Decomposition of graphs into cycles of length seven and single edges

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Abstract

Given graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. Let $\phi_H(n)$ be the smallest number ϕ such that any graph G of order n admits an H-decomposition with at most ϕ parts. Here we study the case when $H = C_7$, that is, the cycle of length 7 and prove that $\phi_{C_7}(n) = \lfloor n^2/4 \rfloor$ for all $n \geq 10$.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph is its *order* and is denoted by $v(G)$. The number of edges is denoted by $e(G)$. The *degree of a vertex* v is the number of edges incident with v and will be denoted by $\deg_G v$ or simply by $\deg v$ if it is clear which graph is being considered. The set of neighbors of v is denoted by $N_G(v)$ or briefly by $N(v)$. For $A \subseteq V(G)$ we denote by $\deg(v, A)$ the number of neighbors that v has in the set A. For $U \subseteq V(G)$, the *induced subgraph* $G[U]$ is the subgraph of G with vertex set U and the edges of G with both endpoints in U. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ defined by $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

Given two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms an H-subgraph, i.e., a graph isomorphic to H. We allow partitions only, that is, every edge of G appears in precisely one part. Let $\phi_H(G)$ be the smallest possible number of parts in an H-decomposition of G.

It is easy to see that, for non-empty H, $\phi_H(G) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint H-subgraphs that can be packed into G. Building upon a body of previous research, Dor and Tarsi [3] showed that if H has a component with at least 3 edges then the problem of checking whether an input graph G is perfectly decomposable into H-subgraphs is NP-complete. Hence, it is NP-hard to compute the function $\phi_H(G)$ for such H.

Here we study the function

$$\phi_H(n) = \max\{\phi_H(G) \mid v(G) = n\},$$

which is the smallest number such that any graph G of order n admits an H-decomposition with at most $\phi_H(n)$ parts. Motivated by the problem of representing graphs by set intersections, Erdös, Goodman and Pósa [4] proved that $\phi_{K_3}(n) = t_2(n)$, where K_r denotes the complete graph (clique) of order r, and $t_r(n)$ is the maximum number of edges in an r-partite graph on n vertices. This result was extended by Bollobás [1], who proved that

$$\phi_{K_r}(n) = t_{r-1}(n), \quad \text{for all } n \geq r \geq 4.$$

In general, for any fixed graph H the exact value of the function $\phi_H(n)$ is still unknown. However, Pikhurko and Sousa [5] determined the asymptotic of $\phi_H(n)$ for any fixed graph H as n tends to infinity. In particular, for a non-bipartite graph H they proved the following.

Theorem 1.1. *Let H be any fixed graph with chromatic number $r \geq 3$. Then,*

$$\phi_H(n) = t_{r-1}(n) + o(n^2).$$

Therefore,

$$\phi_{C_{2t+1}}(n) = \left\lfloor \frac{n^2}{4} \right\rfloor + o(n^2),$$

where C_{2t+1} denotes the odd cycle on $2t+1$ vertices, for $t \geq 1$.

Unfortunately, for $t \geq 4$ the exact value of the function $\phi_{C_{2t+1}}(n)$ is still unknown. The author [6] proved that

$$\phi_{C_5}(n) = \left\lfloor \frac{n^2}{4} \right\rfloor, \quad \text{for all } n \geq 6.$$

Using the same ideas as in [6] we can determine the exact value of the function $\phi_{C_7}(n)$ for all $n \geq 10$. Unfortunately, it seems difficult to extend this method to give us the exact value of the function $\phi_{C_{2t+1}}(n)$ for all $t \geq 4$. We prove the following theorem.

Theorem 1.2.

$$\phi_{C_7}(n) = \left\lfloor \frac{n^2}{4} \right\rfloor, \text{ for all } n \geq 10.$$

The upper bound will be proved in Section 2 and the lower bound follows from the trivial inequality

$$\phi_{C_7}(n) \geq \phi_{C_7}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \left\lfloor \frac{n^2}{4} \right\rfloor,$$

where $K_{t,s}$ denotes the complete bipartite graph with parts of size t and s.

2 Proof of Theorem 1.2

In this section we prove the upper bound of Theorem 1.2. Before presenting the proof we need to state and prove some results that will be needed later. The first observation is that the complete graph on 7 vertices contains 3 edge disjoint C_7's (see Figure 1).

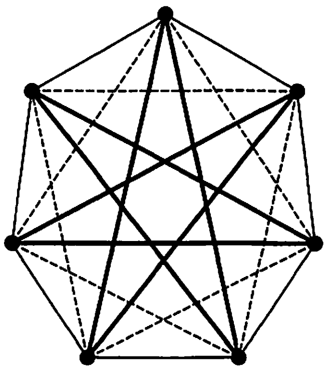

Figure 1: K_7 and the 3 edge disjoint C_7's.

Recall that the *Turán function*, denoted by $\text{ex}(n, H)$, is the maximum number of edges that a graph on n vertices can have without containing H as a subgraph.

The following result was obtained by Yang Yuansheng using the same computer algorithm as in [7].

Lemma 2.3. $\text{ex}(10, C_7) = 25$ *and the only graphs with 10 vertices, 25 edges and no copy of* C_7 *are the complete bipartite graph* $K_{5,5}$ *and a* K_5 *plus a* K_6 *sharing a vertex, denoted by* $K_5 \bullet K_6$.

Lemma 2.4. $\phi_{C_7}(10) = 25$.

Proof. The lower bound follows from $\phi_{C_7}(10) \geq \text{ex}(10, C_7) = 25$. We will now prove the upper bound. Let G be a graph with 10 vertices. Our aim is to prove that $\phi_{C_7}(G) \leq 25$. We have to consider a few cases.

If $e(G) \leq 25$ then it suffices to decompose G into single edges.

Assume $26 \leq e(G) \leq 43$. Suppose first that $e(G) \neq 32, 38, 39$. The upper bound follows since we can greedily remove copies of C_7 and then remove the remaining edges. Suppose $e(G) = 32$ (resp. $e(G) = 39$) and suppose that G contains exactly one C_7 (resp. two C_7's). Let G^* be the graph obtained from G after deleting the edges of the C_7('s). Then, $e(G^*) = 25$ and G^* contains no C_7. By Lemma 2.3 G^* is either $K_{5,5}$ or $K_5 \bullet K_6$. Therefore, the complement of G^* must contain a C_7, which is a contradiction since the complement of G^* is either $K_{4,5}$ or 2 vertex disjoint K_5's. Therefore, G contains at least two (resp. three) edge-disjoint C_7's and the result follows.

Consider the case $e(G) = 38$. It suffices to find 3 edge disjoint C_7's in G. This is true if G contains a K_7 (see Figure 1). Since $e(G) \geq t_4(10) = 37$ it follows that G contains a K_5. We now have to consider two cases.

Case 1: G contains a K_6 and no K_7.

Let $V(K_6) = \{1, 2, 3, 4, 5, 6\}$ and $A = V(G) - V(K_6)$. Observe that $\deg(y, V(K_6)) \leq 5$ for all $y \in A$, since G contains no K_7. Then, $e(G[A]) \geq 3$. Suppose first that $e(G[A]) = 3$, then $\deg(y, V(K_6)) = 5$ for all $y \in A$. Let y_1 and y_2 be adjacent vertices in $G[A]$ and suppose that y_1 is adjacent to $1, 2, 3, 4, 5$. Then,

$$y_1, 2, 1, 6, 5, 4, 3, y_1 \quad \text{and} \quad y_1, 1, 3, 5, 2, 6, 4, y_1$$

form two edge disjoint C_7's. If the vertex y_2 is adjacent to 3 we have $y_1, 5, 1, 4, 2, 3, y_2, y_1$, otherwise we have $y_2, 5, 1, 4, 2, 3, 6, y_2$. We have found 3 edge disjoint C_7's as wanted.

Assume that $e(G[A]) = 4$. Then, there are $y_1, y_2, y_3 \in A$ such that $\deg(y_i, V(K_6)) = 5$ for all $i = 1, 2, 3$. Without loss of generality assume y_1 and y_2 are adjacent in $G[A]$ and the result holds as before.

Finally, suppose $e(G[A]) \geq 5$. Then, there are $y_1, y_2 \in A$ such that y_1 is adjacent to y_2, $\deg(y_1, V(K_6)) = 5$ and $\deg(y_2, V(K_6)) \geq 4$. In this case $G[A]$ is either a K_4 or a K_4 minus one edge and since y_1 is adjacent to y_2, it follows that the edge y_1y_2 belongs to a C_4 in $G[A]$. Let y_1 be adjacent to $1, 2, 3, 4, 5$. Since y_1 and y_2 must have at least 3 common neighbors in K_6, we can assume, without loss of generality, that y_2 is adjacent to vertices $1, 2, 3$ of K_6. Then, Figure 2 shows that G contains 3 edge disjoint C_7's as required.

Case 2: G contains a K_5 and no K_6.

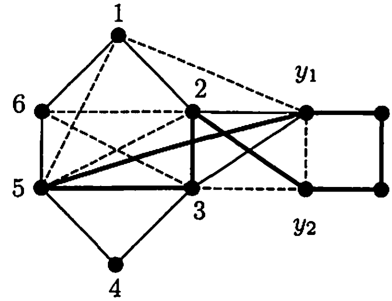

Figure 2: $e(G[A]) \geq 5$.

Let $V(K_5) = \{1,2,3,4,5\}$ and let $A = V(G) - V(K_5)$. Observe that $\deg(y, V(K_5)) \leq 4$ for all $y \in A$, since G contains no K_6. Therefore, $e(G[A]) \geq 8$. Suppose first that $e(G[A]) = 8$, then $\deg(y, V(K_5)) = 4$ for all $y \in A$ and there are only two possible graphs $G[A]$. Let y_1 and y_2 be adjacent vertices in $G[A]$ such that $\deg_{G[A]}(y_1) = 4$ and $\deg_{G[A]}(y_2) = 3$. Without loss of generality let y_1 be adjacent to 1, 2, 3, 4. If y_1 and y_2 have at least 3 common neighbors in $V(K_5)$, say 1,2,3, then Figure 3 shows that G contains 3 edge disjoint C_7's for the two possible graphs $G[A]$. Otherwise, y_2 is adjacent to 1, 2, 4, 5 or to 1, 3, 4, 5. Suppose the first case holds, the second follows by symmetry. Then, Figure 3 holds with $y_1, 2, 1, 4, 5, 3, y_2, y_1$ replaced by $y_1, 2, 1, 4, 3, 5, y_2, y_1$.

If $e(G[A]) = 9$, then there are vertices $y_1, y_2, y_3, y_4 \in A$ such that $\deg(y_i, V(K_5)) = 4$ for $i = 1,2,3,4$. A similar case analysis shows that the results obtained in Figure 3 also hold. Let $e(G[A]) = 10$. Thus, there exist $y_1, y_2 \in A$ such that $\deg(y_i, V(K_5)) = 4$ for $i = 1,2$ and we are done as before.

To finish the proof suppose $e(G) = 44$ or $e(G) = 45$. Then, we can easily find 4 edge disjoint C_7's in G. Let $V(G) = \{v, y, v_1, v_2, v_3, v_4, x_1, x_2, x_3, x_4\}$ and without loss of generality we can suppose that the edge $\{v, y\}$ is not present if $G \neq K_{10}$. Then, for $i = 1,2,3,4$ with indices taken cyclically,

$$v, v_i, x_i, y, v_{i+1}, x_{i+2}, x_{i+3}, v$$

are 4 edges edge disjoint C_7's in G. □

We are now able to prove the upper bound in Theorem 1.2.

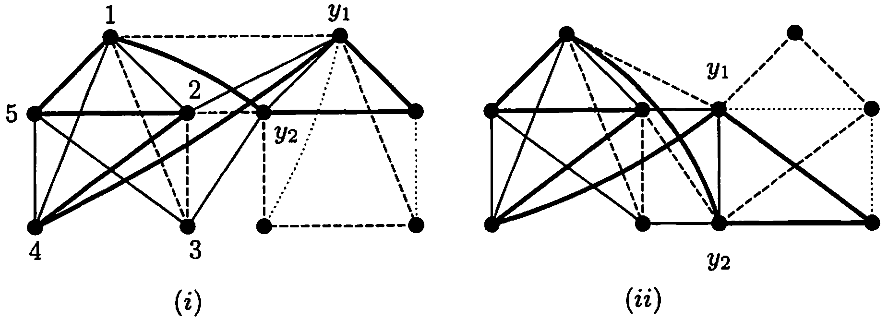

Figure 3: $e(G[A]) = 8$.

Proof of the upper bound in Theorem 1.2. By induction on the number of vertices. Lemma 2.4 proves the result for $n = 10$. Assume that it is true for all graphs of order $n-1$ and note that for any positive integer n

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .$$

Let G be a graph of order $n \geq 11$. Let v be a vertex of minimum degree, say $\deg v = d + m$ where $d = \lfloor \frac{n}{2} \rfloor$ and m is an integer. If $m \leq 0$ then going from $G - v$ to G we only need to use the edges joining v to the other vertices of G and there are at most $\lfloor \frac{n}{2} \rfloor$ of these, so the induction hypothesis implies the result.

Let $m \geq 1$. If there are m edge disjoint C_7's containing v, then the $d+m$ edges incident with v can be decomposed into at most $m + (d + m - 2m) = d$ edge disjoint C_7's and single edges and the result follows by induction. To complete the proof, it remains to show that we can always find m edge disjoint C_7's containing v.

Assume first that G is not the complete graph. Recall that $\deg(y, X)$ denotes the number of neighbors that y has in the set X. Let $x \in N(v)$ and $y \in \overline{N}(v)$, where $\overline{N}(v) := V(G) - (N(v) \cup \{v\})$. We have

$$\deg(x, N(v)) \geq 2m - 1, \tag{2.1}$$

$$\deg(y, N(v)) \geq 2m + 1. \tag{2.2}$$

Let $x_1, \ldots x_m \in N(y) \cap N(v)$, $X = \{x_1, \ldots x_m\}$ and $Y = N(v) - X$. Consider the bipartite graph $G[X, Y]$ with bipartition (X, Y) and all the edges of G between X and Y. Using (2.1) it is easy to see that $G[X, Y]$ has an X-perfect matching, say $M = \{x_i, v_i\}_{i=1,\ldots,m}$.

We first consider the case when $\overline{N}(v)$ contains another element different from y, call it y'. Observe that $\delta(G) \geq d + m$ easily implies the following claim.

Claim 1. *Let $y, y' \in \overline{N}(v)$. Then, y and y' have at least $2m$ common neighbors if they are adjacent and at least $2m + 2$ otherwise.*

Without loss of generality we assume that $x_{j_1}, \dots, x_{j_t}$ and $v_1, \dots, v_\ell$ are common neighbors of y and y', where t and ℓ integers between 0 and m.

Let $a_{\ell+1}, \dots, a_m$ be elements in $(N(y) \cap N(y')) - \{x_{j_1}, \dots, x_{j_t}, v_1, \dots, v_\ell\}$, which exist in view of Claim 1 and the fact that $m \geq t$. Let $w_{j_{t+1}}, \dots, w_{j_m} \in (N(y') \cap N(v)) - \{x_{j_1}, \dots, x_{j_t}, v_1, \dots, v_\ell, a_{\ell+1}, \dots, a_m\}$, which exist in view of (2.2). For $i \in \{1, \dots, m\}$, we define $w_i = x_i$ whenever y' is adjacent to x_i, and for the sake of simplicity we relabel the vertices $w_{j_{t+1}}, \dots, w_{j_m}$ so that we have a set of vertices $w_1, \dots w_m$. For $1 \leq j \leq \ell$ we set $a_j := v_j$.

Finally, for $1 \leq i \leq m$, with indices taken cyclically,

$$v, v_i, x_i, y, a_{i+1}, y', w_{i+1}, v$$

are m edge disjoint C_7's, if $m \geq 2$.

Let $m = 1$. Then $|\overline{N}(v)| \geq 3$. Assume first that y and y' are non-adjacent vertices. By Claim 1 there are $a_1, a_2 \in N(y') \cap N(y) - \{v_1, x_1\}$. If there is $w \in N(y') \cap N(v) - \{x_1, v_1, a_1\}$ then $v, v_1, x_1, y, a_1, y', w, v$ is a C_7. Otherwise, $N(y') \cap N(v) = \{x_1, v_1, a_1\}$, $a_1 \in N(v)$ and we have $v, v_1, x_1, y, a_2, y', a_1, v$. Now assume that all vertices in $\overline{N}(v)$ are pairwise adjacent. Let $y, y', y'' \in \overline{N}(v)$ and $w \in N(y'') \cap N(v) - \{x_1, v_1\}$, then $v, v_1, x_1, y, y', y'', w, v$ is a C_7.

Suppose that y is the only element in $\overline{N}(v)$. Then, y is adjacent to all vertices in $N(v)$ and $m \geq 4$. Let z be an element in $N(v) - \{x_1, \dots, x_m, v_1, \dots, v_m\}$. Since $\delta(G) = n - 2$ it follows that z must have at least $2m - 1$ neighbors in $\{x_1, \dots, x_m, v_1, \dots, v_m\}$, so without loss of generality we assume that z is adjacent to every vertex in $\{x_1, \dots, x_m, v_1, \dots, v_m\}$ except perhaps x_m. Again because of the minimum degree constraint there is $a \in (N(x_m) \cap N(v_m)) - \{v, v_1, z, y\}$. Therefore, for $1 \leq i \leq m - 2$,

$$v, v_i, x_i, y, v_{i+1}, z, x_{i+1}, v$$

are $m - 2$ edge disjoint C_7's, that together with

$$v, v_{m-1}, x_{m-1}, y, v_m, z, x_1, v$$

$$v, v_m, a, x_m, y, v_1, z, v$$

form m edge disjoint C_7's.

To conclude the proof of the theorem it remains to consider the case $G = K_n$. Recall that our goal it to find m edge disjoint C_7's incident with v.

Let n be even and $v, y, x_1, \ldots, x_m, v_1, \ldots, v_m$ be the vertices of G. Then, for $i = 1, \ldots, m$, with indices taken cyclically,

$$v, v_i, x_i, y, v_{i+1}, x_{i+2}, x_{i+3}, v,$$

are m edges disjoint C_7's since $m \geq 5$.

Let n be odd and $v, y, x_1, \ldots, x_m, v_1, \ldots, v_{m-1}$ be the vertices of G. Consider first the case $m \geq 6$, that is $n \geq 13$. Then, for $i = 1, \ldots, m-2$, with indices taken cyclically,

$$v, v_i, x_i, y, v_{i+1}, x_{i+2}, x_{i+3}, v,$$

together with

$$v, v_{m-1}, v_{m-2}, v_{m-3}, x_{m-1}, y, x_m, v$$
$$v, x_{m-1}, x_{m-4}, v_{m-3}, v_{m-4}, v_{m-1}, y, v$$

are m edge disjoint C_7's.

If $n = 11$ then $m = 5$ and

$$v, v_1, x_1, y, v_2, x_3, x_4, v$$
$$v, v_2, x_2, y, v_3, x_4, x_5, v$$
$$v, v_3, x_3, y, v_4, x_5, x_1, v$$
$$v, v_4, x_4, y, v_1, v_3, x_2, v$$
$$v, x_3, v_4, v_3, v_2, x_5, y, v$$

are 5 edge disjoint C_7's. □

Remark: Let $K_p \bullet K_t$ denote a K_p plus a K_p sharing a vertex. For $n = 7, 8, 9$ the graphs $K_6 \bullet K_2$, $K_6 \bullet K_3$ and $K_6 \bullet K_4$ show that $n = 2, 10$ are the smallest values of n for which Theorem 1.2 holds.

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