On the Laplacian integral $(k-1)$-cyclic graphs*

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Abstract A graph is called Laplacian integral if its Laplacian spectrum consists of integers. Let $\theta(n_1, n_2 \ldots, n_k)$ be a generalized θ-graph (see Figure 1). Denote by $\mathcal{G}_{k-1}$ the set of $(k-1)$-cyclic graphs each of them contains some generalized θ-graph $\theta(n_1, n_2, \ldots, n_k)$ as its induced subgraph. In this paper, we give an edge subdividing theorem for Laplacian eigenvalues of a graph (Theorem 2.1), from which we identify all the Laplacian integral graphs in the class $\mathcal{G}_{k-1}$ (Theorem 3.2).

Keywords: Laplacian spectrum; Laplacian integral graph; generalized θ-graph

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1 Introduction

The graph G, considered in this paper, is a simple and undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $D(G) = diag(d(v_1), \ldots, d(v_n))$ is the diagonal matrix of the vertex degrees in G and $A(G)$ is the adjacency matrix of G. It is well known that $L(G)$ is positive semidefinite so that its eigenvalues can be arranged as follows: $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$, where $\mu_{n-1}(G) > 0$ if and only if G is connected and hence is called the *algebraic connectivity* of G.

Let $x_1 > x_2 > \cdots > x_t$ be t distinct eigenvalues of $L(G)$ with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. Denote by $Spec_L(G) = [x_1^{k_1}, x_2^{k_2}, \ldots, x_t^{k_t}]$ the Laplacian spectrum of G. A graph is called a Laplacian integral graph if its Laplacian spectrum consists of integers. Sometimes, we say that G is Laplacian integral if G is a Laplacian integral graph.

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The research of integral graphs started in 1974 [1]. It has been discovered recently [16] that integral graphs can play a role in the so called perfect state transfer in quantum spin networks. Up to now, there are a few classes of integral graphs are characterized. One can consult [8] for a survey. Some class of Laplacian integral graphs have been identified (such as the degree maximal graphs, see [6], the unicyclic and bicyclic graphs, see [7]). Besides, some inductive constructions for Laplacian integral graphs are given in [8–11]. In this paper, we pay attention to a class of $(k-1)$-cyclic graphs: $\mathcal{G}_{k-1}$, which is defined in [15]. We identify all the Laplacian integral graphs in the class $\mathcal{G}_{k-1}$.

Let P_{n_i+2} be the paths on n_i+2 vertices where $n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq 1$, $n_k \geq 0$ and $k \geq 2$. The *generalized θ-graph*, denoted by $\theta(n_1, n_2, \ldots, n_k)$ (see Figure 1), is the graph obtained from these paths by identifying the k initial vertices as u_0 and terminal vertices as v_0, respectively. Denoted by Θ_k the set of all $\theta(n_1, n_2, \ldots, n_k)$.

A connected graph with n vertices and m edges is said to be a k-cyclic graph if $k = m - n + 1$. A k-cyclic graph is said to be a *k-cyclic base graph* if it contains no pendant vertices. Clearly, all the graphs in Θ_k are $(k-1)$-cyclic base graph. Denote by $\mathcal{G}_{k-1}$ all the $(k-1)$-cyclic graphs each of them contains some $\theta(n_1, n_2, \ldots, n_k)$ as its induced $(k-1)$-cyclic base subgraph.

The rest of this paper is organized as follows. In Section 2, we give an edge subdividing theorem for Laplacian eigenvalues of a graph. In Section 3, we identify all the Laplacian integral graphs in $\mathcal{G}_{k-1}$.

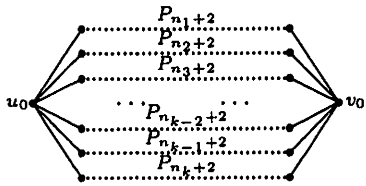


Figure 1: The graph $\theta(n_1, n_2, \ldots, n_k)$

2 Elementary

In this section, we list some useful lemmas and give an edge subdividing theorem for Laplacian eigenvalues of a graph.

Lemma 2.1 ([2, 3]). *Let G be a graph with at least one edge. Then* $\mu_1(G) \geq \Delta(G) + 1$. *Moreover, if G is connected, then equality holds if and only if* $\Delta(G) = n - 1$.

Lemma 2.2 ([5, 13]). *Let G be a connected graph. Then*

$$\mu_1(G) \leq \max\{d(v) + m(v) : v \in V\},$$

where $m(v) = \sum_{u \in N(v)} d(u)/d(v)$. Moreover, the equality holds if and only if G is a regular bipartite graph or a semiregular bipartite graph.

The *join* of two vertex disjoint graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from their union by including all edges between the vertices in G_1 and the vertices in G_2.

Lemma 2.3 ([4, 11]). *If G_1 and G_2 are graphs on k and m vertices respectively, with Laplacian eigenvalues $0 = \mu_k(G_1) \leq \mu_{k-1}(G_1) \leq \cdots \leq \mu_1(G_1)$ and $0 = \mu_m(G_2) \leq \mu_{m-1}(G_2) \leq \cdots \leq \mu_1(G_2)$ respectively, then the Laplacian eigenvalues of $G_1 \vee G_2$ are given by $0, \mu_{k-1}(G_1) + m, \ldots, \mu_1(G_1) + m, \mu_{m-1}(G_2) + k, \ldots, \mu_1(G_2) + k$, and $m + k$.*

Denote by $G - u$ the graph obtained from G by deleting the vertex $u \in V(G)$ along with the edges adjacent to u. Let v be a vertex of a connected graph G, if $G - v$ is disconnected, then v is called a *cutpoint* of G.

Lemma 2.4 ([12]). *If G is a connected graph with a cutpoint v, then $\mu_{n-1}(G) \leq 1$, where equality holds if and only if v is adjacent to every vertex of G.*

Lemma 2.5 ([14]). *Let A and B be Hermitian matrices of order n, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then*

$$\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A + B), i + j \geq n + 1,$$
$$\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B), i + j \leq n + 1.$$

In either of these inequalities equality holds if and only if there exists a nonzero n-vector that is an eigenvector to each of the three involved eigenvalues.

To *subdivide* an edge of G is to add a new vertex v to this edge. Note that the Laplacian matrix $L(G)$ is real symmetric, by Lemma 2.5 we obtain the following edge subdividing theorem for Laplacian eigenvalues of a graph.

Theorem 2.1. *Let G be a graph of order n, and let G' be the graph obtained by subdividing the edge uv of G. Then we have $\mu_i(G') \leq \mu_{i-1}(G)$ for $i = 2, \ldots, n+1$ and $\mu_i(G') \geq \mu_{i+1}(G)$ for $i = 1, \ldots, n-1$.*

Proof. Let x be the inserted vertex on the edge uv. By assumption, $G' = G - uv + ux + vx$. Let $L(G)$ and $L(G')$ be the Laplacian matrix of graphs G and G' respectively, where the rows and columns of $L(G')$ are labeled by $x, u, v, \ldots$. It is easy to verify that $L(G')$ can be decomposed into $L(G') = L_1 + L_2$, where

$$L_1 = \begin{pmatrix} 0_{1\times 1} & 0_{1\times n} \\ 0_{n\times 1} & L(G) \end{pmatrix}, L_2 = \begin{pmatrix} H & 0_{3\times(n-2)} \\ 0_{(n-2)\times 3} & 0_{(n-2)\times(n-2)} \end{pmatrix} \text{ and } H = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

By direct computation, we know that H has eigenvalues: $\lambda_1(H) = 3$, $\lambda_2(H) = 0$ and $\lambda_3(H) = -1$, and so L_2 has eigenvalues: $\lambda_1(L_2) = 3, \lambda_2(L_2) = \cdots = \lambda_n(L_2) = 0$ and $\lambda_{n+1}(L_2) = -1$.

Regarding $A = L_1$ and $B = L_2$, and note that $(i-1)+2 = i+1 \leq n+2$ for $i = 2, \ldots, n+1$, by Lemma 2.5 we have

$$\lambda_i(L(G')) = \lambda_i(L_1 + L_2) \leq \lambda_{i-1}(L_1) + \lambda_2(L_2) = \lambda_{i-1}(L_1) \qquad (1)$$

where $i = 2, \ldots, n+1$. Clearly, $\lambda_i(L_1) = \mu_i(G)$ for $i = 1, \ldots, n$ and $\lambda_{n+1}(L_1) = 0$; $\lambda_i(L(G')) = \mu_i(G')$ for $i = 1, 2, ..., n+1$. Eq. (1) gives that

$$\mu_i(G') \leq \mu_{i-1}(G), \quad i = 2, \ldots, n+1.$$

Note that $(i+1)+n = n+1+i \geq n+2$ for $i = 1, \ldots, n$. Similarly, by Lemma 2.5, we have

$$\lambda_i(L(G')) = \lambda_i(L_1 + L_2) \geq \lambda_{i+1}(L_1) + \lambda_n(L_2) = \lambda_{i+1}(L_1), \quad i = 1, \ldots, n,$$

which leads to

$$\mu_i(G') \geq \mu_{i+1}(G), \quad i = 1, \ldots, n-1.$$

We complete this proof. □

Let G be a simple graph and H be the graph obtained from G by subdividing some edges recursively. H is said to be the *recursive subdivision graph* of G and in turn G is said to be the *subdividing underline graph* of H. By definition, the recursive subdivision graph H of G is just the graph that is obtained by replacing some edges of G by some paths, respectively. It immediately obtains the following result by Theorem 2.1.

Corollary 2.1. *Let G be a graph with n vertices, and let H be a recursive subdivision graph of G with n' vertices. We have*
(a) $\mu_{i+n'-n-1}(H) \leq \mu_{i-1}(G)$ where $2 \leq i \leq n+1$;
(b) $\mu_{i-n'+n+1}(H) \geq \mu_{i+1}(G)$ where $1 \leq i \leq n-1$ and $n' \leq n+i$.

Taking $i = n$, we have $\mu_{n'-1}(H) \leq \mu_{n-1}(G)$ by Corollary 2.1 (a). Thus $\mu_{n'-1}(H) < 1$ if $\mu_{n-1}(G) < 1$, which can be used to exclude non-integral graphs in the next section.

3 The Laplacian integral graphs in $\mathcal{G}_{k-1}$

In order to identify all the Laplacian integral graphs in $\mathcal{G}_{k-1}$, first we give the Laplacian characteristic polynomials of some special generalized θ-graphs.

Lemma 3.1. *Let $G = \theta(3, \overbrace{1, \ldots, 1}^{k-2}, 0) \in \Theta_k$ and $k \geq 3$. Then the Laplacian characteristic polynomial of G is*

$$\begin{aligned} \Psi_L(G) &= \mu(\mu-2)^{k-3}\Phi_1(\mu), \quad \textit{where} \\ \Phi_1(\mu) &= \mu^5 - (2k+8)\mu^4 + (k^2+14k+23)\mu^3 - (6k^2+32k+32)\mu^2 \\ &\quad +(10k^2+30k+25)\mu - (4k^2+14k+6). \end{aligned}$$

Proof. Here G consists of one P_5, $(k-2)$'s P_3 and one P_2, and all these paths have the common ends $\{u_0, v_0\} = V_0$. We denote by V_1 the set of vertices of degree two in P_5 and V_2 the set of vertices of degree two in the latter $(k-2)$'s P_3. Now we arrange the rows and columns of $L(G)$ in the order of V_0, V_1 and V_2, respectively. Thus $\mu I - L(G)$ can be represented by

$$\mu I - L(G) = \begin{pmatrix} B_1 & C & J_{2\times(k-2)} \\ C^T & B_2 & 0_{3\times(k-2)} \\ J_{(k-2)\times 2} & 0_{(k-2)\times 3} & B_3 \end{pmatrix}, \text{ where}$$

$B_1 = \begin{pmatrix} \mu-k & 1 \\ 1 & \mu-k \end{pmatrix}$, $B_2 = \begin{pmatrix} \mu-2 & 1 & 0 \\ 1 & \mu-2 & 1 \\ 0 & 1 & \mu-2 \end{pmatrix}$, $B_3 = (\mu-2)I_{k-2}$, $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $J_{2\times(k-2)}$ denotes the matrix with all entries equal to 1. Set

$$P_1(\mu) = \begin{pmatrix} I_2 & & \frac{-J_{2\times(k-2)}}{\mu-2} \\ & I_3 & \\ & & I_{k-2} \end{pmatrix}, P_2(\mu) = \begin{pmatrix} I_2 & -CB_2^{-1} & \\ & I_3 & \\ & & I_{k-2} \end{pmatrix}.$$

We have

$$P_2(\mu)P_1(\mu)(\mu I - L(G)) = \begin{pmatrix} B_1 - \frac{(k-2)J_{2\times 2}}{\mu-2} - CB_2^{-1}C^T & & \\ C^T & B_2 & \\ J_{(k-2)\times 2} & & B_3 \end{pmatrix}.$$

Note that $B_2^{-1} = \frac{1}{\mu^3-6\mu^2+10\mu-4} \begin{pmatrix} \mu^2-4\mu+3 & 2-\mu & 1 \\ 2-\mu & (\mu-2)^2 & 2-\mu \\ 1 & 2-\mu & \mu^2-4\mu+3 \end{pmatrix}$, we obtain

$$\begin{aligned} \Psi_L(G) &= |P_2(\mu)P_1(\mu)(\mu I - L(G))| \\ &= |B_1 - \frac{(k-2)J_{2\times 2}}{\mu-2} - CB_2^{-1}C^T| \times |B_2| \times |B_3| \\ &= \mu(\mu-2)^{k-3}\Phi_1(\mu), \end{aligned}$$

where

$$\begin{aligned} \Phi_1(\mu) = {} & \mu^5 - (2k+8)\mu^4 + (k^2+14k+23)\mu^3 - (6k^2+32k+32)\mu^2 \\ & +(10k^2+30k+25)\mu - (4k^2+14k+6). \end{aligned}$$

We complete this proof. □

Lemma 3.2. *Let* $G = \theta(\overbrace{2,\ldots,2}^{k-1},0) \in \Theta_k$ *and* $k \geq 2$. *Then the Laplacian characteristic polynomial of* G *is*

$$\Psi_L(G) = \mu(\mu-3)^{k-2}(\mu-1)^{k-2}(\mu-2)(\mu-k)(\mu-k-2).$$

Proof. Here G consists of $(k-1)$'s P_4 and one P_2, and all these paths have the common ends $\{u_0, v_0\} = V_0$. We denote by V_1 the set of vertices of degree two in the $(k-1)$'s P_4. Now we arrange the rows and columns of $L(G)$ in the order of V_0 and V_1. Thus $\mu I - L(G)$ can be represented by

$$\mu I - L(G) = \begin{pmatrix} B_1 & C \\ C^T & B_2 \end{pmatrix},$$

where $B_1 = \begin{pmatrix} \mu-k & 1 \\ 1 & \mu-k \end{pmatrix}$, $C = \begin{pmatrix} I_2 & I_2 & \cdots & I_2 \end{pmatrix}_{2\times(2k-2)}$, $B = \begin{pmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{pmatrix}$ and $B_2 = \begin{pmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{pmatrix}_{(2k-2)\times(2k-2)}$. Set $P(\mu) = \begin{pmatrix} I_2 & -CB_2^{-1} \\ & I_{2k-2} \end{pmatrix}$, we have

$$P(\mu)(\mu I - L(G)) = \begin{pmatrix} B_1 - CB_2^{-1}C^T & \\ C^T & B_2 \end{pmatrix}.$$

Note that $B_2^{-1} = \begin{pmatrix} B^{-1} & & & \\ & B^{-1} & & \\ & & \ddots & \\ & & & B^{-1} \end{pmatrix}$, where $B^{-1} = \frac{1}{\mu^2-4\mu+3}\begin{pmatrix} \mu-2 & -1 \\ -1 & \mu-2 \end{pmatrix}$.

Thus

$$\begin{aligned} \Psi_L(G) &= |P(\mu)(\mu I - L(G))| \\ &= |B_1 - CB_2^{-1}C^T| \times |B_2| \\ &= |B_1 - (k-1)B^{-1}| \times |B|^{k-1} \\ &= \mu(\mu-3)^{k-2}(\mu-1)^{k-2}(\mu-2)(\mu-k)(\mu-k-2). \end{aligned}$$

We complete this proof. □

Lemma 3.3. *Let* $G = \theta(\overbrace{2,\ldots,2}^{k_1}, \overbrace{1,\ldots,1}^{k-k_1-1}, 0) \in \Theta_k$, $k \geq 3$ *and* $1 \leq k_1 \leq k-2$. *Then the Laplacian characteristic polynomial of* G *is*

$$\begin{aligned} \Psi_L(G) &= \mu(\mu-1)^{k_1-1}(\mu-3)^{k_1-1}(\mu-2)^{k-k_1-2}\Phi_2(\mu), \quad \textit{where} \\ \Phi_2(\mu) &= \mu^4 - (2k+6)\mu^3 + (k^2+10k+12)\mu^2 - (4k^2+14k+2k_1+10)\mu \\ &\quad +3k^2 + 2kk_1 + 6k - k_1^2 + 2k_1 + 3. \end{aligned}$$

Proof. Here G consists of k_1's P_4, $k - k_1 - 1 = k_2$'s P_3 and one P_2, and all these paths have the common ends $\{u_0, v_0\} = V_0$. We denote by V_1 the set of vertices of degree two in the first k_1's P_4 and V_2 the set of vertices of degree two in the latter k_2's P_3. Now we arrange the rows and columns vertices in $L(G)$ in the order of vertices V_0, V_1 and V_2. Thus $\mu I - L(G)$ can be represented by

$$\mu I - L(G) = \begin{pmatrix} B_1 & C & J_{2\times k_2} \\ C^T & B_2 & 0_{2k_1\times k_2} \\ J_{k_2\times 2} & 0_{k_2\times 2k_1} & B_3 \end{pmatrix},$$

where $B_1 = \begin{pmatrix} \mu-k & 1 \\ 1 & \mu-k \end{pmatrix}$, $C = (I_2\ I_2\ \cdots\ I_2)_{2\times 2k_1}$, $B = \begin{pmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{pmatrix}$, $B_2 = \begin{pmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{pmatrix}_{2k_1\times 2k_1}$, $B_3 = (\mu-2)I_{k_2}$ and $J_{2\times k_2}$ denotes the matrix with all entries equal to 1. Set

$$P_1(\mu) = \begin{pmatrix} I_2 & & \frac{-J_{2\times k_2}}{\mu-2} \\ & I_{2k_1} & \\ & & I_{k_2} \end{pmatrix}, P_2(\mu) = \begin{pmatrix} I_2 & -CB_2^{-1} & \\ & I_{2k_1} & \\ & & I_{k_2} \end{pmatrix}.$$

We have

$$P_2(\mu)P_1(\mu)(\mu I - L(G)) = \begin{pmatrix} B_1 - \frac{k_2 J_{2\times 2}}{\mu-2} - CB_2^{-1}C^T & & \\ C^T & B_2 & \\ J_{k_2\times 2} & & B_3 \end{pmatrix}.$$

Note that $B_2^{-1} = \begin{pmatrix} B^{-1} & & & \\ & B^{-1} & & \\ & & \ddots & \\ & & & B^{-1} \end{pmatrix}$, where $B^{-1} = \frac{1}{\mu^2-4\mu+3}\begin{pmatrix} \mu-2 & -1 \\ -1 & \mu-2 \end{pmatrix}$.

Thus

$$\begin{aligned} \Psi_L(G) &= |P_2(\mu)P_1(\mu)(\mu I - L(G))| \\ &= |B_1 - \frac{k_2 J_{2\times 2}}{\mu-2} - CB_2^{-1}C^T| \times |B_2| \times |B_3| \\ &= |B_1 - \frac{k_2 J_{2\times 2}}{\mu-2} - k_1 B^{-1}| \times |B_2| \times |B_3| \\ &= \mu(\mu-1)^{k_1-1}(\mu-3)^{k_1-1}(\mu-2)^{k-k_1-2}\Phi_2(\mu), \end{aligned}$$

where

$$\begin{aligned} \Phi_2(\mu) &= \mu^4 - (2k+6)\mu^3 + (k^2+10k+12)\mu^2 - (4k^2+14k+2k_1+10)\mu \\ &\quad +3k^2 + 2kk_1 + 6k - k_1^2 + 2k_1 + 3. \end{aligned}$$

We complete this proof. □

The following theorem completely determine all the Laplacian integral generalized θ-graphs.

Theorem 3.1. *Let $G = \theta(n_1, n_2, \ldots, n_k) \in \Theta_k$ where $n_1 \geq n_2 \geq \cdots \geq n_k$. Then G is Laplacian integral if and only if G is one of $\theta(\overbrace{1,\ldots,1}^{k})$, $\theta(2,2)$, $\theta(\overbrace{1,\ldots,1}^{k-1},0)$ and $\theta(\overbrace{2,\ldots,2}^{k-1},0)$ (see Fig.2).*

Proof. Suppose that u_0 and v_0 are the initial and terminal vertices of G, respectively. We distinguish two cases.

Case 1. $n_k \geq 1$, that is, $u_0 \nsim v_0$;

By assumption, $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$, and so $\Delta(G) = k < n-1$. By Lemma 2.1 we get $\mu_1(G) > k+1$. If G is Laplacian integral, then $\mu_1(G) \geq k+2$. On the other hand, G is a connected graph, by Lemma 2.2 we have $\mu_1(G) \leq k+2$. Thus, $\mu_1(G) = k+2$ and so G is a regular bipartite graph or a semiregular bipartite graph.

If G is a regular bipartite graph then $k = 2$, and so G must be an even cycle C_n where n is an even number no less than 4. It is well know that $\mu_j(C_n) = 2(1-\cos(\frac{2\pi j}{n}))$ for $j = 0, 1, ..., n-1$. For $n \geq 7$, we have

$$0 < \mu_{n-1}(C_n) = 2(1-\cos(\frac{2\pi}{n})) \leq 2(1-\cos(\frac{2\pi}{7})) < 0.7530.$$

Thus C_n is not Laplacian integral. For $4 \leq n \leq 6$, it is routine to verify that C_n is Laplacian integral if and only if $n \in \{4, 6\}$. Hence, $G = C_4 = \theta(1,1)$ or $G = C_6 = \theta(2,2)$ in this situation.

If G is a semiregular bipartite graph then $k \geq 3$, and so $G = \theta(1, \ldots, 1)$ is $K_{2,k}$ that is indeed Laplacian integral with $Spec_L(G) = [(k+2)^1, k^1, 2^{n-3}, 0^1]$.

Case 2. $n_k = 0$, that is, $u_0 \sim v_0$.

According to our assumption, $n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq 1$. We distinguish three cases.

Subcase 2.1. $n_1 \geq 3$. First we suppose that $k \geq 4$ and consider $F = \theta(3, \overbrace{1, \ldots, 1}^{k-2}, 0)$. By Lemma 3.1, the Laplacian characteristic polynomial of F is

$$\begin{aligned} \Psi_L(F) &= \mu(\mu-2)^{k-3}\Phi_1(\mu), \\ \Phi_1(\mu) &= \mu^5 - (2k+8)\mu^4 + (k^2+14k+23)\mu^3 - (6k^2+32k+32)\mu^2 \\ &\quad +(10k^2+30k+25)\mu - (4k^2+14k+6). \end{aligned}$$

Clearly, $\Phi_1(0) = -(4k^2+14k+6) < 0$ and $\Phi_1(1) = k^2-4k+3 > 0$. Thus F has a Laplacian eigenvalue $\mu^* \in (0,1)$, and so $0 < \mu_{n(F)-1}(F) \leq \mu^* < 1$. Clearly, $G = \theta(n_1, n_2, \ldots, n_{k-1}, 0)$ are recursive subdivision graphs of F if $n_1 \geq 3$ and $k \geq 4$. Hence, by Corollary 2.1, we have $0 < \mu_{n-1}(G) \leq \mu_{n(F)-1}(F) < 1$ and so G is not Laplacian integral.

Next we suppose that $k = 3$. Then $G = \theta(n_1, n_2, 0)(n_1 \geq 3)$. By direct computation we know that the graphs $\theta(3,1,0)$, $\theta(3,2,0)$ and $\theta(4,1,0)$ are not Laplacian integral with $Spec_L(\theta(3,1,0)) = [4.41, 4, 3, 1.59, 1, 0]$, $Spec_L(\theta(3,2,0)) = [4.88, 3.80, 2.65, 2.45, 1.47, 0.75, 0]$ and $Spec_L(\theta(4,1,0)) = [4.53, 3.80, 3.35, 2.45, 1.12, 0.75, 0]$, respectively. Observe that both $\theta(3,2,0)$ and $\theta(4,1,0)$ have a Laplacian eigenvalue $\mu^* = 0.75 < 1$. It is easy to see that $G = \theta(n_1, n_2, 0)$ $(n_1 \geq 3)$ would be recursive subdivision graphs of $\theta(3,2,0)$ or $\theta(4,1,0)$. By Corollary 2.1, we have $0 < \mu_{n-1}(G) \leq \mu^* < 1$, and so G is not Laplacian integral.

At last, if $k = 2$ then G is a cycle. We claim, in the case of $n_1 \geq 3$, that G is Laplacian integral if and only if $G = C_6 = \theta(4,0) = \theta(2,2)$ by the arguments in Case 1.

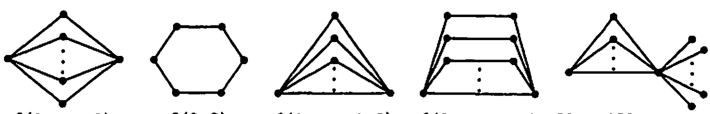
Subcase 2.2. $n_1 = 2$. If $G = \theta(\overbrace{2,\dots,2}^{k-1},0)$ then G is Laplacian integral by Lemma 3.2. Now we consider $G = \theta(\overbrace{2,\dots,2}^{k_1},\overbrace{1,\dots,1}^{k-k_1-1},0)$ where $k \geq 3$ and $1 \leq k_1 \leq k-2$. By Lemma 3.3, the Laplacian characteristic polynomial of G is

$$\begin{aligned}
\Psi_L(G) &= \mu(\mu-1)^{k_1-1}(\mu-3)^{k_1-1}(\mu-2)^{k-k_1-2}\Phi_2(\mu),\\
\Phi_2(\mu) &= \mu^4-(2k+6)\mu^3+(k^2+10k+12)\mu^2-(4k^2+14k+2k_1+10)\mu\\
&\quad +3k^2+2kk_1+6k-k_1^2+2k_1+3.
\end{aligned}$$

Since $1 \leq k_1 \leq k-2$, and so $\Phi(1) = k_1(2k-k_1) > 0$ and $\Phi(2) = -(k+1-k_1)^2 < 0$, then $\Phi_2(\mu)$ has a root $\mu^* \in (1,2)$. Hence, G has a Laplacian eigenvalue $\mu^* \in (1,2)$, and so G is not Laplacian integral.

Subcase 2.3. $n_1 = 1$. In this situation, $G = \theta(\overbrace{1,\dots,1}^{k-1},0) = K_1 \vee K_{1,k-1}$. We know that $Spec_L(K_{1,k-1}) = [k^1, 1^{k-2}, 0^1]$, and so $Spec_L(G) = [(k+1)^2, 2^{k-2}, 0^1]$ by Lemma 2.3. Hence $G = \theta(1,\dots,1,0)$ is Laplacian integral.

We complete this proof. □



$\theta(1,\dots,1)$ $\theta(2,2)$ $\theta(1,\dots,1,0)$ $\theta(2,\dots,2,0)$ $K_1 \vee (K_{1,k-1} \cup sK_1)$

Figure 2: The Laplacian integral graphs in Γ_{k-1}

Now we come to the stage to prove our main result.

Theorem 3.2. *Let $G \in \mathcal{G}_{k-1}$. Then G is Laplacian integral if and only if G is one of* $\theta(\overbrace{1,\dots,1}^{k})$, $\theta(2,2)$, $\theta(\overbrace{1,\dots,1}^{k-1},0)$, $\theta(\overbrace{2,\dots,2}^{k-1},0)$ *and* $K_1 \vee (K_{1,k-1} \cup sK_1)$ *(see Fig.2).*

Proof. If G has no pendant vertices, then $G \in \Theta_k$. By Theorem 3.1, G is Laplacian integral if and only if G is one of the following graphs: $\theta(1,\dots,1)$, $\theta(2,2)$, $\theta(1,\dots,1,0)$ and $\theta(2,\dots,2,0)$.

If G has some pendant vertices, then it has a cutpoint v. Then by Lemma 2.4, we have $\mu_{n-1}(G) \leq 1$. Now suppose that G is Laplacian integral. Then $\mu_{n-1}(G) = 1$, and thus G has a cutpoint v adjacent to all the other vertices again by Lemma 2.4. Since G belongs to $\mathcal{G}_{k-1}$, G must have an induced subgraph $\theta(\overbrace{1,...,1}^{k-1},0)$ and the cutpoint v is adjacent to other $s \geq 1$ pendant vertices. Thus

$G = K_1 \vee (K_{1,k-1} \cup sK_1)$. Conversely, $G = K_1 \vee (K_{1,k-1} \cup sK_1)(s \geq 1)$ is a graph in $\mathcal{G}_{k-1}$. Since $Spec_L(K_{1,k-1} \cup sK_1) = [k^1, 1^{k-2}, 0^{s+1}]$, by Lemma 2.3 we get $Spec_L(G) = [(k+s+1)^1, (k+1)^1, 2^{k-2}, 1^s, 0^1]$. Hence $G = K_1 \vee (K_{1,k-1} \cup sK_1)$ is Laplacian integral.

We complete this proof. □

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