A new hemisystem of $\mathcal{H}(3,49)$

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Abstract

A new hemisystem of the generalized quadrangle $\mathcal{H}(3,49)$ admitting the linear group $PSL_2(7)$ has been found.

Keywords: Generalized quadrangle, hermitian surface, regular system, hemisystem.

1 Introduction

A *finite generalized quadrangle* (GQ) is an incidence structure (P, B,I) in which P and B are disjoint nonempty sets of objects called points and lines (respectively), and for which I is a symmetric point–line incidence relation satisfying the following axioms:

1. Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

2. each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

3. if x is a point and ℓ is a line not incident with x, then there exists a unique pair $(y, M) \in \mathrm{P} \times \mathrm{B}$ for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}\ell$.

The integers s and t are the *parameters* of the GQ and the GQ is said to have *order* (s,t); if $s = t$, the GQ is said to have *order* s.
Here we are interested in the generalized quadrangle $\mathcal{H}(3, q^2)$, the incidence structure of all points and lines of a non-singular Hermitian surface in $\mathrm{PG}(3, q^2)$, a generalized quadrangle of order (q^2, q), with automorphism group $\mathrm{P\Gamma U}(4, q^2)$; and its dual $\mathcal{Q}^-(5, q)$.
In this paper we construct a new hemisystem of the generalized quadrangle $\mathcal{H}(3, 49)$ with full stabiliser $\mathrm{PSL}_2(7)$.
We recall that a *regular system of order* m [9] on $\mathcal{H}(3, q^2)$ is a set $\mathcal{R}$ of lines of $\mathcal{H}(3, q^2)$ with the property that every point lies on exactly m lines of $\mathcal{R}$, $0 < m < q + 1$. Segre proved that, if q is odd, such a system must have $m = (q+1)/2$, and called a regular system on $\mathcal{H}(3, q^2)$ of order $(q+1)/2$ a *hemisystem* on $\mathcal{H}(3, q^2)$. A simple proof that a regular system on $\mathcal{H}(3, q^2)$ is a hemisystem (and so q is odd) was given by Thas in [10], by showing that the concurrency graph of the lines of a regular system on $\mathcal{H}(3, q^2)$ of order m is a strongly regular graph $srg(v, k, \lambda, \mu)$, with $v = (q^3 + 1)(q + 1) - m$, $k = (q^2 + 1)(q - m)$, $\lambda = q - m - 1$ and $\mu = q^2 + 1 - m(q + 1)$, and by applying the fact that in an $srg(v, k, \lambda, \mu)$, $(v - k - 1)\mu = k(k - \lambda - 1)$. Even more general was the work of Cameron-Delsarte-Goethals [4], who defined a hemisystem on a generalized quadrangle of order (s, s^2), s odd, to be a set of points meeting every line in $(s + 1)/2$ points and showed that the collinearity graph of such a set is strongly regular. In [11], the conjecture that there are no hemisystems on $\mathcal{H}(3, q^2)$ for $q > 3$ was made. In [7] counterexamples to this conjecture were constructed on $\mathcal{H}(3, q^2)$, for all odd prime powers $q > 3$, admitting $\mathrm{P\Omega}_4^-(q)$, and giving Segre's example for $q = 3$, and on $\mathcal{H}(3, 25)$ admitting $3.A_7.2$. Also, in [1] a hemisystem of the Fisher-Thas-Walker-Kantor generalized quadrangle of order (5, 25), has been constructed that is related to to the $3 \cdot A_7$-hemisystem of H(3,25), constructed by Cossidente and Penttila in [7].
All of this is motivated by the study of partial quadrangles. These were introduced by Cameron [3]. A *partial quadrangle* $\mathrm{PQ}(s, t, \mu)$ is an incidence structure of points and lines with the properties that any two points are incident with at most one line, every point is incident with $t+1$ lines, every line is incident with $s + 1$ points, any two non-collinear points are jointly collinear with exactly μ points, and for any point P and line l which are not incident, there is at most one point Q on l collinear with P. There are not many constructions of partial quadrangles known: most of them arise from a generalized quadrangle of order (s, s^2) by deleting a point, all lines on that point, and all points collinear with that point; this gives a $\mathrm{PQ}(s - 1, s^2, s^2 - s)$. Many generalized quadrangles of order (s, s^2) are

known. The exceptional examples apart from the (thin) partial quadrangles with $s = 1$ (the Moore graphs (the pentagon, the Clebsch graph, and the Hoffman-Singleton graph), the Gewirtz graph and the Higman-Sims graphs on 77 and 100 vertices) are a partial quadrangle PQ(2, 10, 2) arising from Coxeter's 11–cap, a partial quadrangle PQ(2, 55, 20) arising from Hill's 56–cap in PG(5, 3), and a partial quadrangle PQ(3, 77, 14) arising from the Hill's 78-cap PG(5, 4), all via linear representations.
The preceding results imply that a hemisystem on a generalized quadrangle of order (s, s^2) gives a partial quadrangle $PQ((s-1)/2, s^2, (s-1)^2/2)$ (the points of the partial quadrangle being the points of the hemisystem and the lines of the partial quadrangle being the lines of the generalized quadrangle). In our case a new PQ(3, 49, 18) arises.
It should be mentioned that hemisystems of generalized quadrangles also give rise to Q-polynomial association schemes [8]. The GQ has a strongly regular point graph. So we have a 2-class association scheme with relations R_0, R_1 (collinear) and R_2 (non-collinear). If we split the lines into red and blue lines so that each point lies on $(t+1)/2$ red and $(t+1)/2$ blue lines (by using a hemisystem), then we may split R_1 and R_2 into two relations each, yielding another, finer, association scheme on the points. In this case, the association scheme is "Q-polynomial" (or cometric) and there are very few examples of these known which are not distance-regular graphs, nor duals of distance-regular graphs.

2 The new hemisystem

Let $\mathcal{H}(3, q^2)$ be the Hermitian surface of $PG(3, q^2)$, q odd, with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, where X_0, X_1, X_2, X_3 are homogeneous coordinates in $PG(3, q^2)$ and let ρ denote the unitary polarity induced by $\mathcal{H}(3, q^2)$. Let $\{Q_a \mid a \in GF(q^2) \setminus \{0\},\ a^{q+1} = 1\}$ denote a family of $q + 1$ quadrics of $PG(3, q^2)$, where Q_a has equation $aX_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$. Straightforward computations show that each of these quadrics is hyperbolic and any two of them intersect in the conic $\bar{C}$, given by equation $X_1^2 + X_2^2 + X_3^2 = 0$, lying in the plane $\bar{\pi}$ with equation $X_0 = 0$. Let π denote the Baer subplane of $\bar{\pi}$ whose normalized point coordinates lie in the subfield $GF(q)$, and let $C = \bar{C} \cap \pi$ denote the associated subconic of $\bar{C}$ in π. Furthermore, let $\mathcal{U} = \mathcal{H}(3, q^2) \cap \bar{\pi} \cong \mathcal{H}(2, q^2)$ be the Hermitian curve, given by equation $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$, that one obtains by intersecting the Hermitian surface $\mathcal{H}(3, q^2)$ with the plane $\bar{\pi}$. Then, by [6, Lemma 3.2], $C = \mathcal{H}(3, q^2) \cap \pi = \mathcal{U} \cap \bar{C} = \mathcal{H}(3, q^2) \cap \bar{C}$.
From [9, p. 146] each quadric Q_a is permutable with $\mathcal{H}(3, q^2)$. In particular, $(q + 1)/2$ of them, say $Q_{a_1}, Q_{a_2}, Q_{a_{(q+1)/2}}$, are such that $\mathcal{H}(3, q^2) \cap Q_{a_i}$ is an elliptic quadric $\mathcal{E}_i$ embedded in a Baer subgeometry $B_i \cong PG(3, q)$ of

PG$(3,q^2)$, for $i = 1, 2, \ldots, (q+1)/2$ and with $a_i^{(q+1)/2} = -1$. Also, from [9, Section 75] $Q_{a_i} \cap \mathcal{H}(3,q^2) = B_i \cap \mathcal{H}(3,q^2) = \mathcal{E}_i$ for each i. The remaining $(q+1)/2$ quadrics, say $Q_{b_1}, Q_{b_2}, Q_{b_{(q+1)/2}}$, are such that $\mathcal{H}(3,q^2) \cap Q_{b_i}$ is a hyperbolic quadric $\mathcal{I}_i$ embedded in a Baer subgeometry $S_i \cong$ PG$(3,q)$ of PG$(3,q^2)$, for $i = 1, 2, \ldots, (q+1)/2$ and with $b_i^{(q+1)/2} = 1$. Also, from [9, Section 75] $S_i \cap \mathcal{H}(3,q^2) = \mathcal{I}_i$ for each i, whereas $Q_{b_i} \cap \mathcal{H}(3,q^2)$ consists of $2q^3 + q^2 + 1$ points lying on $2(q+1)$ generators. These generators are partitioned into two (extended) subreguli, actually the subregulus and its opposite of the hyperbolic quadric $\mathcal{I}_i$. Notice that all Baer subgeometries B_i and S_i, where $i \in \{1, 2, \ldots, (q+1)/2\}$, share the Baer subplane π containing the Baer conic C and the point $P = \bar{\pi}^{\rho} = (1,0,0,0)$.
With the notation introduced above, set

$$\mathcal{B} = \left(\bigcup_{i=1}^{(q+1)/2} \mathcal{I}_i \cup \bigcup_{i=1}^{(q+1)/2} \mathcal{E}_i \right) \setminus \{C\}.$$

The following lemma proved in [5] plays a crucial role.

Lemma 2.1. *Let P be a point of $\mathcal{B}$, then*

i) if P belongs to some elliptic quadric $\mathcal{E}_i$, then no generator of $\mathcal{H}(3,q^2)$ through P can meet the set $\mathcal{B}$ in a further point;

ii) if P belongs to some hyperbolic quadric $\mathcal{I}_i$, then no generator of $\mathcal{H}(3,q^2)$ through P can meet the set $\mathcal{B}$ in a point which does not lie on the same hyperbolic quadric $\mathcal{I}_i$.

We are interested in the orbits of the group $L \simeq \mathrm{PSL}_2(q)$ stabilizing the Baer subconic C on the generators of $\mathcal{H}(3,q^2)$. We distinguish two cases.
Certainly L has $q+1$ orbits of size $q+1$ corresponding to reguli of the hyperbolic quadrics $\mathcal{I}_i$'s, for any $i \in \{1, 2, \ldots, (q+1)/2\}$.
Let $P \in \mathcal{E}_i$. There are $q+1$ generators through P and P^{ρ} intersects π in a Baer subline external to the conic C. Since L acts transitively on $\mathcal{E}_i \setminus C$, then $|Stab_L(P)| = (q+1)/2$ and in this case $Stab_L(P)$ contains no central involution. Hence, the set of $q+1$ generators through P splits under L into two orbits of size $(q+1)/2$. As we have $(q+1)/2$ elliptic quadrics $\mathcal{E}_i$, from Lemma 2.1, we have $q+1$ orbits of size $(q^2-q)(q+1)/2$.
Let $P \in \mathcal{I}_i$. Then P^{ρ} meets π in a Baer subline which is secant to the conic C. There are $q-1$ generators through P distinct from the two lines of the reguli of $\mathcal{Q}_{b_i}$ through P. Again, since L acts transitively on $\mathcal{I}_i \setminus C$, then $|Stab_K(P)| = 2$ and in this case $Stab_L(P)$ contains a central involution. It follows that the set of $q-1$ generators through P splits under L into four orbits each of size $(q-1)/4$. In this case we have $2(q+1)$ orbits of size $(q^2+q)(q-1)/4$.

Summing up, we have $(q+1)(q^3+1)$ lines, i.e., the total number of generators of $\mathcal{H}(3,q^2)$.
Specializing to the case $q=7$ we found that generators for L are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 5 \\ 0 & 4 & 5 & 4 \\ 0 & 3 & 5 & 3 \end{bmatrix}.$$

The new $L_2(7)$–hemisystem is obtained using Magma [2] by gluing together 4 orbits of size 8, 8 orbits of size 84 and 4 orbits of size 168. Representative lines of the orbits of size 8 are:

$$L_1 = \langle (1,0,3,5),(1,3,2,0)\rangle, L_2 = \langle (0,1,2,4),(1,6,\omega^7,\omega^{20})\rangle,$$

$$L_3 = \langle (0,1,5,4),(1,0,\omega^{14},\omega^{46})\rangle, L_4 = \langle (1,6,\omega^3,\omega^{39}),(0,1,4,5)\rangle.$$

Representative lines of the orbits of size 84 are:

$$L_5 = \langle (1,3,4,\omega^{44}),(1,5,\omega^6,\omega^6)\rangle, L_6 = \langle (0,1,1,\omega^{47}),(1,1,\omega^4,1)\rangle,$$
$$L_7 = \langle (1,0,\omega^9,0),(0,1,0,\omega^{39})\rangle, L_8 = \langle (1,3,\omega^{31},\omega^{30}),(1,2,5,\omega^{23})\rangle,$$
$$L_9 = \langle (1,\omega,\omega^2,\omega^{12}),(1,3,4,\omega^{44})\rangle, L_{10} = \langle (1,0,2,\omega^{26}),(0,1,\omega^{26},\omega^{28})\rangle,$$
$$L_{11} = \langle (0,1,1,\omega^{23}),(1,\omega^3,2,\omega)\rangle, L_{12} = \langle (1,0,2,\omega^{20}),(1,\omega^2,0,\omega^{28})\rangle,$$

Representative lines of the orbits of size 168 are

$$L_{13} = \langle (1,3,1,\omega^{25}),(1,0,\omega^{14},\omega^{22})\rangle, L_{14} = \langle (1,4,4,3),(1,0,\omega^5,\omega^{36})\rangle,$$

$$L_{15} = \langle (0,1,\omega^2,\omega^{10}),(1,0,\omega^4,\omega^{20})\rangle, L_{16} = \langle (1,\omega,2,\omega 33),(1,0,\omega^{31},\omega^{37})\rangle,$$

where $\omega^7+6\omega+3=0$. The full stabilizer of the new hemisystem is $\mathrm{PSL}_2(7)$ [2].

Remark 2.2. When $q = 5$ gluing together $PSL_2(5)$–orbits we found again using Magma [2] just the hemisystems admitting $P\Omega_4^-(5)$ and $3.A_7.2$ from [7]. When $q = 9$, gluing together $PSL_2(9)$–orbits we just found the $P\Omega_4^-(9)$–hemisystem from [7]. It might be hoped that focussing on the group $PSL_2(7)$ may equally in the future lead to a generalization to another infinite family of hemisystems of $\mathcal{H}(3, q^2)$, at least when $q = 7^h$.

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