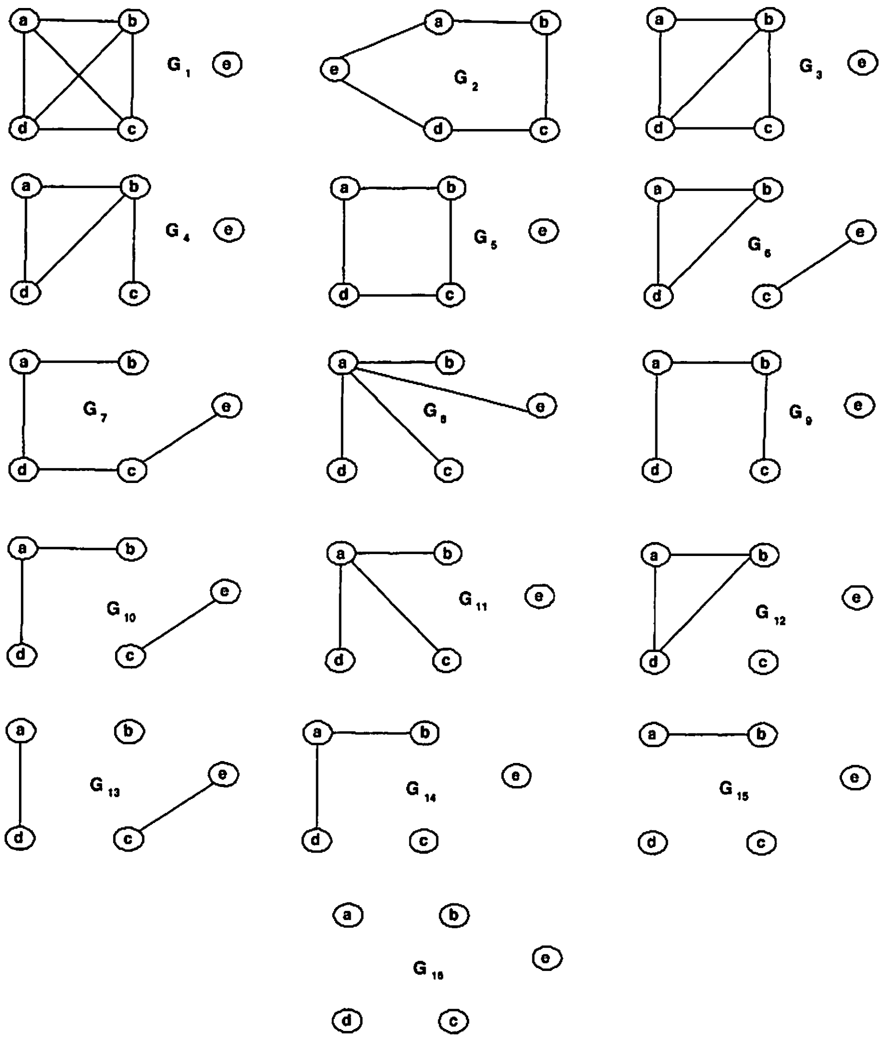

Figure 2: Nonisomorphic spanning subgraphs of $F[D]$ with $|E(F[D])| \leq 6$ and contains no $S_{2,3}$.

Note that it is still not known whether or not the exact Ramsey numbers $R(S_{2,5}, K_{2,q})$ attain the upper bound in Lemma 4 if $q \not\equiv 2 (\text{mod}\, 3) \geq 5$.

To concludes this section, we give the Ramsey numbers $R(tS_{2,3}, K_{2,2})$ and $R(S_{2,3}, sK_{2,2})$ for $s \geq 2, t \geq 1$.

Lemma 5 $R(S_{2,3}, 2K_{2,2}) = 9$.

Proof. Consider the graphs $\mathcal{C} \simeq K_{1,7}$ and $\overline{\mathcal{C}} \simeq K_7 \cup K_1$. Clearly, $\mathcal{C}$ is a $(S_{2,3}, 2K_{2,2}, 8)$-good graph and hence $R(S_{2,3}, 2K_{2,2}) \geq 9$. Now, let F be a graph on nine vertices that contains no $S_{2,3}$. We will show that $\overline{F}$ contains $2K_{2,2}$. Since $|F| = 9 > 6$ then By Lemma 1 we have $\overline{F} \supseteq K_{2,2}$. Let $D = V(F) \backslash V(K_{2,2})$. Then, $|D| = 5$ and call $D = \{a, b, c, d, e\}$. Since $F \not\supseteq S_{2,3}$ then $|E(F[D])| \leq 6$. Without loss of generality, we obtain all nonisomorphic spanning subgraphs of $F[D]$ which do not contain $S_{2,3}$ as presented in Figure 2. It can be verified that each graph in Figure 2 ensures $\overline{F} \supseteq 2K_{2,2}$. □

Theorem 4 *Let $s, t \geq 1$ be integers. Then,*

$$R(tS_{2,3}, sK_{2,2}) = \begin{cases} 5t + 1, & \textit{for } t \geq 1 \textit{ and } s = 1, \\ 4s + 1, & \textit{for } s \geq 2 \textit{ and } t = 1. \end{cases}$$

Proof. We separate the proof of the theorem into two cases.

Case 1. $t \geq 1$ and $s = 1$.

By Lemma 2, we get $R(tS_{2,3}, K_{2,2}) \leq 5t + 1$. Now, consider the graphs $\mathcal{B} \simeq K_{5t-1} \cup \overline{K}_1$ and $\overline{\mathcal{B}} \simeq \overline{K}_{5t-1,1}$. Immediately, we obtain that $\mathcal{B}$ is a $(tS_{2,3}, K_{2,2})$-good graph on $5t$ vertices. Thus $R(tS_{2,3}, K_{2,2}) \geq 5t + 1$. This concludes that $R(tS_{2,3}, K_{2,2}) = 5t + 1$.

Case 2. $s \geq 2$ and $t = 1$.

We will show the inequality $R(S_{2,3}, sK_{2,2}) \leq 5s + 1$ by induction on s. For $s = 2$, the theorem holds by Lemma 5. Let F be a graph on $4s + 1$ vertices and contains no $S_{2,3}$. We shall show that $\overline{F}$ contains $sK_{2,2}$. By induction hypothesis, $\overline{F}$ contains $(s-1)K_{2,2}$. Write $C = V(F) \backslash V((s-1)K_{2,2})$ and clearly $|C| = 5$. Since $F \not\supseteq S_{2,3}$ then $|E(F[C])| \leq 6$. Let Q be the vertex set of one $K_{2,2}$ in $(s-1)K_{2,2}$. By Lemma 5, we obtain that the set $C \cup Q$ induces a $2K_{2,2}$ in $\overline{F}$, which together with the vertex set $V((s-1)K_{2,2}) \backslash Q$ forms a $sK_{2,2}$ in $\overline{F}$. The desired lower bound $R(S_{2,3}, sK_{2,2}) \geq$

$5s+1$ follows from the fact that the graphs $K_{1,4s-1}$ and $K_1 \cup K_{4s-1}$ do not contain $S_{2,3}$ and $sK_{2,2}$, respectively. □

4 The Ramsey Numbers of sK_2 Versus K_s+C_n

Before presenting of the results let us give some notations used in this part. Let G and H be two graphs. The graph $G+H$ is formed by appending to $G \cup H$ edges that have one end vertex in G and the other in H. The notation C_n is a cycle on $n \geq 3$ vertices and $W_n \simeq K_1 + C_n$ is a wheel on $n+1$ vertices. The graphs $f_n \simeq K_1 + P_n$ and $F_n \simeq K_1 + nK_2$ are called a *friendship* on $n+1$ vertices and a *fan* on $2n+1$ vertices, respectively. A *Jahangir graph*, J_n for even $n \geq 4$, is a graph consisting of a cycle C_n with one additional vertex adjacent alternatingly to $\frac{n}{2}$ vertices of C_n.

In 2009, Lin and Li [12] have proved that $R(sK_2, F_n) = \max\{s,n\}+n+s$ for $s,n \geq 1$. Furthermore, we determine the Ramsey numbers $R(sK_2, \mathcal{H})$, where $\mathcal{H}$ is the union graphs which each component is isomorphic to the connected spanning subgraph of $K_s + C_n$ for $n \geq 3$ and $s \geq 1$.

Lemma 6 *Let $s \geq 1$ and $n \geq 2$ be integers. Then, $R(sK_2, K_n) = n+2s-2$.*

Proof. By Lemma 2, we have $R(sK_2, K_n) \leq 2s+n-2$. Now, consider the graphs $\mathcal{J} \simeq K_{2s-1} \cup \overline{K}_{n-2}$ and $\overline{\mathcal{J}} \simeq \overline{K}_{2s-1} + K_{n-2}$. It can be verified that $\mathcal{J}$ is a (sK_2, K_n)-good graph on $n+2s-3$ vertices, and hence $R(nK_2, K_n) \geq n+2s-2$. □

Theorem 5 *Let C_n and K_s be a cycle of order $n \geq 3$ and a complete graph of order $s \geq 1$, respectively. If $\lfloor \frac{n}{2} \rfloor \geq s$ then $R(sK_2, K_s+C_n) = n+2s-1$.*

Proof. We prove the inequality $R(sK_2, K_s+C_n) \leq n+2s-1$ by induction on s. For $s=1$, the assertion is trivial. Take an arbitrary graph F on $n+2s-1$ vertices and suppose that $\overline{F}$ contains no K_s+C_n. We will show that F contains sK_2. By induction on s, F contains $(s-1)K_2$. Write $B = \{a_1, b_1, ..., a_{s-1}, b_{s-1}\}$ the vertex set of $(s-1)K_2$ in F, where a_ib_i are the independent edges in $E(F)$, $i = 1, 2, ..., s-1$. By a contrary, we suppose that F contains no sK_2. Let $A = V(F)\backslash B$. Clearly, $|A| = n+1$. We assume that $\overline{F}[A]$ forms a K_{n+1} in $\overline{F}$ since otherwise $\overline{F}[A]$ contains independent vertices in $\overline{F}$, which together with B gives an sK_2 in F.

Next, if a_i (b_i) is adjacent to one vertex in A then b_i (a_i) must not be adjacent to all other vertices in A since otherwise we will get two independent edges between $\{a_i, b_i\}$ and A, which together with B forms an

sK_2 in F. Without loss of generality, we may assume that each b_i is not adjacent to all but at most one vertex in A, call c_i, if it exists. Let A' be the set of all such c_i in A. Then, $|A'| \leq s-1$. Let $C = A \backslash A'$ and clearly $|C| \geq (n-s)+2 \geq \lceil \frac{n}{2} \rceil + 2 \geq s+2$. Thus, we have s vertices, say $S = \{x_1, x_2, ..., x_s\}$, in A, which do not adjacent to each b_i in B. Let us consider $\overline{F}[D]$, where $D = A \cup \{b_1, b_2, ..., b_{s-1}\}$. Thus, $|\overline{F}[D]| = n+s$, in which each element of the set S in A is adjacent to each other vertex in $\overline{F}[D]$. Note that the set S forms a K_s in $\overline{F}[D]$. Now, observe that the subgraph $\overline{F}[D] - K_s$ of $\overline{F}[D]$ has order n with $\delta(\overline{F}[D] - K_s) \geq n - s \geq n - \lfloor \frac{n}{2} \rfloor \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$. By Lemma 1, the subgraph $\overline{F}[D] - K_s$ contains a cycle with order n, which together with K_s forms a graph $K_s + C_n$ in $\overline{F}$, which leads a contradiction that $\overline{F}$ contains no $K_s + C_n$. Thus F contains sK_2.

The inequality $R(sK_2, K_s + C_n) \geq n + 2s - 1$ follows from the fact that the graph $\mathcal{W} \simeq K_{n+s-1,s-1}$ is a $(sK_2, K_s + C_n)$-good graph on $n + 2s - 2$ vertices. This completes the proof. □

Consider the graph $K_s + C_n$, for $s \geq 1$ and $n \geq 3$. The set of graphs $\mathcal{G} = \{\mathcal{U} : \mathcal{U} \subseteq K_s + C_n\}$ consists of all the connected spanning subgraph of $K_s + C_n$. Therefore, by Theorem 5 we obtain $R(sK_2, \mathcal{U}) \leq R(sK_2, K_s + C_n) = n + 2s - 1$ for every $\mathcal{U} \in \mathcal{G}$. Meanwhile, the inequality $R(sK_2, \mathcal{U}) \geq |\mathcal{U}| + s - 1 = n + 2s - 1$ is derived from the fact that the graph $\mathcal{K} \simeq K_{n+s-1,s-1}$ is a $(sK_2, \mathcal{U})$-good graph on $n + 2s - 2$ vertices. Note that the graphs $W_{n+s-1}, f_{n+s-1}, F_{n+s-1}, J_{n+s-1}$ and C_{n+s} are a member of $\mathcal{G}$. Therefore, if the graph $\mathcal{Q}$ is isomorphic to one of these graphs then $R(sK_2, \mathcal{Q}) = n + 2s - 1$. Thus, the following corollary holds.

Corollary 1 *Let C_n and K_s be a cycle of order $n \geq 3$ and a complete graph of order $s \geq 1$, respectively. Let $\mathcal{G} = \{\mathcal{U} : \mathcal{U} \subseteq K_s + C_n\}$ be the set of graphs consisting of all the connected spanning subgraph of $K_s + C_n$. If $\lfloor \frac{n}{2} \rfloor \geq s$ then $R(sK_2, \mathcal{U}) = n + 2s - 1$, for every $\mathcal{U} \in \mathcal{G}$.*

The following theorem gives the exact Ramsey numbers $R(sK_2, \mathcal{H})$ if $\mathcal{H}$ is the union graphs which each component is isomorphic to the connected spanning subgraph of $K_s + C_n$, for $n \geq 3$ and $s \geq 1$.

Theorem 6 *Let $k, s \geq 1$ be integers and $G_i = K_s + C_{n_i}$ with $n_i \geq 3$, for every $i = 1, 2, ..., k$. Let $\mathcal{G}_i = \{\mathcal{U}_i : \mathcal{U}_i \subseteq G_i\}$ be the set of graphs consisting of all the connected spanning subgraph of G_i. If $\mathcal{H} = \bigcup_{i=1}^{k} \mathcal{U}_i$ with $\mathcal{U}_i \in \mathcal{G}_i$, and $\lfloor \frac{n_i}{2} \rfloor \geq s$ for every i, then $R(sK_2, \mathcal{H}) = \sum_{i=1}^{k} n_i + (k+1)s - 1$.*

Proof. Let $\mathcal{H} = \bigcup_{i=1}^{k} \mathcal{U}_i$ with $\mathcal{U}_i \in \mathcal{G}_i$, $i = 1, 2, ..., k$. We will prove $R(sK_2, \mathcal{H}) \leq \sum_{i=1}^{k} n_i + (k+1)s - 1$ by induction on k. For $k = 1$, the theorem holds by Corollary 1. Let $\mathcal{P}$ be an arbitrary graph on $\sum_{i=1}^{k} n_i + (k+1)s - 1$ vertices that contains no sK_2. We will show that $\overline{\mathcal{P}}$ contains $\mathcal{H}$. By induction hypothesis on k, $\overline{\mathcal{P}}$ contains $\bigcup_{i=1}^{k-1} \mathcal{U}_i$. Write $W = V(\mathcal{P}) \backslash V(\bigcup_{i=1}^{k-1} \mathcal{U}_i)$. Clearly, $|W| = n_k + 2s - 1$. Let us consider $\overline{\mathcal{T}} = \overline{\mathcal{P}}[W]$. Since $\mathcal{T}$ does not contain sK_2 then Corollary 1 ensures that $\overline{\mathcal{T}}$ contains $\mathcal{U}_k$ for every $\mathcal{U}_k$ in $\mathcal{G}_k$. Thus, $\overline{\mathcal{P}} \supseteq \mathcal{H}$.

Next, consider the graphs $\mathcal{F} \simeq K_{t-1,s-1}$ and its complement $\overline{\mathcal{F}} \simeq K_{t-1} \cup K_{s-1}$, where $t = \sum_{i=1}^{k} n_i + ks$. It can be verified that $\mathcal{F}$ does not contain sK_2 and $\overline{\mathcal{F}}$ does not contain $\mathcal{H}$. Therefore, $\mathcal{F}$ is a $(sK_2, \mathcal{H})$-good graph on $t + s - 2$ vertices and hence $R(sK_2, \mathcal{H}) \geq \sum_{i=1}^{k} n_i + (k+1)s - 1$. The proof is now complete. □

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