FULL FRIENDLY INDEX SETS OF SPIDERS

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ABSTRACT. We determine the full friendly index sets of spiders and disprove a conjecture by Lee and Salehi [4] that the friendly index set of a tree forms an arithmetic progression.

1. INTRODUCTION

Let G be a finite graph and A be an abelian group. A vertex labelling F is a function $F : V(G) \longrightarrow A$. The induced edge labelling $f : E(G) \longrightarrow A$ is defined by $f(uv) = F(u) + F(v)$ for each $uv \in E(G)$. A vertex labelling F is called *friendly* if $|F^{-1}(a)|$ and $|F^{-1}(b)|$ differ by at most one for all $a, b \in A$; i.e. the vertex classes are all about the same size. A friendly labelling F is called *A-cordial* if $|f^{-1}(a)|$ and $|f^{-1}(b)|$ differ by at most one for all $a, b \in A$; i.e. the edge classes are all about the same size. A graph that admits an A-cordial labelling is called *A-cordial*, or simply *k-cordial* if $A = \mathbb{Z}_k$.

Cordial labelling was first introduced as a weakened version of graceful labelling by Cahit [1], who showed that all trees are 2-cordial. Since then cordial labellings have been extensively studied. In particular, Hovey [2] showed that all trees are k-cordial for $k = 3, 4, 5$ and for each k provided a finite test whose passing shows that all trees are k-cordial. Moreover, Hovey proved that a tree on n vertices is k-cordial for $k \geq 2(n-1)$ and conjectured that any tree is k-cordial for all k.

When $A = \mathbb{Z}_2$, Lee and Salehi [4] generalize the concept of cordiality to friendly index sets. Each friendly labelling gives rise to an *equitable bipartition* of $V(G)$, that is, a bipartition into $V_0 \cup V_1$ such that $||V_0| - |V_1|| \leq 1$. The *full friendly index set* $FFI(G)$ of G is the set of differences $e(V_0, V_1) - (e(V_0) + e(V_1))$ over all equitable bipartitions $V_0 \cup V_1$ of G, whereas the *friendly index set* $FI(G)$ is the set of absolute value of the differences. Thus, G is 2-cordial if 0 or 1 is in $FI(G)$.

Equivalently, we shall work in terms of cutsize, namely, let

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$$C(G) = \{e(V_0, V_1) \mid V_0 \cup V_1 \text{ is an equitable bipartition of } G\}.$$

Then $FFI(G) = 2C(G) - e(G)$. We say that s is *attainable* if there is an equitable bipartition $V_0 \cup V_1$ with $e(V_0, V_1) = s$, i.e. $s \in C(G)$. While it is easy to get a cut of any size in a tree, there need not be an equitable cut of every size.

As an example, we consider the simplest tree. See Lee and Salehi [4].

Proposition 1. $C(P_n) = [n-1]$.

Proof. Let $E(P_n) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$. It is clear that $1, n-1 \in C(P_n)$, so that we need only to show that $2 \le k \le n-2$ is attainable.

We first prove by induction that $C(P_n) = [n-1]$ for even n. There is nothing to prove for $n = 2$. Consider P_{n+2}. For each $2 \le k \le n$, we extend the equitable bipartition of P_n attaining $k-1$ by putting v_{n+1} in the same class as v_n and v_{n+2} to an equitable bipartition of P_{n+2} with k cross-edges.

For odd n, since $1 \le k \le n-2$ is attainable in the even P_{n-1}, each i between 2 and $n-1$ is attainable in P_n. □

By Proposition 1, $FI(P_n)$ forms an arithmetic progression. More generally, Lee and Salehi [4] (see also [3]) conjectured the following.

Conjecture 2. *For any tree T, $FI(T)$ forms an arithmetic progression.*

The stronger statement that $FFI(T)$ forms an arithmetic progression is clearly false. For example, consider H obtained by joining the centres of two $K_{1,3}$ by an edge. Then $C(H) = \{1, 3, 4, 5, 7\}$ and $FFI(H) = \{-5, -1, 1, 3, 7\}$ but $FI(H) = \{1, 3, 5, 7\}$.

In Section 2, we determine the full friendly index sets of spiders. In particular, we show that if G is a spider then $FI(G)$ is an arithmetic progression with difference 2. In Section 3, we disprove Conjecture 2 by providing an infinite family of counterexamples.

2. Full friendly index sets of spiders

In this section, we compute the friendly index sets of subdivided stars otherwise known as spiders. For $k \in \mathbb{N}$ and positive integers $n_1 \ge n_2 \ge \cdots \ge n_k$ (we also write $n(i)$ for n_i), let $N = 1 + \sum_{i=1}^{k} n_i$ and

$$m = \min\left\{ j \mid \sum_{i=1}^{j} n_i \ge \lfloor \frac{N}{2} \rfloor \right\}, \quad M = N - 1 - \lfloor \frac{|l-1|}{2} \rfloor,$$

where $l = |\{i \mid n_i \text{ is odd}\}|$. A spider $G = S(n_1, \cdots, n_k)$ is obtained by identifying the leaves of the star $K_{1,k}$ with endvertices of paths. More precisely, let $E(K_{1,k}) = \{vv_1, vv_2, \cdots, vv_k\}$. The edge vv_i is replaced by a path on $n_i + 1$ vertices, namely, $(v, x_i = x_{i1}, x_{i2}, \cdots, x_{in(i)} = y_i)$. We shall refer to this path as *leg* i and denote the path from x_i to y_i by $P(i)$. Thus, $|G| = N$. An equitable bipartition $V_0 \cup V_1$ is called *maximum* (resp. *minimum*) if $e(V_0, V_1) = \max C(G)$ (resp. $\min C(G)$) and an edge in $E(V_0, V_1)$ is called a *cross-edge*, otherwise, it is an *in-class edge.*

Proposition 3. $\min C(G) = m$ *and* $\max C(G) = M$.

Proof. Let $v \in V_0$. Suppose $\min C(G) < m$. Then at most $m - 1$ legs have vertices in V_1. Hence, for some $S \subset [k], |S| = m - 1$,

$$|V_1| \leq \sum_{i \in S} n_i \leq \sum_{i=1}^{m-1} n_i < \lfloor \frac{N}{2} \rfloor.$$

Thus, $V_0 \cup V_1$ cannot be an equitable bipartition. To attain m, we put $P(i)$ in V_1 for $1 \leq i \leq m - 1$, and the last $a > 0$ vertices of $P(m)$ in V_1 where $n_1 + \cdots + n_{m-1} + a = \lfloor N/2 \rfloor$.

Next, consider a maximum equitable bipartition of G. Let $A, B \subset [k]$ such that $i \in A$ if $P(i)$ has $a_i > 0$ more vertices in V_1 than V_0, $j \in B$ if $P(j)$ has $b_j > 0$ more vertices in V_0. Since we have an equitable bipartition of G, $|\sum_{i \in A} a_i - (1 + \sum_{j \in B} b_j)| \leq 1$. Moreover, we may assume $x_i \in V_1$ and $x_j \in V_0$ to maximize the size of the cut. Thus, we have

$$\begin{aligned} e(V_0, V_1) &\leq \sum_{i \in A} (n_i - a_i + 1) + \sum_{j \in B} (n_j - b_j) + \sum_{p \notin A \cup B} n_p \\ &= N - 1 - \sum_{i \in A} (a_i - 1) - \sum_{j \in B} b_j \\ &\leq N - 1 - \sum_{j \in B} b_j \leq N - 1 - |B|. \end{aligned}$$

As there are l odd paths, at least l paths have a positive imbalance. Hence, we minimize $|B|$ subject to $0 \leq |A| - |B| \leq 2$ and $|A| + |B| \geq l$ which has the solution $|B| = \lfloor |\frac{l-1}{2}| \rfloor$. On the other hand, M can be attained via greedily attaching legs one at a time while keeping the bipartition equitable at every step. □

Proposition 4. $C(G) = [m, M]$.

Proof. We separate the proof into two cases depending on whether there is a leg of length 1 or not.

(1) $n_k \geq 2$: We first show that $[k, N-1-k] \subset C(G)$ by exhibiting equitable bipartitions of G in which for each i, $P(i)$ is also partitioned equitably and such that leg i contributes c_i cross-edges where $1 \leq c_i \leq n_i - 1$. This relies on the fact that the endvertices of $P(i)$ in an equitable partition attaining odd c_i lie in different classes. If c_i is even, then we take an equitable bipartition of $P(i)$ attaining $c_i - 1$ and make use of the edge incident with v. We add one leg at a time and keep the partition equitable. More precisely, let $V_0^0 = \{v\}, V_1^0 = \emptyset$. For $i \geq 1$, iterate the following:

(a) c_i odd: Let $S^i \cup T^i$ be an equitable bipartition of $P(i)$ attaining c_i such that $|T^i| \geq |S^i|$.
  (i) $|V_0^{i-1}| \geq |V_1^{i-1}|$: Let $V_0^i = V_0^{i-1} \cup S^i$ and $V_1^i = V_1^{i-1} \cup T^i$, where $x_i \in S^i$.
  (ii) $|V_0^{i-1}| < |V_1^{i-1}|$: Let $V_0^i = V_0^{i-1} \cup T^i$ and $V_1^i = V_1^{i-1} \cup S^i$, where $x_i \in T^i$.

(b) c_i even: Let $S^i \cup T^i$ be an equitable bipartition of $P(i)$ attaining $c_i - 1$ such that $|T^i| \geq |S^i|$.
  (i) $|V_0^{i-1}| \geq |V_1^{i-1}|$: Let $V_0^i = V_0^{i-1} \cup S^i$ and $V_1^i = V_1^{i-1} \cup T^i$, where $x_i \in T^i$.
  (ii) $|V_0^{i-1}| < |V_1^{i-1}|$: Let $V_0^i = V_0^{i-1} \cup T^i$ and $V_1^i = V_1^{i-1} \cup S^i$, where $x_i \in S^i$.

To cover $[m, k]$, we alter the minimum equitable bipartition in Proposition 3 by forcing each leg to contribute an edge. Introduce an ordering on the set of $\lfloor \frac{N}{2} \rfloor - m$ non-leaf-vertices in V_1 as

$$(x_1, \cdots, z_1, x_2, \cdots, z_2, \cdots, x_{m-1}, \cdots, z_{m-1}, x_{m,b}, \cdots, z_m),$$

where $z_i = x_{i,n(i)-1}$ and $x_{m,b}$ is the first vertex of leg m in V_1. Let $L = (y_k, y_{k-1}, \cdots, y_{m+1})$ be an ordering of the leaves in V_0. For $1 \leq a \leq k - m$, exchange the first a leaves in L with the first a non-leaf-vertices in V_1. This gives an equitable bipartition attaining $m + a$. We would not run out of vertices in V_1, since $n_k \geq 2$, which in turn implies $a \leq k - m \leq \lfloor \frac{N}{2} \rfloor - m$.

To cover $[N-k, M]$, we alter the maximum equitable bipartition in Proposition 3. It suffices to construct equitable bipartitions where legs that contribute n_i cross-edges in the maximum bipartition now contribute an edge fewer. If n_i is even, we join v to y_i instead of x_i. If n_i is odd, then $|P(i) \cap V_1| = (n_i + 1)/2$. Replace the equitable partition of $P(i)$ with the one attaining $n_i - 2$ with $x_i, x_{i,2}$ in V_1. Then we still have $|P(i) \cap V_1| = (n_i + 1)/2$, so that the resulting partition of G remains equitable. Moreover, with vx_i, leg i now contributes $n_i - 1$ cross-edges.

(2) $n_{k-g} > n_{k-g+1} = 1$ for some $g > 0$: Let $G' = S(n_1, \cdots, n_{k-g})$ and suppose there are l odd values among these n_i's. We construct equitable bipartitions of G as follows.

Given an equitable bipartition of G' with cutsize s, we evenly distribute the legs of length 1 to V_0 and V_1, namely, put the $h = \lfloor (g+1)/2 \rfloor$ leaves $L_1 = \{x_k, x_{k-1}, \cdots, x_{k-h+1}\}$ to V_1 and the rest $\{x_{k-g}, \cdots, x_{k-h}\}$ to V_0. Note that there are $s + h$ cross-edges in this bipartition.

Clearly, if g is even, we have an equitable bipartition of G. On the other hand, observe that in Case (1), all values $s \in C(G')$ are attained with partitions such that v is never in the smaller class unless $l > 0$ is even and $s = \max C(G')$. Thus, unless g is odd and $l > 0$ is even, our partition is equitable for all $s \in C(G')$. Moreover, by checking the parities of g and l, we have $\max C(G') + h = \max C(G)$. Thus, we have $[\min C(G') + h, \max C(G') + h] \subset C(G)$.

When $l > 0$ is even and g is odd, the same construction yields an equitable bipartition of G for s up to $\max C(G') - 1$. Note that in this case, $\max C(G) = \max C(G') - 1 + h$. Hence, we also have $[\min C(G') + h, \max C(G)] \subset C(G)$.

We now proceed to show that cutsizes from $\min C(G') + h$ down to $\min C(G)$ are attainable. Starting from our minimum equitable bipartition of G', we transfer the legs of length 1 from V_1 to V_0 one by one. Indeed, suppose $j = \min C(G')$ is the maximum index for which $P(j)$ has vertices in V_1 and $x_{j,q}$ is the last vertex in V_0. Introduce an ordering of the vertices in V_0 as

$$(x_{j,q}, x_{j,q-1}, \cdots, x_j, y_{j+1}, \cdots, x_{j+1}, \cdots, y_{k-g}, \cdots, x_{k-g}, v).$$

For any $0 \leq a \leq h$, we exchange a vertices from L_1 with the first a vertices of $V_0 \cap V(G')$. Whenever it is the turn of y_i to be put into V_1, we have the same number of cross-edges. Otherwise, we decrease the number of cross-edges by one. Proceeding all the way to our minimum equitable bipartition, we attain all values down to $\min C(G)$.

□

Corollary 5. *Let $a = N - 2\lfloor N/2 \rfloor$. Then $FI(G) = \{a,\ a+2,\ a+4,\ \cdots,$ $\max\{|2m - N + 1|, |2M - N + 1|\}\}$.*

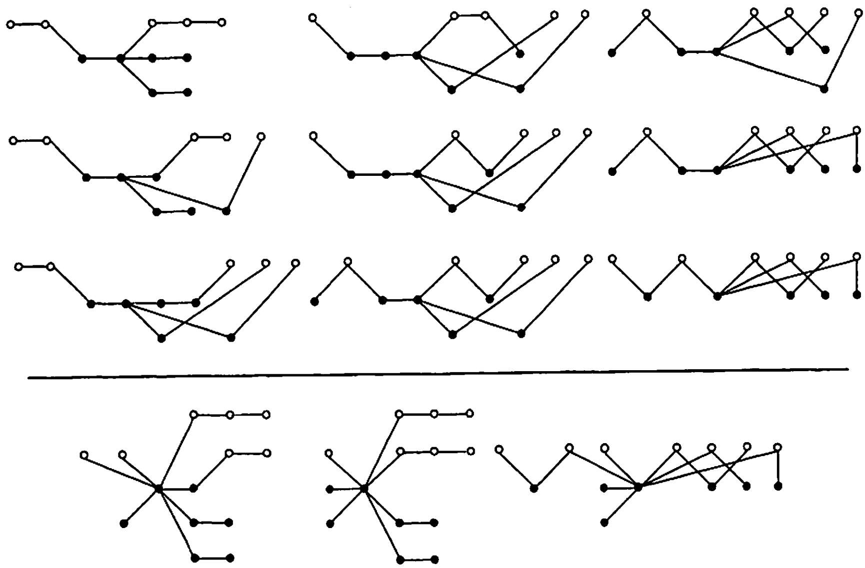

FIGURE 1. Above: equitable bipartitions of $S(3,3,2,2)$ attaining 2 to 10 cross edges; below: equitable bipartitions of $S(3,3,2,2,1,1,1)$ at the extremal ends from bipartitions of $S(3,3,2,2)$. $\bullet$ – V_0; $\circ$ – V_1

3. COUNTEREXAMPLES TO CONJECTURE 2

The conjecture of Lee and Salehi [4] says that the friendly index set of a tree forms an arithmetic progression. We shall construct an infinite family of caterpillars as counterexamples.

A caterpillar is a graph formed by identifying the vertices of a path by the centres of stars. Given a path $(v_1, v_2, \cdots, v_k)$ and integers $n_1, n_2, \cdots, n_k \geq 0$, a caterpillar is constructed by identifying v_i with the centre of the star $K_{1,n(i)}$.

Our family of caterpillar is defined as follows. For an integer $k > 3$, let $R(k)$ be the caterpillar obtained from the path P_{2k} on $2k$ vertices with $n_2 = n_{k-1} = n_k = n_{2k-1} = 1$ and $n_i = 0$ for all other i. That is $V(R(k)) = P \cup A$ where $P = \{v_1, v_2, \cdots, v_{2k}\}$ and $A = \{r_2, r_{k-1}, r_k, r_{2k-1}\}$ and $E(R(k)) = \{v_i v_{i+1} \mid i = 1, 2, \cdots, 2k-1\} \cup \{v_j r_j \mid j = 2, k-1, k, 2k-1\}$. Let $B = \{v_2, v_{k-1}, v_k, v_{2k-1}\}$. The following proposition disproves Conjecture 2.

Proposition 6. *For $k > 4$, $2k-3, 2k-1, 2k+3 \in FI(R(k))$ and $2k+1 \notin FI(R(k))$.*

Proof. Let $R = R(k)$. To show that $2k-3, 2k-1, 2k+3 \in FI(R)$, it suffices to construction equitable bipartitions attaining $2k, 2k+1, 2k+3$ cross-edges respectively. Before that, we observe that in any bipartition of $V_0' \cup V_1'$ of P_{2k} with $v_1 \in V_0'$ and exactly one in-class edge e, we have $v_{2k} \in V_0'$ and either

$$(3.1) \qquad \begin{array}{rl} |V_0'| = k+1, |V_1'| = k-1 & \text{if } e \in E(V_0') \text{ or} \\ |V_0'| = |V_1'| = k & \text{otherwise.} \end{array}$$

We now show that $2k+3, 2k+1, 2k \in C(R)$.

(1) $2k+3 \in C(R)$: Take a maximum equitable bipartition $V_0' \cup V_1'$ of P_{2k} with $v_1 \in V_0'$. Hence, $v_2 \in V_1'$ and $v_{2k-1} \in V_0'$. Moreover, as v_{k-1}, v_k lie in different classes, it follows that $|V_0' \cap B| = 2$. Hence, we may extend the partition by attaching the leaves in A to attain $2k+3$ cross-edges.

(2) $2k+1 \in C(R)$: Let $V_0' \cup V_1'$ be a partition of P_{2k} with $2k-2$ cross-edges and both $v_{2k-1}, v_{2k} \in V_0'$. By (3.1), $|V_0'| = k+1$. Then $V_0 = V_0' \cup \{r_2\}$ and $V_1 = V_1' \cup \{r_{k-1}, r_k, r_{2k-1}\}$ is an equitable bipartition of R with $2k+1$ cross-edges.

(3) $2k \in C(R)$: Let j be the smallest odd integer at least k. Let $V_0' \cup V_1'$ be a partition of P_{2k} with $2k-2$ cross-edges such that $\{v_1, v_j, v_{j+1}\} \subset V_0'$. By (3.1), $|V_0'| = k+1$. It is easy to check that $V_0 = V_0' \cup \{r_{k-1}\}$ and $V_1 = V_1' \cup \{r_2, r_k, r_{2k-1}\}$ for k odd; $V_0 = V_0' \cup \{r_k\}$ and $V_1 = V_1' \cup \{r_2, r_{k-1}, r_{2k-1}\}$ for k even is an equitable bipartition of R with $2k$ cross-edges.

It remains to show that $2k+1 \notin FI(R)$ which is equivalent to $1, 2k+2 \notin C(R)$. As there is no edge disconnecting R to components of equal orders, $1 \notin C(R)$. Suppose $2k+2 \in C(R)$. Let $V_0 \cup V_1$ be an equitable bipartition with e as the only in-class edge. Note that e cannot be incident with a vertex in A, for the only way to have $e = v_i r_i$ is to change the maximum equitable bipartition of R by putting r_i to the same partition as v_i which clearly upsets the bipartition. Hence, P must be partitioned with $2k-2$ cross-edges, so that v_1, v_{2k} lie in the same class, say, V_0.

(1) If $e \in E(V_0)$, then $|V_0 \cap P| = k+1$. As $v_1, v_{2k} \in V_0$, v_2, v_{2k-1} cannot both be in V_0. If $v_2, v_{2k-1} \in V_1$, then $r_2, r_{2k-1} \in V_0$ so that $|V_0| \geq k+3 > k+2 = |R|/2$ which is a contradiction. On the other hand, if v_2, v_{2k-1} lie in different classes, then either v_1v_2 or $v_{2k-1}v_{2k}$ is the in-class edge. Hence, r_2, r_{2k-1} are in different classes. The same is true for the pairs (v_{k-1}, v_k) and (r_{k-1}, r_k). Hence, $|V_0 \cap A| = 2$, so that $|V_0| = k+3$.

(2) If $e \in E(V_1)$, then $|V_0 \cap P| = k$. Hence, $v_2, v_{2k-1} \in V_1$ and $r_2, r_{2k-1} \in V_0$. Moreover, r_{k-1}, r_k cannot both be in V_1 because v_{k-1}, v_k cannot both be in V_0. Again, we have $|V_0| = k + 3$ which is not allowed by an equitable bipartition.

□

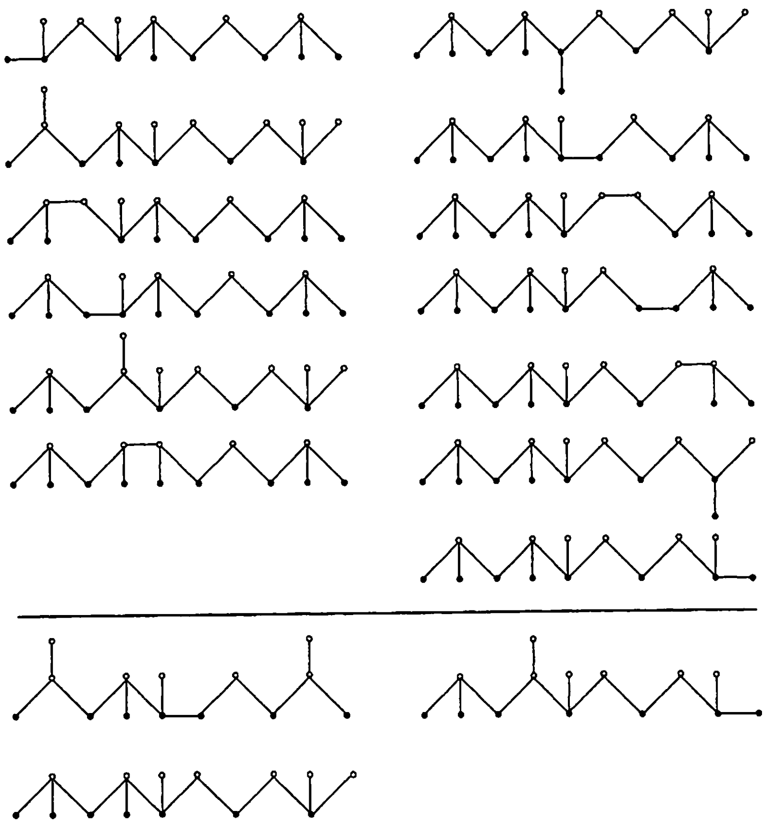

FIGURE 2. Counterexample $R(5)$ to Conjecture 2: above, nonequitable partition with cutsize 12; below, equitable partitions with cutsize 10, 11, 13. $\bullet$ – V_0; $\circ$ – V_1

4. CONCLUSION

We have shown that the friendly index sets of spiders form an arithmetic progression with difference 2. This is also true for some other families of trees whose friendly index sets form arithmetic progressions, see [4]. This suggests the following.

Question 7. *Suppose that the friendly index set of a tree forms an arithmetic progression. What could the difference be?*

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