On the sum of out-domination number and in-domination number of digraphs *

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Abstract: A vertex subset S of a digraph $D = (V, A)$ is called an out-dominating (resp., in-dominating) set of D if every vertex in $V - S$ is adjacent from (resp., to) some vertex in S. The out-domination (resp., in-domination) number of D, denoted by $\gamma^+(D)$ (resp., $\gamma^-(D)$), is the minimum cardinality of an out-dominating (resp., in-dominating) set of D. In 1999, Chartrand et al. proved that $\gamma^+(D) + \gamma^-(D) \leq 4n/3$ for every digraph D of order n with no isolated vertices. In this paper, we determine the values of $\gamma^+(D) + \gamma^-(D)$ for rooted trees and connected contrafunctional digraphs D, based on which we show that $\gamma^+(D)+\gamma^-(D) \leq (2k+2)n/(2k+1)$ for every digraph D of order n with minimum out-degree or in-degree no less than 1, where $2k + 1$ is the length of a shortest odd directed cycle in D. Our result partially improves the result of Chartrand et al. In particular, if D contains no odd directed cycles, then $\gamma^+(D) + \gamma^-(D) \leq n$.

Keywords: Out-domination number; In-domination number; Rooted tree; Contrafunctional digraph

1 Introduction and notations

Now domination has become one of the major areas in graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems, such as facility location problems [5]. Various types of domination problems in undirected graphs, such as total domination [6], k-tuple domination [9], connected domination [3], perfect domination [8], and rainbow domination [12] have been widely studied. The concept of domination in undirected graphs is naturally extended to directed graphs (digraphs). In fact, domination in digraphs comes up more naturally in modeling

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real world problems. Compared to undirected graphs, domination in digraphs has not yet gained the same amount of attention, although it has several useful applications as well. For example, domination in digraphs has been used in the study of the routing problems in networks [13] and answering skyline query in database [7]. Domination in digraphs is well studied [1, 2, 10, 11].

Throughout this paper, $D = (V, A)$ denotes a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed). For two vertices $u, v \in V(D)$, we use uv to denote the arc with direction from u to v, that is, u is adjacent to v, or equivalently, v is adjacent from u. For $u \in V(D)$, the *out-neighborhood* and *in-neighborhood* of u are $N_D^+(u) = \{v \in V(D) : uv \in A(D)\}$ and $N_D^-(u) = \{v \in V(D) : vu \in A(D)\}$, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of u are $N_D^+[u] = \{u\} \cup N_D^+(u)$ and $N_D^-[u] = \{u\} \cup N_D^-(u)$, respectively. The *out-degree* and *in-degree* of a vertex u of D are defined by $d_D^+(u) = |N_D^+(u)|$ and $d_D^-(u) = |N_D^-(u)|$, respectively. The *minimum out-degree* and the *minimum in-degree* among the vertices of D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. For two vertices u and v of D, the *distance* $d_D(u,v)$ *from* u *to* v is the length of a shortest u-v directed path in D. If D contains no u-v directed path, then $d_D(u,v) = \infty$. For a subdigraph H of D and $u \in V(D)$, the *distance from* H *to* u in D is $d_D(H,u) = \min\{d_D(v,u) : v \in V(H)\}$.

Given two vertices u and v of D, we say u *out-dominates* v (or v *in-dominates* u) if $u = v$ or $uv \in A(D)$. A subset S of $V(D)$ is called an *out-dominating* (resp., *in-dominating*) *set*, abbreviated as OD-set (resp., ID-set), of D if every vertex in $V(D) - S$ is is out-dominated (resp., in-dominated) by at least a vertex in S. The *out-domination* (resp., *in-domination*) *number* of a digraph D, denoted by $\gamma^+(D)$ (resp., $\gamma^-(D)$), is the minimum cardinality of an OD-set (resp., ID-set) of D. An OD-set (resp., ID-set) of D of cardinality $\gamma^+(D)$ (resp., $\gamma^-(D)$) is called a $\gamma^+(D)$*-set* (resp., $\gamma^-(D)$*-set*). Clearly a vertex with in-degree (resp., out-degree) 0 belongs to every OD-set (resp., ID-set).

If the underlying graph of a digraph D is connected, then we say that D is *connected.* A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. A digraph D is *contrafunctional* if each vertex of D has in-degree 1. The *converse* D' of a digraph D is the digraph obtained from D by reversing the orientation of each arc of D. Clearly $\gamma^+(D) = \gamma^-(D')$ and $\gamma^-(D) = \gamma^+(D')$. For two subdigraphs D_1 and D_2 of a digraph D with $V(D_1) \cup V(D_2) = V(D)$, it is easy to see that $\gamma^+(D) \le \gamma^+(D_1) + \gamma^+(D_2)$ and $\gamma^-(D) \le \gamma^-(D_1) + \gamma^-(D_2)$.

2 Main results

We begin with the following two lemmas, which are the key points for our further discussion.

Lemma 2.1. *Let y be a vertex of a digraph D such that $d_D^-(y) = 1, d_D^+(y) \ge 1$ and $d_D^-(x) = 1, d_D^+(x) = 0$ for each $x \in N_D^+(y)$ (see Figure 1). Then*

(1) *There exists a $\gamma^+(D)$-set S^+ such that $y \in S^+$;*
(2) *There exists a $\gamma^-(D)$-set S^- such that $y \notin S^-$.*

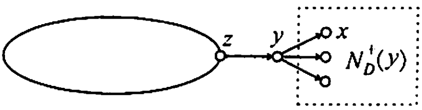

Figure 1: A digraph D and its local structure at a vertex y.

Proof. (1) Let S^+ be a $\gamma^+(D)$-set. If $y \notin S^+$, then $N_D^+(y) \subseteq S^+$ since y is the unique vertex adjacent to each vertex in $N_D^+(y)$. Note that y out-dominates each vertex in $N_D^+(y)$. Therefore, $(S^+ - N_D^+(y)) \cup \{y\}$ is an OD-set of D. Moreover, $|(S^+ - N_D^+(y)) \cup \{y\}| \leq |S^+|$. This implies that $(S^+ - N_D^+(y)) \cup \{y\}$ is also a $\gamma^+(D)$-set.

(2) Let S^- be a $\gamma^-(D)$-set. Clearly, $N_D^+(y) \subseteq S^-$ since each vertex in $N_D^+(y)$ has out-degree 0. Therefore, if $y \in S^-$, then $(S^- - \{y\}) \cup \{z\}$ is an ID-set of D since each vertex in $N_D^+(y)$ in-dominates y, where z is the unique vertex adjacent to y. Moreover, $|(S^- - \{y\}) \cup \{z\}| \leq |S^-|$. This implies that $(S^- - \{y\}) \cup \{z\}$ is also a $\gamma^-(D)$-set. □

Lemma 2.2. *Let D and y be defined as in Lemma 2.1. Then*
(1) *If S^+ is a $\gamma^+(D)$-set with $y \in S^+$, then $S^+ - \{y\}$ is a $\gamma^+(D_y)$-set, where and herein after, $D_y = D - N_D^+[y]$;*
(2) *If S^- is a $\gamma^-(D)$-set with $y \notin S^-$, then $S^- - N_D^+(y)$ is a $\gamma^-(D_y)$-set.*

Proof. (1) Let S^+ be a $\gamma^+(D)$-set with $y \in S^+$. Since S^+ is minimum and $y \in S^+$, $N_D^+(y) \cap S^+ = \emptyset$. Moreover, $S^+ - \{y\}$ is an OD-set of D_y since y out-dominates no vertex in D_y. Therefore, $|S^+(D_y)| \leq |S^+| - 1$, where $S^+(D_y)$ is a $\gamma^+(D_y)$-set. On the other hand, it is easy to see that $S^+(D_y) \cup \{y\}$ is an OD-set of D and hence $|S^+| \leq |S^+(D_y)| + 1$. As a result, we have $|S^+(D_y)| = |S^+| - 1$. Thus, if S^+ is a $\gamma^+(D)$-set with $y \in S^+$, then $S^+ - \{y\}$ is a $\gamma^+(D_y)$-set.

(2) Let S^- be a $\gamma^-(D)$-set with $y \notin S^-$. Since each vertex in $N_D^+(y)$ has out-degree 0, $N_D^+(y) \subseteq S^-$. Moreover, $S^- - N_D^+(y)$ is an ID-set of D_y since each vertex in $N_D^+(y)$ in-dominates no vertex in D_y. Therefore, $|S^-(D_y)| \leq |S^-| - |N_D^+(y)|$, where $S^-(D_y)$ is a $\gamma^-(D_y)$-set. On the other hand, it is easy to see that $S^-(D_y) \cup N_D^+(y)$ is an ID-set of D and hence $|S^-| \leq |S^-(D_y)| + |N_D^+(y)|$. As a result, we have $|S^-(D_y)| = |S^-| - |N_D^+(y)|$. Thus, if S^- is a $\gamma^-(D)$-set with $y \notin S^-$, then $S^- - N_D^+(y)$ is a $\gamma^-(D_y)$-set. □

We now consider a rooted tree T of order n with root r. Let x be a vertex of T such that the distance from r to x, i.e., $d_T(r, x)$, is maximum, and let y be the unique vertex adjacent to x. We note that $T_y = T - N_T^+[y]$ is still a rooted tree with root r or empty. This means that we can take a sequence of such operations on T until the resulting subdigraph is empty or the isolated vertex r. If the former happens, then we call T the type I. Otherwise, we call T the type II.

Theorem 2.3. *Let T be a rooted tree of order n. Then*

$$\gamma^+(T) + \gamma^-(T) = \begin{cases} n, & \text{if } T \text{ is of type I}, \\ n+1, & \text{if } T \text{ is of type II}. \end{cases}$$

Proof. We prove by induction on n. If $n = 1$, then T is an isolated vertex and therefore, is of type II. On the other hand, $\gamma^+(T) + \gamma^-(T) = 2 = n + 1$. If $n = 2$, then T is a directed path of order two and therefore, is of type I. On the other hand, $\gamma^+(T) + \gamma^-(T) = 2 = n$. Hence we may assume that $n \geq 3$. Let x be a vertex of T such that $d_T(r, x)$ is maximum, where r is the root of T, and let y be the unique vertex adjacent to x.

If $d_T(r, x) = 1$, then T is of type I. On the other hand, it is easy to verify that $\gamma^+(T) + \gamma^-(T) = |\{r\}| + |N_T^+(r)| = n$. We now assume that $d_T(r, x) \geq 2$. Let S^+ and S^- be a $\gamma^+(T)$-set and a $\gamma^-(T)$-set, respectively. Note that y satisfies the two conditions in Lemma 2.1. So by Lemma 2.1, we may choose S^+ and S^- to be such that $y \in S^+$ and $y \notin S^-$. Then by Lemma 2.2, $S^+ - \{y\}$ and $S^- - N_T^+(y)$ are a $\gamma^+(T_y)$-set and a $\gamma^-(T_y)$-set, respectively. On the other hand, we notice that T is of type i ($i \in \{\text{I}, \text{II}\}$) if and only if T_y is of type i. Thus, by the induction hypothesis, if T is of type I, then

$$\gamma^+(T) + \gamma^-(T) = |S^+| + |S^-|$$

$$= |S^+ - \{y\}| + |\{y\}| + |S^- - N_T^+(y)| + |N_T^+(y)|$$

$$= \gamma^+(T_y) + \gamma^-(T_y) + |N_T^+(y)| + 1 = |V(T_y)| + |N_T^+(y)| + 1 = n.$$

The discussion for the case when T is of type II is analogous, which completes our proof. □

In [4], Harary et al. showed that every connected contrafunctional digraph has a unique directed cycle and the removal of any one arc of the directed cycle results in a rooted tree. The simplest connected contrafunctional digraph of order n is the *directed cycle* $\overrightarrow{C_n}$ of length n, for which we have the following result.

$$\gamma^+(\overrightarrow{C_n}) + \gamma^-(\overrightarrow{C_n}) = \begin{cases} n, & \text{if } n \text{ is even}, \\ n+1, & \text{if } n \text{ is odd}. \end{cases} \quad (1)$$

Let D be a connected contrafunctional digraph. We define the *height* of D, denoted by $h(D)$, to be the maximum distance from its unique directed cycle C to all vertices of D, i.e., $h(D) = \max\{d_D(C, v) : v \in V(D)\}$. In particular, the height of a directed cycle is exactly equal to 0.

Lemma 2.4. *Let D be a connected contrafunctional digraph of order n. If $h(D) = 1$, then*

$$\gamma^+(D) + \gamma^-(D) = n.$$

Proof. Without loss of generality, we may assume that C is the unique directed cycle of D, $u, v \in V(C)$, $x \notin V(C)$ and $vu, ux \in A(D)$. Let $T = D - vu$. It is clear that T is a rooted tree with the root u of order n [4]. Further, one can verify that T is a rooted tree of type I. So by Theorem 2.3, $\gamma^+(T) + \gamma^-(T) = |V(T)| = n$. Since T is a spanning subdigraph of D, we have

$$\gamma^+(D) + \gamma^-(D) \leq \gamma^+(T) + \gamma^-(T) = n.$$

We now prove that $\gamma^+(D) + \gamma^-(D) \geq n$. Let S^+ and S^- be a $\gamma^+(D)$-set and a $\gamma^-(D)$-set, respectively, and let

$$S = V(D) - (S^+ \cup S^-).$$

The assertion holds directly if S is empty. Now suppose that $S = \{u_i : 1 \leq i \leq s\}$, where $s \geq 1$. Since $u_i \notin S^-$ for each $i \in \{1, 2, \cdots, s\}$, there must be at least one vertex $v_i \in N_D^+(u_i)$ such that v_i in-dominates u_i, that is, $v_i \in S^-$. On the other hand, since $u_i \notin S^+$ and u_i is the unique vertex adjacent to the vertices in $N_D^+(u_i)$ for each $i \in \{1, 2, \cdots, s\}$, we must have $v_i \in N_D^+(u_i) \subseteq S^+$. As a result, $v_i \in S^+ \cap S^-$ for each $i \in \{1, 2, \cdots, s\}$. Recalling that each vertex in D has in-degree 1, we have $v_i \neq v_j$ for $1 \leq i < j \leq s$ (for otherwise, both u_i and u_j are adjacent to v_i, a contradiction). The above argument means that $|S| \leq |S^+ \cap S^-|$. Thus,

$$\gamma^+(D) + \gamma^-(D) = |S^+| + |S^-| = |S^+ \cup S^-| + |S^+ \cap S^-|$$
$$= |V(D)| - |S| + |S^+ \cap S^-| \geq n,$$

which completes the proof. □

We now consider a connected contrafunctional digraph D with its unique directed cycle $\overrightarrow{C_k}$. Let x be a vertex of D such that $d_D(\overrightarrow{C_k}, x) = h(D) \geq 2$ and let y be the unique vertex adjacent to x. We note that D_y is still a connected contrafunctional subdigraph with the unique directed cycle $\overrightarrow{C_k}$. Similar to the discussion for a rooted tree, we can take a sequence of such operations on D until the resulting subdigraph is the directed cycle $\overrightarrow{C_k}$ or a connected contrafunctional digraph of height 1. If the former happens and k is odd, then we call D the type II. Otherwise, we call D the type I.

Theorem 2.5. *Let D be a connected contrafunctional digraph of order n. Then*

$$\gamma^+(D) + \gamma^-(D) = \begin{cases} n, & \text{if } D \text{ is of type I}, \\ n+1, & \text{if } D \text{ is of type II}. \end{cases}$$

Proof. Let $\overrightarrow{C_k}$ be the unique directed cycle of D. We fix k and proceed by induction on n. If $n = k$ is even (resp., odd), then D is of type I (resp., type II). On the other hand, by (1), we have $\gamma^+(D) + \gamma^-(D) = n$ (resp., $n+1$). Hence we may assume that $n \geq k+1$. Let $x \in V(D) - V(\overrightarrow{C_k})$ such that $d_D(\overrightarrow{C_k}, x) = h(D)$ and let y be the unique vertex adjacent to x.

If $d_D(\overrightarrow{C_k}, x) = 1$, then D is of type I. On the other hand, by Lemma 2.4, $\gamma^+(D) + \gamma^-(D) = n$. We now assume that $d_D(\overrightarrow{C_k}, x) \geq 2$. Let S^+ and S^- be a $\gamma^+(D)$-set and a $\gamma^-(D)$-set, respectively. Noticing that y satisfies the two conditions in Lemma 2.1, we may choose S^+ and S^- to be such that $y \in S^+$ and $y \notin S^-$. Then by Lemma 2.2, $S^+ - \{y\}$ and $S^- - N_D^+(y)$ are a $\gamma^+(D_y)$-set and a $\gamma^-(D_y)$-set, respectively. On the other hand, we notice that D is of type i ($i \in \{\text{I}, \text{II}\}$) if and only if D_y is of type i. Thus, by the induction hypothesis, if D is of type I, then

$$\begin{aligned}\gamma^+(D) + \gamma^-(D) &= |S^+| + |S^-| \\ &= |S^+ - \{y\}| + |\{y\}| + |S^- - N_D^+(y)| + |N_D^+(y)| \\ &= \gamma^+(D_y) + \gamma^-(D_y) + |N_D^+(y)| + 1 = |V(D_y)| + |N_D^+(y)| + 1 = n.\end{aligned}$$

The discussion for the case when D is of type II is analogous, which completes our proof. □

Using Theorem 2.5, we can derive the following result.

Theorem 2.6. *Let D be a digraph of order n with $\delta^+(D) \geq 1$ or $\delta^-(D) \geq 1$. Then*

$$\gamma^+(D) + \gamma^-(D) \leq \frac{2k+2}{2k+1}n,$$

where $2k+1$ is the length of a shortest odd directed cycle in D. In particular, if D contains no odd directed cycles, then

$$\gamma^+(D) + \gamma^-(D) \leq n.$$

Proof. Assume $\delta^-(D) \geq 1$. We choose an arbitrary incoming arc of x for each vertex $x \in V(D)$. Then all such arcs induce a spanning subdigraph H of D consisting of some connected components, say $H_1, H_2, \cdots, H_l$. Moreover, H_i ($i \in \{1, 2, \cdots, l\}$) is a connected contrafunctional digraph since each vertex in H_i has in-degree 1. Note that the length of a shortest odd directed cycle in D is $2k+1$. Hence if $|V(H_i)| \leq 2k$ for some $i \in \{1, 2, \cdots, l\}$, then the length of the unique directed cycle of H_i is even and hence H_i is of type I. So by Theorem 2.5,

$$\gamma^+(H_i) + \gamma^-(H_i) = |V(H_i)| < \frac{2k+2}{2k+1}|V(H_i)|.$$

If $|V(H_i)| \geq 2k+1$ for some $i \in \{1, 2, \cdots, l\}$, then again by Theorem 2.5, we have

$$\gamma^+(H_i) + \gamma^-(H_i) \leq |V(H_i)| + 1 \leq \frac{2k+2}{2k+1}|V(H_i)|.$$

Therefore,

$$\begin{aligned}\gamma^+(D) + \gamma^-(D) \leq \gamma^+(H) + \gamma^-(H) &= \sum_{i=1}^{l} [\gamma^+(H_i) + \gamma^-(H_i)] \\ &\leq \frac{2k+2}{2k+1}\sum_{i=1}^{l} |V(H_i)| = \frac{2k+2}{2k+1}n.\end{aligned}$$

In particular, if D contains no odd directed cycles, then for each $i \in \{1, 2, \cdots, l\}$, the length of the unique directed cycle of H_i is even and hence H_i is a connected contrafunctional digraph of type I. So by Theorem 2.5, we have $\gamma^+(H_i) + \gamma^-(H_i) = |V(H_i)|$ for each $i \in \{1, 2, \cdots, l\}$. Therefore,

$$\gamma^+(D) + \gamma^-(D) \leq \sum_{i=1}^{l} [\gamma^+(H_i) + \gamma^-(H_i)] = \sum_{i=1}^{l} |V(H_i)| = n.$$

The discussion for the case when $\delta^+(D) \geq 1$ is analogous, which completes our proof. □

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