A Note on Edge Coloring of Graphs

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Abstract

Let G be a graph of minimum degree $\delta(G)$. R.P. Gupta proved the two following interesting results: 1) A bipartite graph G has a δ-edge-coloring in which all δ colors appear at each vertex. 2) If G is a simple graph with $\delta(G) > 1$, then G has a $(\delta - 1)$-edge-coloring in which all $(\delta - 1)$ colors appear at each vertex. Let t be a positive integer. In this paper, we extend the first result by showing that for every bipartite graph, there exists a t-edge coloring such that, at each vertex v, $\min\{t, d(v)\}$ colors appear. Also, we show that if G is a graph, then the edges of G can be colored using t colors in which for each vertex v, the number of colors appear at v is at least $\min\{t, d(v) - 1\}$, which generalizes the second result.

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Introduction.

Throughout this paper, all graphs are simple. Let G be a graph and let $E(G)$ and $V(G)$ denote the edge set and vertex set of G, respectively. We denote the minimum and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. For every $v \in V(G)$, $d_G(v)(= d(v))$ denotes the degree of v. A *k-edge-coloring* of a graph G is a mapping $c : E \longrightarrow S$, where S is a set of k colors. An edge coloring of G is called *proper* if the adjacent edges receive distinct colors. The edge chromatic number, $\chi'(G)$, of a graph G is the minimum positive integer k for which G has a proper k-edge- coloring. Consider an edge coloring of G. For a vertex $v \in V(G)$, let $s_G(v)$ $(= s(v))$ denote the number of different colors appearing on the edges incident with v. A celebrated theorem due to Vizing says that for every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ (see Theorem 17.4 of [1]).

R.P. Gupta proved that a bipartite graph G has a (not necessarily proper) $\delta(G)$-edge-coloring in which all $\delta(G)$ colors appear at each vertex (see Exercise 17.1.16 of [1]). The following theorem generalizes this result.

Theorem 1. *Let G be a bipartite graph and let t be a positive integer. Then all edges of G can be colored using t colors such that for each vertex v, $s(v) \geq \min\{t, d(v)\}$.*

Proof. We apply induction on $m = |E(G)|$. For $m = 1$, the assertion is trivial. If $\Delta(G) > t$, then consider a vertex of degree $\Delta(G)$, say u. Let v be one of its neighbors and let e be the edge between u and v. By the induction hypothesis, $H = G\backslash\{e\}$ has an edge coloring with t colors such that $s_H(w) \geq \min\{t, d_H(w)\}$ for each vertex w. Since $\min\{\Delta(G)-1, t\} = t$, by the induction hypothesis, t colors appear at u. If $\min\{t, d(v) - 1\} = t$, then by an arbitrary coloring of e we obtain the result. Now, if $\min\{t, d(v) - 1\} = d(v) - 1$, then we can color e by a color not appearing at v. Obviously, $s(w) \geq \min\{t, d(w)\}$ for each vertex w of G. So assume that $\Delta(G) \leq t$. By Theorem 17.2 of [1], since G is a bipartite graph, $\chi'(G) = \Delta(G)$. Now, by considering a proper $\Delta(G)$-edge-coloring of G we are done. □

We note that this theorem is not true for a non-bipartite graph. Consider the complete graph $G = K_{t+1}$ for even t. Then G has no t-edge-coloring with the desired property (see Exercise 17.1.15 [1]).

Theorem 2. *Let G be a graph with no odd cycle component. Then all edges of G can be colored by 2 colors such that for each vertex v with degree at least 2, $s(v) \geq 2$.*

Proof. Without loss of generality, we may assume that G is a connected graph. First, suppose that G is an Eulerian graph. If G is a cycle, then consecutively color the edges of G by the colors 1 and 2. If G is not a cycle, then it has a vertex v of degree at least 3. It suffices to consider an Eulerian circuit in G with starting vertex v and color its edges alternatively by colors 1 and 2. If G is not Eulerian, then we add a new vertex u and join u to all vertices of odd degree. Then the new graph is an Eulerian graph and we proceed as before. Now, by removing vertex v the proof is complete. □

R.P. Gupta proved that if G is a simple graph with $\delta(G) > 1$, then G has a $(\delta(G) - 1)$-edge-coloring (necessarily improper) in which all $\delta(G) - 1$ colors appear at each vertex (see [1, p.461]). The next theorem generalizes Gupta's result.

Theorem 3. *If G is a graph and t is a positive integer, then all edges of G can be colored using t colors such that for each vertex v, $s(v) \geq \min\{t, d(v) - 1\}$.*

Proof. First, assume that $t \leq \delta(G) - 1$. Obviously, by removing $\delta(G) - t+1$ edges of G, we find a spanning subgraph G' of G with $\delta(G') = t+1$. By Gupta's result, we conclude that there exists a $(\delta(G') - 1)$-edge-coloring of G' such that $s(v) = t$ for every vertex v. Now, we arbitrary color all edges of $E(G) \setminus E(G')$ and in this case we are done. So assume that $t \geq \delta(G)$. There exists a graph H such that G is an induced subgraph of H and $\delta(H) = t + 1$. To see this, consider two copies of G and join the corresponding vertices of degree $\delta(G)$ and repeat this procedure $t+1-\delta(G)$ times. By Gupta's result, there exists a $(\delta(H) - 1)$-edge-coloring of H such that $s(v) = t$ for every $v \in V(H)$. Now, consider the restriction of this coloring to the graph G. In this coloring of G, for every $v \in V(G)$ we have $s(v) \geq t - \max\{t - d(v) + 1, 0\}$, which implies that $s(v) \geq \min\{t, d(v) - 1\}$ and the proof is complete. □

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