HARMONIOUS COLORINGS OF DIGRAPHS[1]

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Abstract

Let D be a directed graph with n vertices and m edges. A function $f : V(D) \rightarrow \{1, 2, 3, ...k\}$ where $k \leq n$ is said to be *harmonious coloring* of D if for any two edges xy and uv of D, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If the pair (i, i) is not assigned, then f is said to be a *proper harmonious coloring* of D. The minimum k is called the *proper harmonious coloring number* of D. We investigate the proper harmonious coloring number of graphs such as unidirectional paths, unicycles, inspoken (outspoken) wheels, n-ary trees of different levels etc.

Keywords: Harmonious coloring, proper harmonious coloring number, digraphs

1. INTRODUCTION

In this paper, we consider only finite simple graphs. For all notations in graph theory we follow Harary [4], West [6] and Chartrand [1]. Coloring the vertices and edges of a graph which is required to obey certain conditions, have often been motivated by their utility to various applied fields and their mathematical interest. Various coloring problems such as the vertex coloring and edge coloring problem have been studied in the literature [4].

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Definition 1.1. *A coloring of a graph* G *is a function* $c: V(G) \rightarrow X$ *for some set of colors* X *such that* $c(u) \neq c(v)$ *for each edge* $uv \in E(G)$.

The coloring defined above is the vertex coloring where we color the vertices of a graph such that no two adjacent vertices are colored with the same color. Similarly the edge coloring problem can be defined in such a way that no two adjacent edges are colored the same color. Hopcroft and Krishnamoorthy [5] introduced a type of edge coloring called harmonious coloring.

Definition 1.2. *A harmonious coloring [5] of a graph* G *is an assignment of colors to the vertices of* G *and the color of an edge is defined to be the unordered pair of colors to its end vertices such that all edge colors are distinct. The harmonious coloring number is the least number of colors in such a coloring.*

An enormous body of literature has grown around the subject Harmonious Coloring. The list of articles published on the subject can be found in [2].

The following is an extension of harmonious coloring to directed graphs.

Definition 1.3. *Let* D *be a directed graph with* n *vertices and* m *edges. A function* $f : V(D) \rightarrow \{1, 2, ..., k\}$ *where* $k \leq n$ *is said to be a harmonious coloring of* D *if for any two edges* xy *and* uv *of* D *, the ordered pair* $(f(x), f(y)) \neq (f(u), f(v))$. *If the pair* (i, i) *is not assigned, then* f *is called a proper harmonious coloring of* D. *The minimum* k *for which* D *admits a proper harmonious coloring is called the proper harmonious coloring number of* D *and is denoted by* $\overrightarrow{\chi_h}(\overrightarrow{D})$.

In Figure 1 a proper harmonious coloring of Petersen graph and its oriented graph are given.

2. RESULTS

In this section we present the results on proper harmonious coloring of some classes of digraphs.

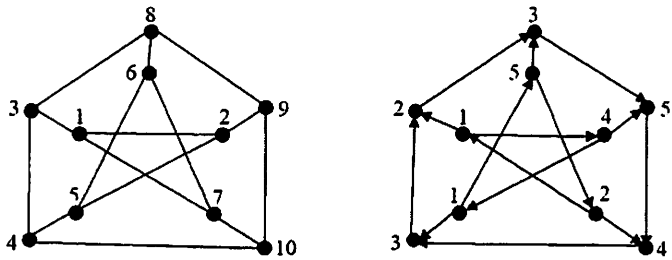

Figure 1: A proper harmonious coloring of Petersen Graph and its oriented graph.

Proposition 2.4. *The proper harmonious coloring number of a symmetric digraph is same as the proper harmonious coloring number of its underlying graph.*

Proposition 2.5. *Let $\overrightarrow{D}$ be a directed graph with p vertices. Then* $\Delta + 1 \leq \overrightarrow{\chi_h}(\overrightarrow{D}) \leq p$.

Proposition 2.6. *For any graph G, $\overrightarrow{\chi_h}(G) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$ where m is the number of edges.*

Proof. Let G be a digraph. Then G is colored with k colors using proper harmonious coloring. Then the possible number of ordered pairs is $k(k-1)$.
$\therefore m \leq k(k-1)$.
$\Rightarrow k^2 - k - m \geq 0$.
$\Rightarrow k \geq \frac{1+\sqrt{4m+1}}{2}$.
$\therefore k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$.

Proposition 2.7. *Let $\overrightarrow{P}_n$ be a unipath with n vertices. Then* $\overrightarrow{\chi_h}(\overrightarrow{P}_n) = \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$.

Proof. Since unipath $\overrightarrow{P}_n$ contains $(n-1)$ edges, $\overrightarrow{\chi_h}(\overrightarrow{P}_n) \geq \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$. Let $k = \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$. Then, $(k-1)(k-2)+1 <$

$n \leq k(k-1)+1$. Now, we shall prove that $\overrightarrow{\chi_h}(\overrightarrow{P}_n) = k$ for $(k-1)(k-2)+1 < n \leq k(k-1)+1$. Consider a complete symmetric digraph $\overleftrightarrow{K}_k$ with k vertices. Then $\overleftrightarrow{K}_k$ contains $k(k-1)$ edges. The proper harmonious coloring of $\overrightarrow{P}_n$ is equivalent to the coloring of an Eulerian path traversing the edges of $\overleftrightarrow{K}_k$ (of length $(n-1)$) where $(k-1)(k-2)+1 < n \leq k(k-1)+1$. We need to prove that there exists an Eulerian path of length $k(k-1)$. We shall prove this by mathematical induction. For $k=2$ the result holds. Assume that the result is true for $k=m$. i.e. there exists an Eulerian path of length $m(m-1)$ in $\overleftrightarrow{K}_m$. Consider $\overleftrightarrow{K}_m$ and a vertex v. Then joining v to all the vertices of $\overleftrightarrow{K}_m$ in both directions, we get $\overleftrightarrow{K}_{m+1}$. Let $u_1, u_2, \ldots, u_m$ be the vertices of $\overleftrightarrow{K}_m$. Let u_m be the end vertex of the Eulerian path of length $m(m-1)$ (Consequently it is the first vertex). Then traverse along the path $u_m\ v\ u_1\ v\ u_2\ v \ldots u_{m-1}\ v\ u_m$ and see that it is the extension of the Eulerian path obtained from $\overleftrightarrow{K}_m$ (of length $m(m-1)$), so that the length of the path obtained is $m(m-1)+2m$

$= m(m+1)$.
Hence by the principle of mathematical induction, the result holds.

Figure 2 is an illustration of the above proof.

Figure 2: A proper harmonious coloring of unipath P_7.

Proposition 2.8. *Let $\overrightarrow{D} = \overrightarrow{P}_1 \bigcup \overrightarrow{P}_2 \bigcup \cdots \bigcup \overrightarrow{P}_i$ be a union of disjoint unipaths, where $\overrightarrow{P}_i$ has i vertices for $i = 1, 2, \cdots$. Then $\overrightarrow{\chi_h}(\overrightarrow{D}) = k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.*

Proof. Let $\overrightarrow{D} = \overrightarrow{P}_1 \bigcup \overrightarrow{P}_2 \bigcup \cdots \bigcup \overrightarrow{P}_i$. Then $\overrightarrow{D}$ has $\frac{i(i+1)}{2}$ vertices and $\frac{i(i-1)}{2}$ edges.

We know that $k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$ where m is the number of edges.

$\implies k \geq \lceil \frac{1+\sqrt{\frac{4i(i-1)}{2}+1}}{2} \rceil$

$\implies k \geq \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.

Now, we shall prove that $k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.

The harmonious coloring number of $\overrightarrow{D}$ is equivalent to the harmonious coloring number of a unipath $\overrightarrow{P_t}$ where $\overrightarrow{P_t}$ is the unipath obtained by adjoining the endvertex of $\overrightarrow{P_j}$ and the starting vertex of $\overrightarrow{P_{j+1}}$ for $j = 1, 2, \cdots, i-1$. Since $\overrightarrow{P_t}$ contains $\frac{i(i-1)}{2}+1$ vertices, $\overrightarrow{P_t}$ can be colored with $k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$ colors (by proposition 2.8). Let $a_1, a_2, \cdots, a_t$ be the minimal sequence of colors assigned to the vertices of unipath $\overrightarrow{P_t}$. Note that $a_1, a_2, \cdots, a_t$ are not distinct. Now assign the colors $a_{\frac{(j-1)(j-2)}{2}+1}, a_{\frac{(j-1)(j-2)}{2}+2}, \cdots, a_{\frac{j(j-1)}{2}+1}$ to the vertices of $\overrightarrow{P_j}$, for $j = 1, 2, \cdots, i$. Note that the color of the end vertex of $\overrightarrow{P_j}$ $(1 \leq j < i)$

$= a_{\frac{j(j-1)}{2}+1}$

$= a_{\frac{(j+1-1)(j+1-2)}{2}+1}$

$=$ the color of the starting vertex of $\overrightarrow{P_{j+1}}$ $(j < j+1 \leq i)$.

Hence, $\overrightarrow{\chi_h}(\overrightarrow{D}) = \overrightarrow{\chi_h}(\overrightarrow{P_t}) = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$. □

Figure 3 is an illustration of the above proof.

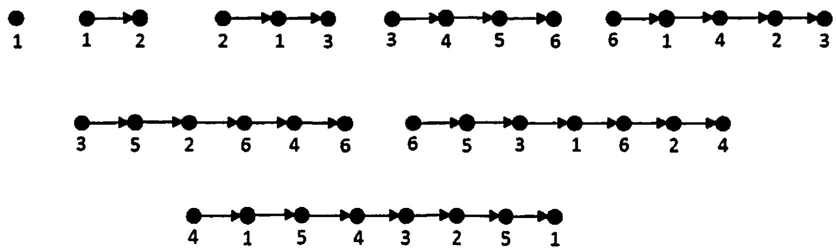

Figure 3: A proper harmonious coloring of union of disjoint unipaths.

Proposition 2.9. *Let $\overrightarrow{C}_n$ be a unicycle with n vertices, then,*
$$\overrightarrow{\chi_h}(\overrightarrow{C}_n)=\begin{cases} k+1 & for \quad n=k(k-1)-1, \\ k & for \quad n=(k-1)(k-2)+1,\ldots,k(k-1)-2,k(k-1), \end{cases}$$
where $k=\lceil\frac{1+\sqrt{4n+1}}{2}\rceil$ *for* $(k-1)(k-2)+1\leq n\leq k(k-1)$.

Proof. Since a unicycle $\overrightarrow{C}_n$ contains n edges, $\overrightarrow{\chi_h}(\overrightarrow{C}_n)=k\geq\lceil\frac{1+\sqrt{4n+1}}{2}\rceil$. Let $k=\lceil\frac{1+\sqrt{4n+1}}{2}\rceil$. Then $(k-1)(k-2)+1\leq n\leq k(k-1)$. Now, we shall prove that
$$\overrightarrow{\chi_h}(\overrightarrow{C}_n)=\begin{cases} k+1 & for \quad n=k(k-1)-1, \\ k & for \quad n=(k-1)(k-2)+1,\ldots,k(k-1)-2,k(k-1). \end{cases}$$
It is equivalent to prove that there exists an Eulerian circuit with n edges and k vertices for $(k-1)(k-2)+1\leq n\leq k(k-1)$ except for $n=k(k-1)-1$. We know that a digraph D has an Eulerian circuit if and only if $id\ (v)=od\ (v)$ for every vertex v. For $n=k(k-1)-1$, the possible degree sequence (k vertices) is $(k-1),(k-1),\ldots,(k-1),(k-2)$. But there exists no such digraph. (For otherwise, let v be the vertex with $id\ (v)=od\ (v)=k-2$. Since degree of each of the other vertex is $k-1$, every other vertex has an edge to v so that $id\ (v)=k-1$, a contradiction.) Now consider a complete symmetric digraph $\overleftrightarrow{K}_k$ with k vertices. In $\overleftrightarrow{K}_k$, $id\ (v)=od\ (v)=k-1$ for all v. Hence $\overleftrightarrow{K}_k$ is Eulerian. Therefore, we have the result for $n=k(k-1)$. Remove an Eulerian cycle of length i where $i=2,3,4,\ldots,(2k-3)$ from $\overleftrightarrow{K}_k$. Then we get an Eulerian cycle of length $k(k-1)-2,k(k-1)-3,\ldots,(k-1)(k-2)+1$. Hence we have the result. Since we are removing a cycle, the equation $id\ (v)=od\ (v)$ remains unchanged for the vertices lying on the cycle. (When we remove an outgoing edge, we remove an incoming edge and vice versa.)
Therefore, the resulting cycle is also an Eulerian circuit. □

Figure 4 is an illustration of the above proof.

Proposition 2.10. *Let $\overrightarrow{D}=\overrightarrow{C}_3\bigcup\overrightarrow{C}_4\bigcup\cdots\bigcup\overrightarrow{C}_i$ be a union of disjoint unicycles, where $\overrightarrow{C}_i$ has i vertices for $i=3,4,\cdots$. Then* $\overrightarrow{\chi_h}(\overrightarrow{D})=k=\lceil\frac{1+\sqrt{2i^2+2i-11}}{2}\rceil$.

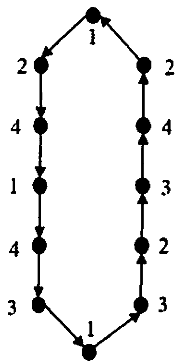

Figure 4: A proper harmonious coloring of unicycle C_{12}.

Proof. Let $\overrightarrow{D} = \overrightarrow{C}_3 \bigcup \overrightarrow{C}_4 \bigcup \cdots \bigcup \overrightarrow{C}_i$. Then $\overrightarrow{D}$ has $\frac{(i-2)(i+3)}{2}$ vertices and $\frac{(i-2)(i+3)}{2}$ edges.

We know that $k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$ where m is the number of edges.

$\implies k \geq \lceil \frac{1+\sqrt{\frac{4(i-2)(i+3)}{2}+1}}{2} \rceil$

$\implies k \geq \lceil \frac{1+\sqrt{2i^2+2i-11}}{2} \rceil = t$.

Now, we shall prove that $k = t$.

Consider the complete symmetric digraph $\overleftrightarrow{K}_t$. Since $\overleftrightarrow{K}_t$ is Eulerian, it can be partitioned into cycles (from **Theorem 4.4** of [1]). It can be proved by induction that $\overleftrightarrow{K}_t$ can be partitioned such that the partition include cycles of length 3,4,..., i. The vertices of these cycles give the harmonious coloring of $\overrightarrow{D}$. Hence the harmonious coloring number of $\overrightarrow{D}$, $k \leq t$.

$\therefore\ k = t$.

i.e. $\overrightarrow{\chi_h}(\overrightarrow{D}) = \lceil \frac{1+\sqrt{2i^2+2i-11}}{2} \rceil$. □

Figure 5 is an illustration of the above proof.

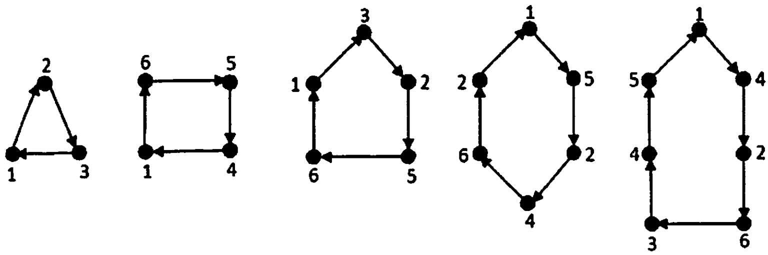

Figure 5: A proper harmonious coloring of union of disjoint unicycles.

Proposition 2.11. *Let $\overleftrightarrow{C}_n$ be a symmetric cycle with n vertices. Then,* $\overleftrightarrow{\chi_h}(\overleftrightarrow{C}_n) \geq k = \lceil \frac{1+\sqrt{8n+1}}{2} \rceil$. *In particular,*

$$\overleftrightarrow{\chi_h}(\overleftrightarrow{C}_n) = \begin{cases} n & for \quad n = 2,3,4. \\ k & for \quad k^2 - 4k + 5 \leq 2n \leq k(k-1) \quad k \geq 5 \text{ and } k \text{ is odd.} \\ k+1 & for \quad 2n = k(k-1) - j, j = 2,4 \quad k \geq 5 \text{ and } k \text{ is odd.} \\ k & for \quad k^2 - 3k + 4 \leq 2n \leq k(k-2) \quad k \geq 6 \text{ and } k \text{ is even.} \end{cases}$$

It is similar to the harmonious coloring number of undirected cycle which is proved by Frank et al [3].

Figure 6 is an illustration of the above result.

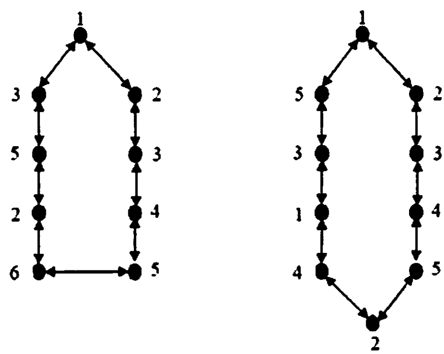

Figure 6: Proper harmonious coloring of symmetric cycles C_9 and C_{10}.

Proposition 2.12. $\overrightarrow{\chi_h}(\overrightarrow{S}_n) = max[id\ (v), od\ (v)] + 1$ *where* $\overrightarrow{S}_n$ *is a directed star with* n *vertices and* v *is the central vertex.*

Proof. Let $\overrightarrow{S}_n$ be a directed star where n is the number of vertices. Let v be the central vertex. Let there be s incoming edges to v and t outgoing edges from v. Label the central vertex v as 1.
Case (i): Let $s > t$. Then the incoming edges to v will be $(2,1),(3,1),...,(s+1,1)$ and the outgoing edges from v will be $(1,2),(1,3),...,(1,t)$.
$\therefore \overrightarrow{\chi_h}(\overrightarrow{S}_n) = s+1$.
Case (ii): Let $t > s$. Then the outgoing edges from v will be $(1,2),(1,3),...,(1,t+1)$ and the incoming edges to v will be $(2,1),(3,1),...,(s,1)$.

$\therefore \overrightarrow{\chi_h}(\overrightarrow{S}_n) = t+1$.
From the above two cases, we can conclude that
$\overrightarrow{\chi_h}(\overrightarrow{S}_n) = max[id\ (v), od\ (v)] + 1$.

Figure 7 is an illustration of the above proof.

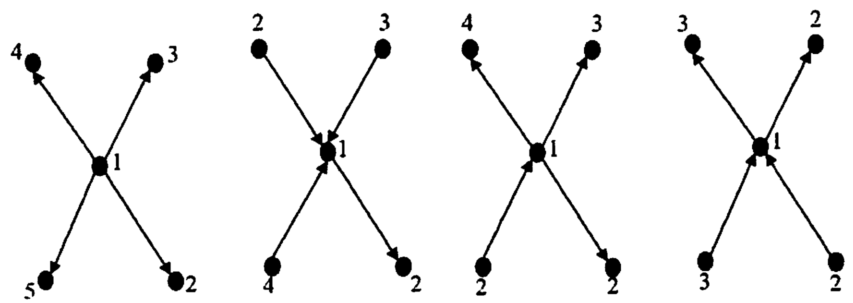

Figure 7: Proper harmonious colorings of S_5.

Proposition 2.13. *Let $\overrightarrow{W_n}$ be a unicyclic wheel with n vertices and let v be the central vertex. Then*
(i) $\overrightarrow{\chi_h}(\overrightarrow{W}_4) = (id\ (v) + od\ (v) + 1)$.
(ii) For $n = 5$ *and* 6

$$\overrightarrow{\chi_h}(\overrightarrow{W}_n) = \begin{cases} n & if\ id\ (v) = 0\ or\ od\ (v) = 0, \\ id\ (v) + od\ (v) & otherwise. \end{cases}$$

(iii) For $n \geq 7$, $\overrightarrow{\chi_h}(\overrightarrow{W}_n) = max[id\ (v), od\ (v)] + 1$.

Proof. (i) and (ii) can be easily verified.
(iii) Let $\overrightarrow{W_n}$ be a unicyclic wheel where $n \geq 7$. The total number of edges of the wheel is $2(n-1)$. Let $v = v_1$ be the central vertex and let $v_2, v_3, ..., v_n$ be the vertices on the circumference of the wheel. Let there be s incoming edges to v and t outgoing edges from v. Label the vertex v_1 as 1.
Case (i): Let $s > t$. Label the tails of the incoming edges to v as $2, 3, ..., s+1$ so that the incoming edges to v will be $(2,1), (3,1), ..., (s+1, 1)$ and also label the heads of the outgoing edges from v as $2, 3, ..., s+1$, $(s+1 > t)$ provided the adjacent vertices on the circumference of the wheel will not get the same color. Hence the outgoing edges from v will be $(1,2), (1,3), ..., (1, s+1)$.
$\therefore \overrightarrow{\chi_h}(\overrightarrow{W}_n) = s + 1$
Case (ii): Let $t > s$. Label the heads of the outgoing edges from v as $2, 3, ..., t+1$ so that the outgoing edges from v will be $(1,2), (1,3), ..., (1, t+1)$ and also label the tails of the incoming edges to v as $2, 3, ..., t+1$, $(t+1 > s)$ provided the adjacent vertices on the circumference of the wheel will not get the same color. Hence the incoming edges to v will be $(2,1), (3,1), ..., (t+1, 1)$.
$\therefore \overrightarrow{\chi_h}(\overrightarrow{W}_n) = t + 1$
From case (i) and case(ii), it follows that
$\overrightarrow{\chi_h}(\overrightarrow{W}_n) = max[id\ (v), od\ (v)] + 1$.

Figure 8,9,10 is an illustration of the above proof.

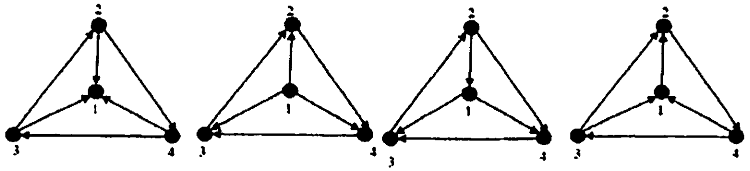

Figure 8: Proper harmonious colorings of unicyclic wheel W_4.

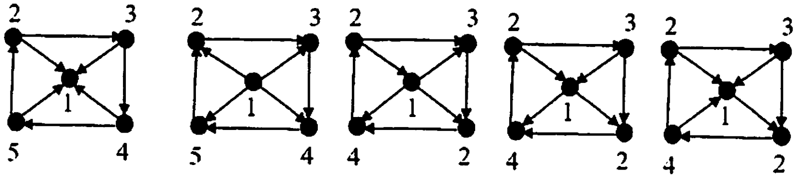

Figure 9: Proper harmonious colorings of unicyclic wheel W_5.

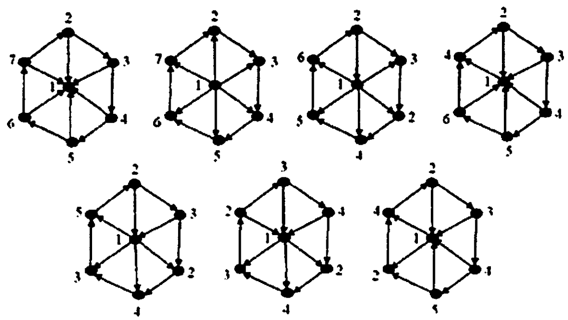

Figure 10: Proper harmonious colorings of unicyclic wheel W_7.

Proposition 2.14. *For any n-ary topdown tree $\overrightarrow{T}_n$, $\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k \le \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n-1}$ where l is the level of the tree.*

Proof. Let $\overrightarrow{T}_n$ be the n-ary tree of level $l, l = 1, 2, \dots$.
It is enough to prove the result for complete topdown n-ary tree $\overrightarrow{T}_n$.
i.e. We shall prove that $\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k = \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n-1}$ for complete topdown n-ary tree, $\overrightarrow{T}_n$. There are $\frac{n^l - 1}{n-1}$ vertices and $\frac{n(n^{l-1}-1)}{n-1}$ edges in $\overrightarrow{T}_n$.
We color the vertices of n-ary tree as follows:

Color the root vertex as 1. In level 2 there are n vertices. Color the vertices as $2, 3, ..., n+1$. Hence the total number of colors used in level 2 is $n+1$. In level 3, color the vertices as follows:

$$L(v_i) = \begin{cases} j & if\ \ j \leq k+1, \\ j+1 & if\ \ j > k+1. \end{cases}$$

where $i = kn + j$, $k = 0, 1, 2, ..., n-1$, $j = 1, 2, ..., n$ and v_i are the vertices of level 3.

Hence in level 3, the total number of colors required is $n+1$. Now in level 4, color the vertices adjacent to 1 as $n+2, n+3, ..., 2n+1$; $2n+2, 2n+3, ..., 3n+1$; ...; $n^2+2, n^2+3, ..., n(n+1)+1$. Use the same colors for the vertices adjacent to $2, 3, ..., n+1$. Hence in level 4, n^2 additional colors are required to color the vertices. In the next level, one can observe that all the vertices adjacent to the vertices colored with $n+2, n+3, ..., n(n+1)+1$ can be colored as $1, 2, ..., n$. In this level, we don't require any additional colors to color the vertices. Continuing in this way, one can observe that the number of colors used in any odd level l is less than or equal to the number of colors used till the $l-1$ level and the graph is harmonious.

Now, we shall prove the theorem by mathematical induction on the level l. For $l = 1$, one can easily see that $k = 1$.

Assume that the result is true for some level $l = m$.

i.e. $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n-1}$.

Now, to prove that the result is true for $l = m+1$,

i.e. to prove that $k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n-1}$,

we shall consider 2 cases.

Case (i): Let $l = m$ be even. Then $m+1$ is odd. Also, we have seen that the number of colors used in any odd level is less than or equal to the number of colors used till the previous level (i.e. even level) except for level 1. Also the number of colors added in level m is equal to $n^{\frac{m}{2}}$. We know that the number of colors sufficient in level $m+1$ is equal to the number of colors used in level m as m is even.

Now, by induction hypothesis, the number of colors used in level m is $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n-1}$.

Here, $\lfloor \frac{m+1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$ as m is even. Hence the number of colors used in level $m+1$ will be , $k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n-1}$.

Case(ii): Let $l = m$ be odd. Then $m+1$ is even. We know that

the number of colors added in level m is $n^{\frac{m-1}{2}}$. Hence the number of colors added in level $m+1$ is $n^{\frac{m+1}{2}}$. Also by induction hypothesis, the number of colors used in level m is $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n-1}$.
Hence the number of colors used in level $m+1$ is

$$k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n-1} + n^{\frac{m+1}{2}}$$

$$\Rightarrow k = \frac{n^{(\frac{m-1}{2}+1)} - 1}{n-1} + n^{\frac{m+1}{2}} \quad (\text{as } m \text{ is odd}, \lfloor \tfrac{m}{2} \rfloor = \tfrac{m-1}{2})$$

$$\Rightarrow k = \frac{n^{\frac{m+3}{2}} - 1}{n-1}$$

$$\Rightarrow k = \frac{n^{\frac{m+1}{2}+1} - 1}{n-1}$$

$$\Rightarrow k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n-1} \quad [\text{as } m \text{ is odd}, \tfrac{m+1}{2} = \lfloor \tfrac{m+1}{2} \rfloor].$$

Therefore, the result is true for level $l = m+1$.
Hence, by the principle of mathematical induction, the result follows for complete topdown n-ary tree $\overrightarrow{T}_n$ of level l. This proves that for any topdown n-ary tree $\overrightarrow{T}_n$

$$\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k \leq \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n-1}. \qquad \square$$

Figure 11 is an illustration of the above proof.

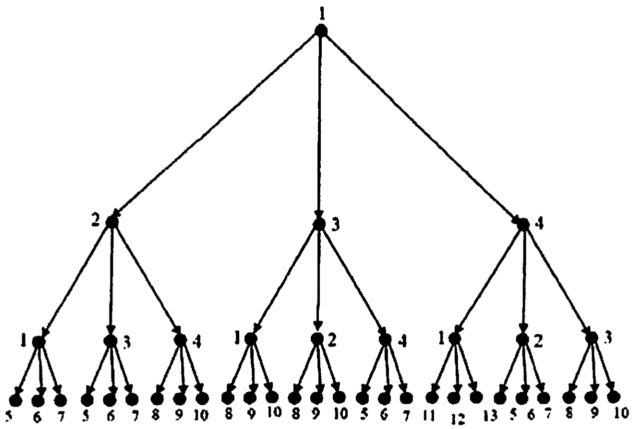

Figure 11: A proper harmonious coloring of complete trinary top-down tree T_3 of level 4.

Acknowledgement

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