$[r, s, t]$-colorings of fans

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Abstract

Given non-negative integers r, s and t, an $[r, s, t]$-coloring of a graph $G = (V(G), E(G))$ is a function c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices v_i and edges e_j. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ is the minimum k such that G admits an $[r, s, t]$-coloring. In this paper, we examine $[r, s, t]$-chromatic numbers of fans for every positive integer r, s and t.
Keywords: $[r, s, t]$-coloring, $[r, s, t]$-chromatic number, wheels, friendship graphs, fans

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A vertex coloring of a graph G is a mapping c from $V(G)$ to the color set $\{0, 1, \dots, k-1\}$ such that no adjacent vertices receive the same color. An edge coloring of a graph G is a mapping c from $E(G)$ to the color set $\{0, 1, \dots, k-1\}$ such that no adjacent edges receive the same color. A total coloring of a graph G is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k-1\}$ such that no adjacent or incident elements receive the same color. For each of these colorings, the minimum k that G admits such a coloring is called the chromatic number $\chi(G)$, the chromatic index $\chi'(G)$ and the total chromatic number $\chi''(G)$, respectively.

Given non-negative integers r, s and t, an $[r, s, t]$-coloring of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges. The minimum k such that G admits an $[r, s, t]$-coloring is called the $[r, s, t]$-chromatic number of G and is denoted by $\chi_{r,s,t}(G)$.

$[r, s, t]$-colorings are obvious generalization of all the classical colorings, because a $[1, 0, 0]$-coloring is a vertex coloring, a $[0, 1, 0]$-coloring is an edge coloring, and a $[1, 1, 1]$-coloring is a total coloring. Due to these relations to the classical colorings, there are different applications for $[r, s, t]$-colorings, as described in [7].

First results on the $[r,s,t]$-coloring are given by Kemnitz and Marangio [7], such as monotonicity properties and general bounds. They also presented exact values and bounds on the $[r,s,t]$-chromatic number for complete graphs, for the cases that at least one of the parameters r,s,t is 0, and for the cases that two of the parameters r,s,t are 1. Further results on the $[r,s,t]$-chromatic number of complete graphs can be found in [6, 9, 13]. Moreover, Kemnitz, Marangio, and Mihók [8] characterized hereditary properties of graphs that have an $[r,s,t]$-chromatic number less than k, for $k=1,2,3$ as well as for $k \geq 3$ and $\max\{r,s,t\}=1$. Other results on the $[r,s,t]$-chromatic number are presented in [1, 2, 5, 11–13], where exact values and bounds are proved for some graphs and graph products.

Let n be a positive integer. A fan F_n of order $n+1$ is a graph obtained by connecting a single vertex to all vertices of a path P_n of order n. A fan F_n is called even or odd if n is even or odd, respectively. Fans play an important role in coloring problems [3, 4, 10, 14].

In this paper, we aim to investigate $[r,s,t]$-chromatic numbers of fans for any positive integers r,s and t.

We first recall the following notations and definitions. The maximum degree of G is denoted by $\Delta(G)$. The three conditions of an $[r,s,t]$-coloring of G between the colors of vertices and edges are denoted by r-condition, s-condition and t-condition, see [1]. A friendship graph $C_3^{(n)}$ is a graph obtained by taking n copies of the cycle graph C_3 with a vertex in common, and a wheel W_n of order $n+1$ is a graph obtained by connecting a single vertex to all vertices of a cycle graph C_n of order n, see [4]. We need the following lemmas from [7] and [11] to study $[r,s,t]$-colorings of fans.

Lemma 1.1. ([7]) *If $H \subseteq G$, then $\chi_{r,s,t}(H) \leq \chi_{r,s,t}(G)$.*

Lemma 1.2. ([7]) $\max\{r(\chi(G)-1)+1, s(\chi'(G)-1)+1, t+1\} \leq \chi_{r,s,t}(G) \leq r(\chi(G)-1)+s(\chi'(G)-1)+t+1$.

Lemma 1.3. ([11]) *If $\Delta(G) \geq 2$ and G is class 1, then*

$$\chi_{r,s,t}(G) \geq \begin{cases} (\Delta(G)-1)s+1 & \text{if } s \geq 2t, \\ (\Delta(G)-2)s+2t+1 & \text{if } t \leq s < 2t, \\ (\Delta(G)-1)s+t+1 & \text{if } s < t. \end{cases}$$

Lemma 1.4. ([11]) *Let $C_3^{(n)}$ be a friendship graph with $\Delta(C_3^{(n)}) \geq 4$. For $\min\{r,s,t\} \geq 1$, the $[r,s,t]$-chromatic number of $C_3^{(n)}$ is given by:*

(1) *if $s \geq 2t$, then*

$$\chi_{r,s,t}(C_3^{(n)}) = \begin{cases} 2r+1 & \text{if } r \geq ns-t, \\ \max\{2r+1, (2n-1)s+1\} & \text{if } (n-1)s+t \leq r < ns-t, \\ (2n-1)s+1 & \text{if } r < (n-1)s+t. \end{cases}$$

(2) *if $t \leq s < 2t$, then*

$$\chi_{r,s,t}(C_3^{(n)}) = \begin{cases} 2r+1 & \textit{if } r \geq (n-1)s+t, \\ (2n-2)s+2t+1 & \textit{if } r < (n-1)s+t. \end{cases}$$

(3) *if* $s < t \leq ns$, *then*

$$\chi_{r,s,t}(C_3^{(n)}) = \begin{cases} 2r+1 & \textit{if } r \geq (n-1)s+t, \\ (\leq)r+(n-1)s+t+1 & \textit{if } ns \leq r < (n-1)s+t, \\ (2n-1)s+t+1 & \textit{if } r < ns. \end{cases}$$

Lemma 1.5. ([11]) *Let* W_{2n} *be an even wheel and* $C_3^{(n)}$ *be a spanning friendship graph of* W_{2n} *such that* $\Delta(W_{2n}) = \Delta(C_3^{(n)}) \geq 6$. *If* $s \geq t$, *then* $\chi_{r,s,t}(W_{2n}) = \chi_{r,s,t}(C_3^{(n)})$.

2 The $[r, s, t]$-chromatic number of fans

In this section we investigate $[r, s, t]$-colorings of fans for any positive integers r, s and t. Let the vertex set and edge set of the fan F_n be defined as follows: $V(F_n) = \{x\} \cup \{y_i | 1 \leq i \leq n\}$ and $E(F_n) = \{xy_i | 1 \leq i \leq n\} \cup \{y_i y_{i+1} | 1 \leq i \leq n-1\}$. Even fans and odd fans are denoted by F_{2n} and F_{2n+1}, respectively. Moreover, let $C_v = [c_{v\min}, c_{v\max}]$ be the color set used for all the vertices of odd fans F_{2n+1}, and $C_e = [c_{e\min}, c_{e\max}]$ the color set used for the edges $\{xy_i | 1 \leq i \leq 2n+1\}$ of odd fans F_{2n+1}.

We first prove that under some conditions on r, s and t, the $[r, s, t]$-chromatic number of an even fan F_{2n} is equal to the $[r, s, t]$-chromatic number of its largest induced friendship graph $C_3^{(n)}$.

Theorem 2.1. *Let* F_{2n} *be an even fan and* $C_3^{(n)}$ *be a spanning friendship graph of* F_{2n} *such that* $\Delta(F_{2n}) = \Delta(C_3^{(n)}) \geq 4$. *If* $s \geq t$ *or,* $s < t \leq ns$ *and* $r \geq (n-1)s+t$, *then* $\chi_{r,s,t}(F_{2n}) = \chi_{r,s,t}(C_3^{(n)})$.

Proof. Since $C_3^{(n)} \subset F_{2n} \subset W_{2n}$, by Lemma 1.1, $\chi_{r,s,t}(C_3^{(n)}) \leq \chi_{r,s,t}(F_{2n}) \leq \chi_{r,s,t}(W_{2n})$. Moreover, according to Lemma 1.5, if $s \geq t$ and $\Delta(W_{2n}) \geq 6$, then $\chi_{r,s,t}(C_3^{(n)}) = \chi_{r,s,t}(W_{2n})$. Therefore $\chi_{r,s,t}(F_{2n}) = \chi_{r,s,t}(C_3^{(n)})$ if $\Delta(F_{2n}) \geq 6$ and $s \geq t$. Now we consider the remaining cases:

Case 1. $\Delta(F_{2n}) \geq 4$, $s < t \leq ns$ and $r \geq (n-1)s+t$.

By Lemma 1.4, $\chi_{r,s,t}(C_3^{(n)}) = 2r+1$. Then we prove $\chi_{r,s,t}(F_{2n}) = 2r+1$. First, we prove $\chi_{r,s,t}(F_{2n}) \leq 2r+1$ by construction. F_{2n} is colored as follows: $c(x) = r$, $c(y_j) = 0$ and $c(y_{j+1}) = 2r$, for $j = 1, 3, \ldots, 2n-1$. $c(xy_1) = r+t, c(xy_3) = r+t+s, \cdots, c(xy_{2n-1}) = r+t+(n-1)s$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_j y_{j+1}) = c(y_{2n-1}y_{2n}) = r, c(y_{j+1}y_{j+2}) = r+s$, for $j = 1, 3, \ldots, 2n-3$.

For this coloring, the r-condition is obviously fulfilled. Since $s < t \leq ns$ and $r \geq (n-1)s+t$, it follows that $|c(y_j y_{j+1}) - c(xy_j)| \geq |r-(r+t)| \geq t > s$, $|c(y_j y_{j+1}) - c(xy_{j+1})| \geq |r-(n-1)s| \geq t > s$, $|c(y_{j+1}y_{j+2}) - c(xy_{j+1})| \geq |r+$

$s-(n-2)s| \geq s$, $|c(y_{j+1}y_{j+2})-c(xy_{j+2})| \geq |r+s-(r+t+s)| \geq s$, as well as $|c(y_{j+1})-c(xy_{j+1})| \geq |2r-(n-1)s| \geq t$, $|c(y_{j+1})-c(y_{j+1}y_{j+2})| \geq |2r-(r+s)| \geq t$, $|c(y_{j+1})-c(y_jy_{j+1})| \geq |2r-r| \geq t$, for any $j = 1, 3, \ldots, 2n-1$. The s-condition and t-condition of remaining cases are also satisfied. Therefore $\chi_{r,s,t}(F_{2n}) \leq 2r+1$. Moreover, the lower bound is proved by Lemma 1.2. Thus $\chi_{r,s,t}(F_{2n}) = 2r+1$.

Case 2. $\Delta(F_{2n}) = 4$ and $s \geq t$.

Lemma 1.4 presents exact values for the $[r,s,t]$-chromatic number of $C_3^{(2)}$ if $s \geq t$. For each case, we color F_4 with $\chi_{r,s,t}(C_3^{(2)})$ colors.

First, if $s \geq 2t$ and $r \geq 2s-t$, or $t \leq s < 2t$ and $r \geq s+t$, F_4 is colored as follows: $c(x) = r, c(y_1) = c(y_3) = 2r, c(y_2) = c(y_4) = 0$; $c(xy_1) = r+t, c(xy_2) = r+t+s, c(xy_3) = 0, c(xy_4) = s$; $c(y_1y_2) = t, c(y_2y_3) = s+t, c(y_3y_4) = 2s+t$. Second, if $s \geq 2t$ and $s+t \leq r < 2s-t$, we color F_4 similar to the first coloring, and only recolor the edges xy_1 and xy_2: $c(xy_1) = 2s$ and $c(xy_2) = 3s$. Third, if $s \geq 2t$ and $r < s+t$, we color F_4 similar to the second coloring, and only recolor the vertices x, y_1 and y_3: $c(x) = s+t$ and $c(y_1) = c(y_3) = 3s$. Finally, if $t \leq s < 2t$ and $r < s+t$, F_4 is colored as follows: $c(x) = s+t, c(y_1) = c(y_3) = 2s+2t, c(y_2) = c(y_4) = 0$; $c(xy_1) = s+2t, c(xy_2) = 2s+2t, c(xy_3) = 0$, $c(xy_4) = s$; $c(y_1y_2) = t, c(y_2y_3) = s+t, c(y_3y_4) = 2s+t$.

For each coloring, the three conditions of an $[r,s,t]$-coloring of F_4 are fulfilled. Therefore $\chi_{r,s,t}(F_4) \leq \chi_{r,s,t}(C_3^{(2)})$. Moreover, by Lemma 1.1, $\chi_{r,s,t}(F_4) \geq \chi_{r,s,t}(C_3^{(2)})$, and therefore $\chi_{r,s,t}(F_4) = \chi_{r,s,t}(C_3^{(2)})$.

Hence, $\chi_{r,s,t}(F_{2n}) = \chi_{r,s,t}(C_3^{(n)})$ in the considered cases. □

We establish an upper bound for the $[r,s,t]$-chromatic number of F_{2n} if $s < t \leq ns$ and $r < (n-1)s+t$.

Theorem 2.2. *Let F_{2n} be an even fan with $\Delta(F_{2n}) \geq 4$. If $\min\{r,s,t\} \geq 1$, $s < t \leq ns$, and $r < (n-1)s+t$, then $\chi_{r,s,t}(F_{2n}) \leq (2n-2)s+2t+1$.*

Proof. We prove the upper bound by contradiction. We color F_{2n} as follows: $c(x) = (n-1)s+t$, $c(y_j) = 0$, $c(y_{j+1}) = (2n-2)s+2t$, for $j = 1, 3, \ldots, 2n-1$. $c(xy_1) = (n-1)s+2t, c(xy_3) = ns+2t, \cdots, c(xy_{2n-1}) = (2n-2)s+2t$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_jy_{j+1}) = c(y_{2n-1}y_{2n}) = (n-1)s+t, c(y_{j+1}y_{j+2}) = ns+t$, for $j = 1, 3, \ldots, 2n-3$. For this coloring, all the required conditions to have an $[r,s,t]$-coloring are fulfilled. Therefore $\chi_{r,s,t}(F_{2n}) \leq (2n-2)s+2t+1$. □

In the rest of this paper we study $[r,s,t]$-colorings of odd fans F_{2n+1} for every positive integer r, s and t. We start by a lemma that is used to determine $[r,s,t]$-chromatic numbers of odd fans F_{2n+1} with $\Delta(F_{2n+1}) \geq 5$.

Lemma 2.1. *Let F_{2n+1} be an odd fan with $\Delta(F_{2n+1}) \geq 5$. If $\min\{r,s,t\} \geq 1$ and $s \geq t$, then*

$$\chi_{r,s,t}(F_{2n+1}) \geq \begin{cases} 2ns+t+1 & \text{if } ns \leq r < ns+t, \\ 2r+t+1 & \text{if } (n-1)s+\max\{s-t,t\} \leq r < ns. \end{cases}$$

Proof. Since $\chi(G) = 3$ and $\chi'(G) = 2n$, by the r-condition between $c_{v\min}$ and $c_{v\max}$, and the s-condition between $c_{e\min}$ and $c_{e\max}$, it follows that $c_{v\max} \geq 2r + c_{v\min}$ and $c_{e\max} \geq 2ns + c_{e\min}$. If $ns \leq r < ns+t$, we prove $\chi_{r,s,t}(F_{2n+1}) \geq 2ns+t+1$ by contradiction. Suppose that there exists an $[r,s,t]$-coloring on the set of colors $\{0,1,\ldots,k\}$ with $k < 2ns+t$. Since $c(x)$ is distinct from $c(xy_i)$ for $1 \leq i \leq 2n+1$, it follows that $c(x) < c_{e\min}$, $c(x) > c_{e\max}$, or $c_{e\min} < c(x) < c_{e\max}$.

First, let $c(x) < c_{e\min}$ or $c(x) > c_{e\max}$. If $c(x) < c_{e\min}$, due to the t-condition between $c(x)$ and $c_{e\min}$, $c_{e\min} \geq c(x)+t \geq t$. Thus, $c_{e\max} \geq c_{e\min}+2ns \geq 2ns+t$. Similarly, if $c(x) > c_{e\max}$, then $c(x) \geq c_{e\max}+t \geq 2ns+t$. Therefore $k \geq 2ns+t$, a contradiction.

Second, let $c_{e\min} < c(x) < c_{e\max}$. Since $\chi(G) = 3$ and $k < 2ns+t < 3r$, it follows that $c(x) = c_{v\min}$, $c(x) = c_{v\max}$, or $c_{v\min}+r \leq c(x) \leq c_{v\max}-r$. If $c(x) = c_{v\min}$, then $c_{v\max} \geq 2r+c(x) \geq 2r+c_{e\min}+t \geq 2r+t$, a contradiction. If $c(x) = c_{v\max}$, then $c_{e\max} \geq c(x)+t \geq 2r+t$, a contradiction. Therefore $c_{v\min}+r \leq c(x) \leq c_{v\max}-r$. Since $c_{v\max} < 2ns+t \leq r+ns+t$, $r \leq c(x) < ns+t$. Due to the t-condition between the colors $c(x)$ and $c(xy_i)$, $c(xy_i) \leq c(x)-t$ or $c(xy_i) \geq c(x)+t$, for any $1 \leq i \leq 2n+1$. Since $r \leq c(x) < ns+t$, at most n colors of $\{c(xy_1), c(xy_2), \ldots, c(xy_{2n+1})\}$ can be less than or equal to $c(x)-t$ and at least $n+1$ colors of $\{c(xy_1), c(xy_2), \ldots, c(xy_{2n+1})\}$ must be greater than or equal to $c(x)+t$. Moreover, by the s-condition among the $n+1$ colors of $\{c(xy_1), c(xy_2), \ldots, c(xy_{2n+1})\}$ which are not less than $c(x)+t$, $c_{e\max} \geq c(x)+t+ns \geq 2ns+t$, a contradiction.

By a proof similar to the above, we can obtain the second result. □

Next we study $[r,s,t]$-colorings of odd fans F_{2n+1} with $\Delta(F_{2n+1}) \geq 5$ for every positive integer r, s, and $t \leq ns$ except in a few cases.

Theorem 2.3. *Let F_{2n+1} be an odd fan with $\Delta(F_{2n+1}) \geq 5$. If $\min\{r,s,t\} \geq 1$, then*

$$\chi_{r,s,t}(F_{2n+1}) = \begin{cases} 2r+1 \text{ if } s \geq t \text{ or } s < t \leq ns, \text{ and } r \geq ns+t, \\ \max\{2r+1, 2ns+t+1\} \text{ if } s \geq 2t \text{ and } ns \leq r < ns+t, \text{or } t \leq s < 2t \text{ and} \\ \qquad 2s \leq r < 3s-t, \text{ or } t \leq s < 2t,\ n \geq 3, \text{ and } ns \leq r < ns+t, \\ (\leq) r+(n-1)s+t+1 \text{ if } s < t \leq ns \text{ and } ns \leq r < ns+t, \\ \max\{2r+t+1, (2n-1)s+\max\{s,2t\}+1\} \\ \quad \text{if } s \geq t \text{ and } (n-1)s+\max\{s-t,t\} \leq r < ns, \\ (2n-1)s+\max\{s,2t\}+1 \text{ if } s \geq t \text{ and } r < (n-1)s+\max\{s-t,t\}, \\ (2n-1)s+t+1 \text{ if } s < t \leq ns \text{ and } r < ns. \end{cases}$$

Proof. We prove these results independently.

Claim 1: $\chi_{r,s,t}(F_{2n+1}) = 2r+1$, if $s \geq t$ or $s < t \leq ns$, and $r \geq ns+t$.

We prove the upper bound by construction. We consider the following coloring of F_{2n+1}: $c(x) = 2r$, $c(y_j) = c(y_{2n+1}) = 0$ and $c(y_{j+1}) = r$, for $j = 1,3,\ldots,2n-1$. $c(xy_1) = ns, c(xy_3) = (n+1)s, \cdots, c(xy_{2n+1}) = 2ns$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_jy_{j+1}) = 2r$ and $c(y_{j+1}y_{j+2}) = ns$, for $j = 1,3,\ldots,2n-1$.

For this coloring, the r-condition and s-condition are obviously fulfilled. Since $r \geq ns+t$, $|c(x) - c(xy_i)| \geq |2r - 2ns| \geq t$, $|c(y_{j+1}) - c(xy_{j+1})| \geq |r-(n-1)s| \geq t$, $|c(y_{j+1}) - c(y_jy_{j+1})| \geq |r-2r| \geq t$, and $|c(y_{j+1}) - c(y_{j+1}y_{j+2})| \geq |r-ns| \geq t$, for any $i = 1,2,\ldots,2n+1$ and $j = 1,3,\ldots,2n-1$. Moreover, since $c(y_j) = 0$ for any $j = 1,3,\ldots,2n+1$, the color difference between the vertex y_j and its incident edges is not less than ns. Thus the t-condition is verified. Therefore $\chi_{r,s,t}(F_{2n+1}) \leq 2r+1$. Moreover, by Lemma 1.2, $\chi_{r,s,t}(F_{2n+1}) \geq 2r+1$. Hence $\chi_{r,s,t}(F_{2n+1}) = 2r+1$.

Claim 2: $\chi_{r,s,t}(F_{2n+1}) = \max\{2r+1, 2ns+t+1\}$, if $s \geq 2t$ and $ns \leq r < ns+t$, or $t \leq s < 2t$ and $2s \leq r < 3s-t$, or $t \leq s < 2t$, $n \geq 3$, and $ns \leq r < ns+t$.

We prove the upper bound by construction. We color F_{2n+1} as follows: $c(x) = \max\{2r, 2ns+t\}$, $c(y_j) = c(y_{2n+1}) = 0$ and $c(y_{j+1}) = r$, for $j = 1,3,\ldots,2n-1$. $c(xy_1) = ns, c(xy_3) = (n+1)s, \cdots, c(xy_{2n+1}) = 2ns$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_jy_{j+1}) = 2r$, $c(y_2y_3) = c(y_4y_5) = \cdots = c(y_{2n-2}y_{2n-1}) = (n-1)s$ and $c(y_{2n}y_{2n+1}) = r+t$, for $j = 1,3,\ldots,2n-1$.

For this coloring, the three conditions of an $[r,s,t]$-coloring are fulfilled. Therefore the upper bound is proved. On the other hand, by Lemma 1.2 and Lemma 2.1, $\chi_{r,s,t}(F_{2n+1}) \geq \max\{2r+1, 2ns+t+1\}$. Hence $\chi_{r,s,t}(F_{2n+1}) = \max\{2r+1, 2ns+t+1\}$.

Claim 3: $\chi_{r,s,t}(F_{2n+1}) \leq r+ns+t+1$, if $s < t \leq ns$ and $ns \leq r < ns+t$.

We give a coloring of F_{2n+1} to prove the upper bound. F_{2n+1} is colored as follows: $c(x) = r+ns+t, c(y_j) = c(y_{2n+1}) = 0$, and $c(y_{j+1}) = ns+t$ for $j = 1,3,\ldots,2n-1$. $c(xy_1) = ns, c(xy_3) = (n+1)s, \cdots, c(xy_{2n+1}) = 2ns$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_jy_{j+1}) = r+ns+t$ and $c(y_{j+1}y_{j+2}) = ns$, for $j = 1,3,\ldots,2n-1$.

For this coloring, the r-condition and s-condition are obviously fulfilled. Since $s < t \leq ns$ and $ns \leq r < ns+t$, $|c(x)-c(xy_i)| \geq |r+ns+t-2ns| \geq t$, $|c(y_j) - c(xy_j)| \geq |0-ns| \geq t$, $|c(y_j) - c(y_{j-1}y_j)| \geq |0-ns| \geq t$, $|c(y_j) - c(y_jy_{j+1})| \geq |0-(r+ns+t)| \geq t$, $|c(y_{j+1})-c(xy_{j+1})| \geq |ns+t-(n-1)s| \geq t$, $|c(y_{j+1})-c(y_jy_{j+1})| \geq |ns+t-(r+ns+t)| \geq t$, and $|c(y_{j+1})-c(y_{j+1}y_{j+2})| \geq |ns+t-ns| \geq t$, for $i = 1,2,\ldots,2n+1$ and $j = 1,3,\ldots,2n+1$. Therefore the t-condition is also verified. Thus $\chi_{r,s,t}(F_{2n+1}) \leq r+ns+t+1$.

Claim 4: $\chi_{r,s,t}(F_{2n+1}) = \max\{2r+t+1, (2n-1)s + \max\{s, 2t\} + 1\}$, if $s \geq t$ and $(n-1)s + \max\{s-t, t\} \leq r < ns$.

We prove the upper bound by construction. We color F_{2n+1} similar to the coloring of Claim 2, and only recolor the vertex x, as well as the edges xy_{2n+1} and y_jy_{j+1} for $j = 1, 3\dots, 2n-1$. First, if $(2n-1)s+\max\{s-t,t\} \leq 2r < 2ns$, then $c(x) = 2r$ and $c(xy_{2n+1}) = c(y_jy_{j+1}) = 2r + t$. Second, if $2(n-1)s + \max\{2s-2t, 2t\} \leq 2r < (2n-1)s + \max\{s-t, t\}$, then $c(x) = (2n-1)s + \max\{s-t,t\}$ and $c(xy_{2n+1}) = c(y_jy_{j+1}) = (2n-1)s + \max\{s, 2t\}$. For each coloring, the three condition of an $[r, s, t]$-coloring are fulfilled. Therefore the upper bound is proved. Moreover, the lower bound is easily obtained by Lemma 1.3 and Lemma 2.1. Hence $\chi_{r,s,t}(F_{2n+1}) = \max\{2r+t+1, (2n-1)s + \max\{s, 2t\} + 1\}$.

Claim 5: $\chi_{r,s,t}(F_{2n+1}) = (2n-1)s + \max\{s, 2t\} + 1$, if $r < (n-1)s + \max\{s-t, t\}$.

First, if $s \geq 2t$ and $r < ns - t$, we color F_{2n+1} as follows: $c(x) = ns - t, c(y_j) = c(y_{2n+1}) = 0$ and $c(y_{j+1}) = 2ns$, for $j = 1, 3, \dots, 2n-1$. $c(xy_1) = ns, c(xy_3) = (n+1)s, \cdots, c(xy_{2n+1}) = 2ns$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_1y_2) = (2n-1)s$ and $c(y_ly_{l+1}) = (l-1)s$, for $l = 2, 3, \dots, 2n$. Second, if $t \leq s < 2t$ and $r < (n-1)s + t$, F_{2n+1} is colored as follows: $c(x) = (n-1)s + t, c(y_j) = c(y_{2n+1}) = 0$ and $c(y_{j+1}) = (2n-1)s + 2t$, for $j = 1, 3, \dots, 2n-1$. $c(xy_1) = (n-1)s + 2t, c(xy_3) = ns + 2t, \cdots, c(xy_{2n+1}) = (2n-1)s + 2t$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_1y_2) = (2n-2)s+2t$ and $c(y_ly_{l+1}) = (l-1)s$, for $l = 2, 3, \dots, 2n$. For each coloring, all the required conditions of an $[r, s, t]$-coloring are fulfilled. Therefore $\chi_{r,s,t}(F_{2n+1}) \leq (2n-1)s + \max\{s, 2t\} + 1$.

On the other hand, the lower bound is easily obtained by Lemma 1.3. Hence $\chi_{r,s,t}(F_{2n+1}) = (2n-1)s + \max\{s, 2t\} + 1$.

Claim 6: $\chi_{r,s,t}(F_{2n+1}) = 2ns + t + 1$, if $r < ns$ and $s < t \leq ns$.

If $r < ns$ and $s < t \leq ns$, F_{2n+1} is colored as follows: $c(x) = 2ns + t$, $c(y_j) = c(y_{2n+1}) = 0$ and $c(y_{j+1}) = ns + t$, for $j = 1, 3, \dots, 2n-1$. $c(xy_1) = ns, c(xy_3) = (n+1)s, \cdots, c(xy_{2n+1}) = 2ns$; $c(xy_2) = 0, c(xy_4) = s, \cdots, c(xy_{2n}) = (n-1)s$; $c(y_jy_{j+1}) = 2ns+t$ and $c(y_{j+1}y_{j+2}) = ns$, for $j = 1, 3, \dots, 2n-1$. For this coloring, the three conditions of an $[r, s, t]$-coloring are fulfilled. That is, $\chi_{r,s,t}(F_{2n+1}) \leq 2ns+t+1$. Moreover, by Lemma 1.3, $\chi_{r,s,t}(F_{2n+1}) \geq 2ns + t + 1$. Hence $\chi_{r,s,t}(F_{2n+1}) = 2ns + t + 1$. □

Note that we give exact values and an upper bound in one case for the $[r, s, t]$-chromatic number of odd fans F_{2n+1} with $\Delta(F_{2n+1}) \geq 5$ for every positive integer r, s, and $t \leq ns$ except the case that $n = 2$, $t \leq s < 2t$ and $3s - t \leq r < 2s + t$. What is $\chi_{r,s,t}(F_5)$ if $t \leq s < 2t$ and $3s - t \leq r < 2s + t$?

In the last part of this paper we study $[r, s, t]$-colorings of fans F_3 which are isomorphic to complete graphs K_4 without an edge, for every positive integer r, s and t. We first establish a lower bound for the results in Theorem 2.4 in the next lemma.

Lemma 2.2. *Let F_3 be an odd fan with $\Delta(F_3) = 3$. If* $\min\{r, s, t\} \geq 1$,

$s \geq t$ *and* $\max\{s-t, t\} \leq r < s+t$, *as well as* $s < t$ *and* $t \leq r < s+t$, *then* $\chi_{r,s,t}(F_3) \geq r+s+t+1$.

Proof. We prove the lower bound by contradiction. Suppose that there exists an $[r,s,t]$-coloring of F_3 on the color set $\{0,1,\ldots,k\}$ with $k < r+s+t$. Since $\chi(F_3) = 3$, there are at least three relations between the color $c(x)$ and the color set C_v: $c(x) = c_{v\min}$, $c(x) = c_{v\max}$, or $c_{v\min} + r \leq c(x) \leq c_{v\max} - r$. Furthermore, if $c(x) \geq c_{v\min} + 3r$, we deduce a contradiction similar to the case that $c(x) = c_{v\max}$. Thus we only consider the three relations mentioned above in the following proof.

We first consider the case that $\max\{s,t\} \leq r < s+t$. Since $c(x)$ is different from $c(xy_i)$ for $1 \leq i \leq 3$, we distinguish the following two cases:
Case 1. $c(x)$ is in C_e (i.e. $c_{e\min} < c(x) < c_{e\max}$).

As argument in Lemma 2.1, if $c(x) = c_{v\min}$ or $c(x) = c_{v\max}$, $k \geq 2r+t \geq r+s+t$, a contradiction. Therefore $c_{v\min} + r \leq c(x) \leq c_{v\max} - r$. Since $k < r+s+t$, $r \leq c(x) < s+t$. By the t-condition between the colors $c(x)$ and $c(xy_i)$ for $1 \leq i \leq 3$, at least two colors of $\{c(xy_1), c(xy_2), c(xy_3)\}$ are not less than $c(x)+t$. Thus $c_{e\max} \geq c(x)+t+s \geq r+s+t$, a contradiction.
Case 2. $c(x)$ is not in C_e (i.e. $c(x) < c_{e\min}$ or $c(x) > c_{e\max}$).

Fact, $c(x) = c_{v\min}$ or $c(x) = c_{v\max}$. If $c_{v\min} + r \leq c(x) \leq c_{v\max} - r$, $c_{e\max} \geq 2s + c_{e\min} \geq 2s + c(x) + t \geq 2s + r + t$ if $c(x) < c_{e\min}$, and $c_{v\max} \geq r + c(x) \geq r + c_{e\max} + t \geq r + 2s + t$ if $c(x) > c_{e\max}$, a contradiction.

First, let $c(x) = c_{v\min}$. If $c(x) > c_{e\max}$, then $c_{v\max} \geq 2r + c_{e\max} + t \geq 2r + 2s + t$, a contradiction. Thus $c(x) < c_{e\min}$. Then $t \leq c(x) + t \leq c_{e\min} < r + t - s$ and $2s + t \leq c_{e\max} < r + s + t$. If $s + t \leq c(xy_1) < r + t$ or $s + t \leq c(xy_3) < r + t$. Without loss of generality, assume that $s + t \leq c(xy_1) < r+t$. Then $c(xy_2) = c_{e\max}$ and $c(xy_3) = c_{e\min}$, or $c(xy_2) = c_{e\min}$ and $c(xy_3) = c_{e\max}$. Since $k < r+s+t < 3s+t$, it follows that $s+t \leq c(y_2y_3) < r+t$. Since $\max\{s,t\} \leq r < s+t$, $c(y_2) \geq s+2t$ or $c(y_3) \geq s+2t$. Thus $c_{v\max} \geq r+s+2t$, a contradiction. If $s+t \leq c(xy_2) < r+t$, without loss of generality, assume that $c(xy_1) = c_{e\min}$ and $c(xy_3) = c_{e\max}$. It follows that $2s+t \leq c(y_1y_2) \leq r+s+t$. Since $\max\{2s, 2t\} \leq 2r < r+s+t$, $c(y_1) \geq 2s+2t$ or $c(y_2) \geq 2s+2t$, a contradiction.

Second, let $c(x) = c_{v\max}$. If $c(x) < c_{e\min}$, then $c_{e\max} \geq 2s + c_{e\min} \geq 2s + c(x) + t \geq 2s + 2r + t$, a contradiction. Thus $c(x) > c_{e\max}$. Since $c(x) < r+s+t$, $2s \leq c_{e\max} < r+s$ and $0 \leq c_{e\min} < r-s$. If $s \leq c(xy_1) < r$ or $s \leq c(xy_3) < r$, without loss of generality, assume that $s \leq c(xy_1) < r$. Then $c(y_1) < c(y_2)$. Otherwise, $c(x) \geq r + c(y_1) \geq r + c(xy_1) + t \geq r+s+t$, a contradiction. If $s \geq t$ and $s \leq r < s+t$, since $r \leq c(y_2) < s+t$, at least two colors of $\{c(xy_2), c(y_1y_2), c(y_2y_3)\}$ are not less than $c(y_2)+t$. Therefore at least one color of $\{c(xy_2), c(y_1y_2), c(y_2y_3)\}$ is not less than $r+s+t$, a contradiction. If $s < t$ and $t \leq r < s+t$, since $c(y_1) < c(y_2)$ and $c(x) < r+s+t < 2r+t$, $0 \leq c(y_1) < t$. Then $t \leq c(xy_1) < r$ and

$s+t \le c(xy_3) < r+s$. Since $c(x) < r+s+t < 3r$, $c(y_3) < c(y_2)$. By the s-condition between the colors $c(y_2y_3)$ and $c(xy_2)$, $c(y_2y_3)$ and $c(xy_3)$, and the t-condition between the colors $c(y_2y_3)$ and $c(y_3)$, $2s+t \le c(xy_3)+s \le c(y_2y_3) < r+s+t$ or $t \le c(y_3)+t \le c(y_2y_3) \le c(xy_3)-s < r$. If $t \le c(y_2y_3) < r$, then $c(x) \ge r+c(y_2) \ge r+c(y_2y_3)+t \ge r+2t$, a contradiction. If $2s+t \le c(y_2y_3) < r+s+t$, then $s+t \le c(xy_1)+s \le c(y_1y_2) \le c(y_2y_3)-s < r+t$. Thus $c(x) \ge r+c(y_2) \ge r+c(y_1y_2)+t \ge r+s+2t$, a contradiction. If $s \le c(xy_2) < r$, without loss of generality, assume that $c(xy_1) = c_{e\min}$ and $c(xy_3) = c_{e\max}$. Then $c(y_2) < c(y_3)$. Otherwise, $c(x) \ge r+c(y_2) \ge r+c(xy_2)+t \ge r+s+t$, a contradiction. If $s \ge 2t$ and $s \le r < s+t$, and $t \le s < 2t$ and $s \le r < 2s-t$, then $k < r+s+t < 3s$. Thus $0 \le c(y_2y_3) < r-s$. If $t \le s < 2t$ and $2s-t \le r < s+t$, then $r+t \le c(y_3)+t \le c(xy_3) < r+s$. Since $k < r+s+t$, $0 \le c(y_2y_3) < r-s$. Since $c(y_2) < c(y_3)$, $c(y_2) \ge t$. Thus $c(x) \ge 2r+t$, a contradiction. If $s < t$ and $t \le r < s+t$, since $r \le c(y_3) < s+t$, $c(xy_3) \ge c(y_3)+t \ge r+t$. Thus $c(x) \ge c(xy_3)+t \ge r+2t > r+s+t$, a contradiction.

Now we consider the case that $s \ge t$ and $\max\{s-t, t\} \le r < s$. As argument in Lemma 2.1, if $c(x)$ is not in C_e, then $k \ge 2s+t > r+s+t$, a contradiction. Therefore $c(x)$ is in C_e. If $c_{v\min}+r \le c(x) \le c_{v\max}-r$, by an argument similar to the case that $\max\{s,t\} \le r < s+t$, we can deduce a contradiction that $c_{e\max} \ge r+s+t$. Therefore $c(x) = c_{v\min}$ or $c(x) = c_{v\max}$.

First, let $c(x) = c_{v\min}$. Since $c(x) > c_{e\min}$ and $k < r+s+t$, it holds that $t \le c(x) < s+t-r$ and $\max\{s+2t, 2s\} \le c_{e\max} < r+s+t$. If $\max\{s,2t\} \le c(xy_1) < r+t$ or $\max\{s,2t\} \le c(xy_3) < r+t$, without loss of generality, assume that $\max\{s,2t\} \le c(xy_1) \le r+t$. Then $c(xy_2) = c_{e\max}$ and $c(xy_3) = c_{e\min}$, or $c(xy_2) = c_{e\min}$ and $c(xy_3) = c_{e\max}$. Since $k < r+s+t < \max\{3s, 2s+2t\}$, $s \le c(y_2y_3) < r+t$. Since $\max\{s,2t\} \le r+t < s+t$, $c(y_2) \ge s+t$ or $c(y_3) \ge s+t$. Therefore $c_{v\max} \ge r+s+t$, a contradiction. If $\max\{s,2t\} \le c(xy_2) < r+t$, without loss of generality, assume that $c(xy_1) = c_{e\max}$ and $c(xy_3) = c_{e\min}$. Then $\max\{2s, s+2t\} \le c(y_2y_3) < r+t+s$. Since $\max\{3t, 2s-t\} \le 2r+t < 2s+t$, $c(y_2) \ge \max\{s+3t, 2s+t\}$ or $c(y_3) \ge \max\{s+3t, 2s+t\}$, a contradiction.

Second, let $c(x) = c_{v\max}$. Then $0 \le c_{e\min} < r+t-s$ and $2s \le c_{e\max} < r+t+s$. If $s \le c(xy_1) < r+t$ or $s \le c(xy_3) < r+t$, by a proof similar to the above case that $c(x) = c_{v\min}$, we can deduce a contradiction that $c(x) \ge r+s+t$. Therefore $s \le c(xy_2) < r+t$. Without loss of generality, assume that $c(xy_1) = c_{e\max}$ and $c(xy_3) = c_{e\min}$. Since $k < r+s+t < 3s$, $0 \le c(y_1y_2) < r+t-s$. Then $c(y_1) \ge t$ or $c(y_2) \ge t$. Therefore $c(x) \ge 2r+t$, and therefore $c_{e\max} \ge c(x)+t \ge 2r+2t > r+s+t$, a contradiction. □

Theorem 2.4. *Let F_3 be an odd fan with $\Delta(F_3) = 3$. If* $\min\{r,s,t\} \ge 1$, *then*

$$\chi_{r,s,t}(F_3)=\begin{cases} 2r+1 & \textit{if } r\geq s+t,\\ r+s+t+1 & \textit{if } s\geq t \textit{ and } \max\{s-t,t\}\leq r<s+t,\\ & \textit{or } s<t \textit{ and } t\leq r<s+t,\\ \max\{s,2t\}+s+1 & \textit{if } s\geq t \textit{ and } r<\max\{s-t,t\},\\ (\leq)s+2t+1 & \textit{if } s<t \textit{ and } r<t. \end{cases}$$

Proof. We prove the following results on fans F_3.

Claim 1: $\chi_{r,s,t}(F_3) = 2r+1$, if $r \geq s+t$.

If $s \geq t$, by a proof similar to Claim 1 in Theorem 2.3, $\chi_{r,s,t}(F_3) = 2r+1$. If $s < t$, by Lemma 1.2, $\chi_{r,s,t}(F_3) \geq 2r+1$. We color F_3 as follows: $c(x) = 2r$, $c(y_1) = c(y_3) = 0$ and $c(y_2) = r$; $c(xy_1) = t, c(xy_2) = 0, c(xy_3) = s+t$; $c(y_1y_2) = r+t$ and $c(y_2y_3) = r+t+s$. We can deduce that the proposed coloring is an $[r,s,t]$-coloring. Thus $\chi_{r,s,t}(F_3) = 2r+1$.

Claim 2: $\chi_{r,s,t}(F_3) = r+s+t+1$, if $s \geq t$ and $\max\{s-t,t\} \leq r < s+t$, or $s < t$ and $t \leq r \leq s+t$.

If $s \geq t$ and $s \leq r < s+t$, then F_3 is colored as follows: $c(x) = r+s+t$, $c(y_1) = c(y_3) = 0$ and $c(y_2) = s+t$; $c(xy_1) = s, c(xy_2) = 0, c(xy_3) = 2s$; $c(y_1y_2) = r+s+t$ and $c(y_2y_3) = s$. If $s \geq t$ and $\max\{s-t,t\} \leq r < s$, then F_3 is colored as follows: $c(x) = r$, $c(y_1) = c(y_3) = 0$ and $c(y_2) = r+s$; $c(xy_1) = r+t, c(xy_2) = 0, c(xy_3) = r+t+s$; $c(y_1y_2) = r+s+t$ and $c(y_2y_3) = s$. If $s < t$ and $t \leq r < s+t$, then F_3 is colored similar to the coloring of Claim 1, and only use the color $r+s+t$ to recolor the vertex x. For each coloring, the three conditions of an $[r,s,t]$-coloring are fulfilled. Therefore $\chi_{r,s,t}(F_3) \leq r+s+t+1$. Moreover, by Lemma 2.2, $\chi_{r,s,t}(F_3) \geq r+s+t+1$. Thus $\chi_{r,s,t}(F_3) = r+s+t+1$.

Claim 3: $\chi_{r,s,t}(F_3) = \max\{s,2t\}+s+1$, if $s \geq t$ and $r < \max\{s-t,t\}$.

If $s \geq t$ and $r < \max\{s-t,t\}$, then F_3 is colored as follows: $c(x) = \max\{s-t,t\}$, $c(y_1) = c(y_3) = 0$ and $c(y_2) = \max\{s-t,t\}+s$; $c(xy_1) = \max\{s,2t\}, c(xy_2) = 0, c(xy_3) = s+\max\{s,2t\}$; $c(y_1y_2) = s+\max\{s,2t\}$ and $c(y_2y_3) = s$. For this coloring, the three conditions of an $[r,s,t]$-coloring are fulfilled. Therefore $\chi_{r,s,t}(F_3) \leq \max\{s,2t\}+s+1$. Moreover, by Lemma 1.3, $\chi_{r,s,t}(F_3) \geq \max\{s,2t\}+s+1$. Thus $\chi_{r,s,t}(F_3) = \max\{s,2t\}+s+1$.

Claim 4: $\chi_{r,s,t}(F_3) \leq s+2t+1$, if $s < t$ and $r < t$.

If $s < t$ and $r < t$, then F_3 is colored as follows: $c(x) = t$, $c(y_1) = c(y_3) = 0$ and $c(y_2) = s+2t$; $c(xy_1) = 2t, c(xy_2) = 0, c(xy_3) = s+2t$; $c(y_1y_2) = t$ and $c(y_2y_3) = s+t$. For this coloring, all the required conditions of an $[r,s,t]$-coloring are satisfied. Thus $\chi_{r,s,t}(F_3) \leq s+2t+1$. □

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