k-Connected Graphs Without K_4^-

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Abstract

Let K_4^- be the graph obtained from K_4 by deleting one edge. If G doesn't contain K_4^- as a subgraph, G is called K_4^--free. K.Kawarabayashi showed that a K_4^--free k-connected graph has a k-contractible edge if k is odd. Further, when k is even, K.Ando et.al showed that every vertex of K_4^--free contraction critical k-connected graph is contained in at least two triangles. In this paper, we extend the result of K.Kawarabayashi and get a new lower bound of k-contractible edges in a K_4^--free k-connected graph when k is odd. In addition, we give some characters and properties to K_4^--free contraction critical k-connected graph, and prove that this graph has at least $\frac{2|G|}{k-1}$ vertices of degree k.

Keywords: K_4^--free ; Contractible edge; Contraction critical k-connected
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1. Introduction

In this paper, all graphs considered are finite, undirected, and with neither loops nor multiple edges. Basically, we follow the terminology of J.A.Bondy [2]. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. For a vertex $v \in V$, we write $N(v)$ for the neighborhood of v, $d(v) = |N(v)|$ denotes the degree of v in G. $E(v)$ denotes the set of the edges incident to v. For a subset $S \subseteq V$, we write $G[S]$ for the induced subgraph of S in G. For subsets S and T of V, $E(S,T)$ denotes the set of edges between S and T, if $S = \{x\}$, we simply write $E(x,T)$ instead of $E(\{x\},T)$. A subset $S \subseteq V(G)$ is said to be a cutset or a separating set of G, if $G - S$ is not connected. A cutset S is said to be a k-cutset if $|S| = k$. For a graph G, let $V_k(G)$ be the set of vertices of G with degree k. We call

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$\{x\} + 2K_2$ a x-bowtie. Denote the cartesian product of two graphs G and H by $G \times H$.

For a subset $F \subseteq V$, let $N(F) = (\bigcup_{x \in F} N(x)) - F$ and $\overline{F} = V - (F \cup N(F))$. The set F or the subgraph induced by F is called a fragment of G if $F \neq \emptyset \neq \overline{F}$ and $|N(F)| = \kappa(G)$, where $\kappa(G)$ denotes the connectivity number of G. We call F a $N(F)$- fragment, we don't distinct $V(F)$ and F if it causes no confusion.

Let K_4^- be the graph obtained from K_4 by deleting one edge. If G doesn't contain K_4^- as a subgraph, G is called K_4^--free. Let G be a k-connected non-complete graph with $k \geq 2$. An edge of G is called k-contractible if its contraction yields again a k-connected graph. If G does not have a k-contractible edge, it is said to be contraction critical k-connected. If an edge is not k-contractible, then it is called a k-non contractible edge. It is easy to see that a k-connected graph G is contraction critical k-connected if and only if every edge of G is contained in some k-cutsets. We denote the set of k-contractible edges in G by $E_c(G)$. If the end vertices of e have a common neighbor of degree k, we call e is trivially k-non contractible, shortly as trivially; if the end vertices of e have no a common neighbor of degree k, we call e is nontrivially. Let E^* denote the set of trivially k-non contractible edge in G.

C.Thomassen[8] proved that every k-connected graph without triangle has a k-contractible edge. Y.Egawa[3] improved the result in the following.

Theorem A. *Every k-connected graph G without triangle has at least $min\{|V(G)| + \frac{2}{3}k^2 - 3k, |E(G)|\}$ k-contractible edges.*

As Theorem A shown, a k-connected graph G without triangle has considerable k-contractible edges. Hence the condition "without triangle" is too strong a condition for a k-connected graph to have k-contractible edge. In fact, K.Kawarabayashi [5] obtained the following result.

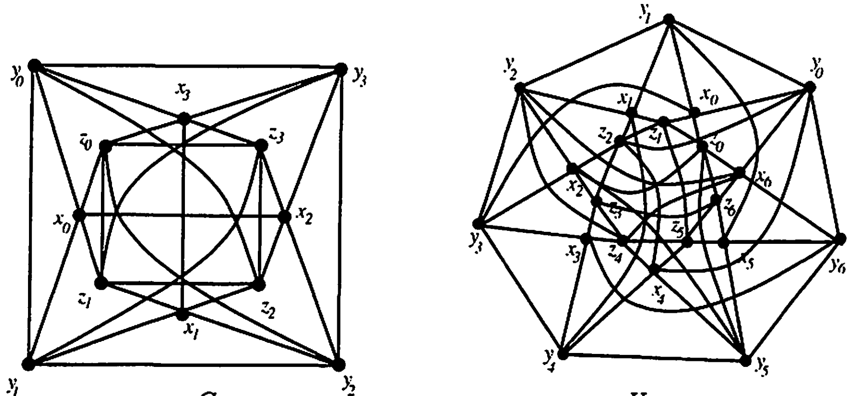
Theorem B. *Let $k \geq 3$ be an odd integer, G be a K_4^--free k-connected graph, then G has a k-contractible edge.*

By Theorem B, we know, when k is odd, a K_4^--free k-connected graph has at least one k-contractible edge. We give a new lower bound of the number of k-contractible edge in a K_4^--free k-connected graph in this case.

Theorem 1. *Let $k \geq 3$ be an odd integer and G be a K_4^--free k-connected graph. Then G has at least $min\{k+1, \frac{|G|}{2}\}$ k-contractible edges.*

We construct a K_4^--free 5-connected graph G_0 which has 6 6-contractible edges(see to Figure 1). Hence, the lower bound in Theorem 1 is sharp.

The same conclusion does not hold when k is even. K.Kawarabayashi[5] constructed a regular graph $G = K_3 \times K_3 \times \cdots K_3 = K_3^{\frac{k}{2}}$ with k being



G_0 $\qquad$ H_1

Figure 1

even. Clearly, G is contraction critical k-connected and doesn't contain K_4^-. Thus there does exists a K_4^--free contraction critical k-connected graph. So it is natural to discuss the property of these graph. M.Fontet [4] and independently N.Martinov [6] gave a complete characterization of contraction critical 4-connected graphs. In view of the result of M.Fontet and N.Matinov, we actually get a complete characterization of K_4^--free contraction critical 4-connected graph.

Theorem C. *Let G be a K_4^--free contraction critical 4-connected graph. Then G is the line graph of a cyclically 4-edge connected cubic graph.*

Further more, K.Ando et.al [1] proved the following result.

Theorem D. *Let G be a K_4^--free contraction critical k-connected graph with $k \geq 4$. Then k is even and every vertex in G is contained in at least two triangles.*

In the following we construct two K_4^-- free contraction critical 6-connected graphs G_1 which isn't 6-regular, and G_2 which exists a vertex contained in exactly 2 triangles(see to Figure 2). The construction of G_1 is in the following 4 steps:

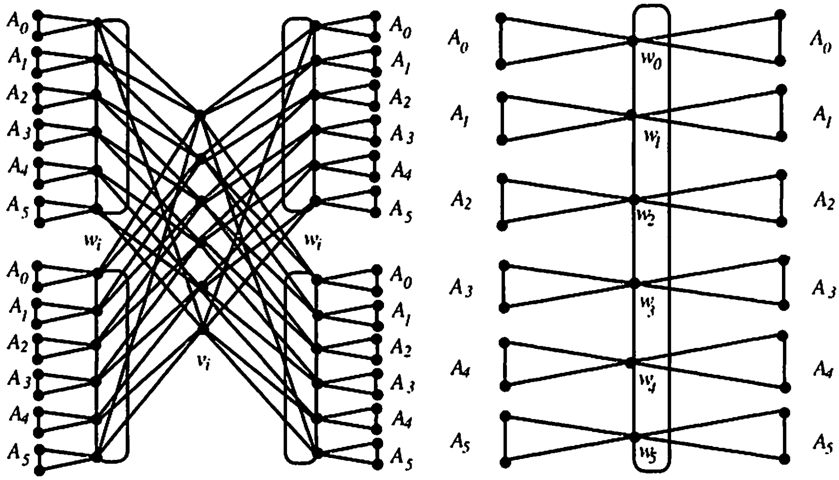
Step 1. Construction of a 4-regular graph H; $V(H) = \{x_i, y_i, z_i | i = 0, 1, \cdots, 6\}$, C^1 is a 7-cycle, $V(C^1) = \{y_0, y_1, \cdots, y_6\}$, C^2 is another 7-cycle, $V(C^2) = \{z_0, z_1, \cdots, z_6\}$, $N_H(x_i) = \{y_i, z_i, y_{i+1}, z_{i+1}\}$, Then $H[\{y_i, z_i, x_i, y_{i+1}, z_{i+1}\}]$ is a x_i-bowtie(The addition of indices is taken mod 7).

Step 2. Construction of H_1(see to Figure 1); add edges to H, let $y_0z_2x_4, y_2z_4x_6, y_5z_0x_2$ be triangles in H_1, $x_0y_3, x_1y_4, x_3y_6, x_5y_1, z_1z_5, z_3z_6 \in E(H_1)$. Let $A_0 = \{x_0, y_3\}, A_1 = \{x_1, y_4\}, A_2 = \{x_3, y_6\}, A_3 = \{x_5, y_1\}, A_4 = \{z_1, z_5\}, A_5 = \{z_3, z_6\}$.

Step 3. Construction of H_2; take a 6-cycle $C^3, V(C^3) = \{w_0, w_1, \cdots, w_5\}$, for $i = 0, 1, \cdots, 5$, join w_i to $V(A_i)$.

Step 4. Construction of G_1; take 6 new vertices $v_0, v_1, \cdots, v_5$, 4 disjoint copies of H_2, join v_i to w_i, w_{i+1} of all copies of H_2(The addition of indices is taken mod 6).

The first two steps of construction of G_2 is same to the construction of G_1, the third step is taking a 6-cycle $C^3, V(C^3) = \{w_0, w_1, \cdots, w_5\}$, 2 disjoint copies of H_1, join w_i to $V(A_i)$ of all copies of H_1.



Structure of G_1 Structure of G_2

Figure 2

It's easy to know that G_1 is a K_4^--free contraction critical 6-connected graph, but $d(v_i) = 8$. In addition, we can see w_i of G_2 is exactly in two triangles. This two examples tell us K_4^-- free contraction critical 6-connected graph need not to be 6-regular, the result '2' in Theorem D can't be improved further. We further study the property of K_4^-- free contraction critical k-connected graph, and get some local properties of each vertex in the following.

Theorem 2. *Let $k \geq 4$ be an even integer, and let G be a K_4^--free contraction critical k-connected graph. Then every vertex x in G is contained in at least two edge disjoint triangles, and such triangles both have a vertex of degree k other than x.*

By Theorem 2, we can estimate the number of vertex degree of k in a K_4^--free contraction critical k-connected graph.

Theorem 3. *Let $k \geq 4$ be an even integer, and let G be a K_4^--free contraction critical k-connected graph. Then $|V_k| \geq \frac{2}{k-1}|G|$.*

2. Properties of fragment

For the fragments, we have the following properties (see in [7]).

Lemma 1. *Let F and F' be two distinct fragments of G, $T = N(F), T' = N(F')$.*

(1) If $F \cap F' \neq \emptyset$, then $|F \cap T'| \geq |\overline{F'} \cap T|$, $|F' \cap T| \geq |\overline{F} \cap T'|$.

(2) If $F \cap F' \neq \emptyset \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cup F') = (T \cap T') \cup (T \cap F') \cup (F \cap T'), N(\overline{F} \cap \overline{F'}) = (T \cap T') \cup (T \cap \overline{F'}) \cup (\overline{F} \cap T')$.

(3)If $F \cap F' \neq \emptyset$ and $F \cap F'$ isn't a fragment, then $\overline{F} \cap \overline{F'} = \emptyset$ and $|F \cap T'| > |\overline{F'} \cap T|, |F' \cap T| > |\overline{F} \cap T'|$.

Mader[7] introduced some new concepts.

Definition 1. *Let $\mathcal{S}$ be a set of nonempty subset of $V(G)$, if minimal separating set T contain an element S in $\mathcal{S}$, then T-fragment F is called $\mathcal{S}$-fragment. An inclusion minimal $\mathcal{S}$- fragment is called an $\mathcal{S}$-end, a minimum $\mathcal{S}$- fragment is called an $\mathcal{S}$-atom.*

Mader[7] proved that $\mathcal{S}$-atom has following property.

Lemma 2. *[7] Let $\mathcal{S}$ be a set of subset of $V(G)$, A be an $\mathcal{S}$-atom in G. If there is a minimal separating set T, such that $T \cap A \neq \emptyset$ and $T \cap (A \cup N(A))$ contain an element of $S \in \mathcal{S}$, then $A \subseteq T$ and $|A| \leq \frac{1}{2}|T - N(A)|$.*

Let $\mathcal{S}_x = \{\{x, y\}|\ y \in N(x)\}$, then, by lemma 2, one can easily obtain the following

Corollary 1. *G be a contraction critical k-connected graph, $x \in V(G)$. If A be an $\mathcal{S}_x$-atom, thus $|A| \leq \frac{k-1}{2}$.*

3. Proof of Theorem 1

Let $k \geq 3$ be an odd integer, G be a K_4^--free k-connected graph. Since G doesn't contain K_4^-, for each $e = xy \in E(G)$, if e is contained in a triangle, $e = xy$ is contained in only one triangle, say xyz, we define $d'(e) := d(z)$. We let $R = \{e \in E(G)|\ e$ isn't contained in any triangle or $d'(e) > k\}$, let $\mathcal{R} = \{\{x, y\}|\ xy \in R\}$.

Assertion 1. *For every vertex $v \in V_k$, there exists an edge $e \in E(v)$ such that e is not contained in any triangle.*

Proof. For any $v \in V_k$, let $H = G[N(v)]$. Then $d_H(x) \leq 1$ for every vertex $x \in H$, since G does not contain K_4^-. Now, as $|H| = d(v) = k$ is odd, H has an isolate vertex, say u, then $e = uv \in E(v)$ is not contained in any triangle.

Take $\mathcal{S} = \mathcal{R}$ in Definition 1, we have several assertions in the following.

Assertion 2. *If $R \backslash E_c(G) \neq \emptyset$, let A be a $\mathcal{R}$-fragment. Then $|A| \geq k-1$.*

Proof. Let A be a $\mathcal{R}$-fragment, $e = uv \in R$ is contained in $N(A)$. If $|A| = 1$, then $e = uv$ is contained in a triangle and $d'(e) = k$, which contradicts to $e \in R$. So $|A| \geq 2$ and there exists an edge xy in A. Now G does not contain K_4^- which implies that $|N(x) \cap N(y)| \leq 1$, thus $|N(x) \cup N(y)| = |N(x)| + |N(y)| - |N(x) \cap N(y)| \geq 2k-1$. So $|A| \geq 2k-1-|N(A)| = k-1$.

Assertion 3. *If $R \setminus E_c(G) \neq \emptyset$, let A be a $\mathcal{R}$-fragment, $T = N(A)$. Then $R' = (E(A,T) \cup E(A)) \cap R \neq \emptyset$.*

Proof. Assume $R' = \emptyset$, then, by the definition of R, we know that every edge $e \in E(A,T) \cup E(A)$ is contained in a triangle and $d'(e) = k$. Now, by Assertion 1, $V_k \cap A = \emptyset$. Thus, for any $e = uv \in E(A)$, we have $d(u) > k, d(v) > k$. However, uv is in a triangle, say uvw, then $vw \in E(A,T) \cup E(A), d'(vw) = d(u) > k$, a contradiction.

Assertion 4. *If $R \setminus E_c(G) \neq \emptyset$, let A be a $\mathcal{R}$-end, $T = N(A)$. Then all the edges in $R' = (E(A,T) \cup E(A)) \cap R$ are k-contractible.*

Proof. By Assertion 3, $R' \neq \emptyset$. Assume an edge $e = uv \in R'$ is k-non contractible. Take a $\mathcal{R}$-fragment B such that $\{u,v\} \subseteq S = N(B)$. By Assertion 2, $|A| \geq k-1, |\overline{A}| \geq k-1, |B| \geq k-1, |\overline{B}| \geq k-1$. Let $H_1 = A \cap B, H_2 = A \cap S, H_3 = A \cap \overline{B}, Q_1 = B \cap T, Q_2 = S \cap T, Q_3 = \overline{B} \cap T, W_1 = \overline{A} \cap B, W_2 = \overline{A} \cap S, W_3 = \overline{A} \cap \overline{B}$. If $H_1 \neq \emptyset$, since A is a $\mathcal{R}$-end and $H_2 \cup Q_2 \cup Q_1$ contain some element of R, it follows that $|H_2 \cup Q_2 \cup Q_1| > k$. By Lemma 1(3), $|H_2| > |Q_3|, W_3 = \emptyset$. Similarly, if $H_3 \neq \emptyset$, we have $|H_2| > |Q_1|, W_1 = \emptyset$.

Now if $H_1 \neq \emptyset \neq H_3$, then $W_1 = \emptyset = W_3$. Thus $|W_2| = |\overline{A}| \geq k-1$, then $|Q_1| \geq k, |Q_3| \geq k$. So we get $k = |T| = |Q_1| + |Q_2| + |Q_3| \geq 2k$, a contradiction.

So, without loss of generality, we assume $H_1 \neq \emptyset, H_3 = \emptyset$. Then, similarly, we have $W_3 = \emptyset$, then $|Q_3| = |\overline{B}| \geq k-1$. By Lemma 1(3), $|H_2| > |Q_3| \geq k-1$, then $|H_2| \geq k$. By $|S| = |H_2| + |Q_2| + |W_2|$, we know that $|H_2| = k, Q_2 = \emptyset = W_2$. Lemma 1(1) show that $W_1 = \emptyset$. So $\overline{A} = W_1 \cup W_2 \cup W_3 = \emptyset$, a contradiction.

So we have $H_1 = \emptyset$ and $H_3 = \emptyset$, that's to say $A \subseteq S$. If $W_1 \neq \emptyset$, by Lemma 1(1), $|Q_1| \geq |H_2| \geq k-1$, if $W_1 = \emptyset$, then $|Q_1| = |B| \geq k-1$, so we always have $|Q_1| \geq k-1$ and, similarly, $|Q_3| \geq k-1$. Hence, $|T| = |Q_1| + |Q_2| + |Q_3| \geq 2k-2$, a contradiction. So our assumption is absurd, then all the edges in $R' = (E(A,T) \cup E(A)) \cap R$ are k-contractible.

Now we are ready to complete the proof of Theorem 1.

Let $R_1 = \{e|\ e \in E(G)$ is not contained in any triangle $\}$, $R_2 = \{e|e$ is contained in a triangle and $d'(e) > k\}$, then $R = R_1 \cup R_2$. Let $V_k = \{v|d_G(v) = k\}$, $V_{>k} = \{v|d_G(v) > k\}$.

We consider two cases whether $R \setminus E_c(G) \neq \emptyset$ or $R \subseteq E_c(G)$.

Case 1. $R \setminus E_c(G) \neq \emptyset$

Thus there exists a $\mathcal{R}$-fragment, take A as a $\mathcal{R}$-end such that $N(A) = T$ contain some element of $R \setminus E_c(G)$. So Assertion 2 implies that $|A| \geq k-1$. By Assertion 4, all edges in $R' = (E(A,T) \cup E(A)) \cap R$ are k-contractible.

For $v \in A$, let $\gamma_1(v) = |\{e|e \in E(v) \cap R_1)\}|$, let $\gamma_2(v) = |E(G[N(v)]) - E(T)|$.

For any $v \in V_k \cap A$, then, by Assertion 1, $\gamma_1(v) \geq 1$ and, hence, $\gamma_2(v) + \frac{1}{2}\gamma_1(v) \geq \frac{1}{2}$. For any $v \in V_{>k} \cap A$, then there exists a vertex $u \in A \cap N(v)$. Now if uv is contained in a triangle uvw such that $uw \in R_2$, then $\gamma_2(v) \geq 1$. So we also have $\gamma_2(v) + \frac{1}{2}\gamma_1(v) \geq \frac{1}{2}$. Hence $\gamma_2(v) + \frac{1}{2}\gamma_1(v) \geq \frac{1}{2}$ always holds for every vertex in A.

To estimate $|R'|$, we have

$$|R'| \geq \frac{1}{2}\sum_{v \in A} \gamma_1(v) + \sum_{v \in V_{>k} \cap A} \gamma_2(v)$$

$$= \frac{1}{2}\sum_{v \in V_k \cap A} \gamma_1(v) + \sum_{v \in V_{>k} \cap A} (\gamma_2(v) + \tfrac{1}{2}\gamma_1(v)).$$

In the next, we shall prove $|R'| \geq \frac{k+1}{2}$.

Subcase 1.1 $\quad A \setminus V_k \neq \emptyset$

That is to say $V_{>k} \cap A \neq \emptyset$ and, hence $|A| \geq k$. If for any $v \in V_{>k} \cap A, \gamma_2(v) \geq 1$, then $|R'| \geq \frac{1}{2}\sum_{v \in V_k \cap A} \gamma_1(v) + \sum_{v \in V_{>k} \cap A} 1 \geq \frac{|V_k \cap A|}{2} + \frac{|V_{>k} \cap A|}{2} = \frac{|A|}{2} \geq \frac{k}{2}$.

If there exists a vertex $v \in V_{>k} \cap A, \gamma_2(v) = 0$, then $|R'| \geq \frac{1}{2}\sum_{v \in V_k \cap A} \gamma_1(v) + \sum_{v \in V_{>k} \cap A} (\gamma_2(v) + \frac{1}{2}\gamma_1(v)) \geq \frac{1}{2}\sum_{v \in V_k \cap A} 1 + \sum_{v \in V_{>k} \cap A} \frac{1}{2} \geq \frac{|A|}{2} \geq \frac{k}{2}$.

Thus we always have $|R'| \geq \frac{k}{2}$ when $A \setminus V_k \neq \emptyset$. Since $|R'|$ is an integer and k is odd, we get $|R'| \geq \frac{k+1}{2}$.

Subcase 1.2 $\quad A \subseteq V_k$

Now for each vertex $v \in A, \gamma_1(v) \geq 1$. If $E(A) \cap R_1 = \emptyset$, then $E(v) \cap R_1 \subseteq E(A,T)$ and, hence, $|R'| \geq \sum_{v \in A} \gamma_1(v) \geq k-1 \geq \frac{k+1}{2}$. If there are $e = uv \in E(A) \cap R_1$, then $N(u) \cap N(v) = \emptyset$, $|A| \geq |N(u) \cup N(v)| - |N(A)| = k$. So $|R'| \geq \frac{1}{2}\sum_{v \in A} \gamma_1(v) \geq \frac{|A|}{2} \geq \frac{k}{2}$, at the same time, since $|R'|$ is an integer and k is odd, $|R'| \geq \frac{k+1}{2}$.

Thus we can say that $|R'| \geq \frac{k+1}{2}$. Take another $\mathcal{R}$-end A' in $\overline{A}$, in a similar way,we get all the edges in $R'' = (E(A',T) \cup E(A')) \cap R$ are

k-contractible, $|R''| \geq \frac{k+1}{2}$. Hence $|E_c(G)| \geq |R'| + |R''| \geq k+1$.

Case 2. $R \subseteq E_c(G)$

Now all the edges in R are k-contractible. For every vertex $v \in V$, let $\theta_1(v) = |\{e|e \in E(v) \cap R_1)\}|$, let $\theta_2(v) = |E(G[N(v)])|$.

By Assertion 1, for any $v \in V_k, \theta_1(v) \geq 1$. For any $v \in V_{>k}$, either $\theta_1(v) \geq 1$ or $\theta_2(v) \geq 2$. So, in a word, $\theta_1(v) + \theta_2(v) \geq 1$ for each vertex v. Considering the graph $(G, E(G) \cap R)$, we have

$2|R| \geq \sum_{v \in V_k} \theta_1(v) + \sum_{v \in V_{>k}} (\theta_1(v) + \theta_2(v)) \geq \sum_{v \in V_k} 1 + \sum_{v \in V_{>k}} 1 = |G|$. So $|R| \geq \frac{|G|}{2}$, thus $|E_c(G)| \geq \frac{|G|}{2}$.

From Case 1 and Case 2, we obtain $|E_c(G)| \geq min\{k+1, \frac{|G|}{2}\}$ and complete the proof of Theorem 1.

4. Proof of Theorem 2

Let G be a K_4^--free contraction critical k-connected graph with $k \geq 4$ and k is even, $\beta(x) = |E(x) \cap E^*|, S'_x = \{\{x,y\}|\ xy \in E(x) \setminus E^*\}$.

Assertion 5. *G be a K_4^--free contraction critical k-connected graph with $k \geq 4$ and k is even. Then $\beta(x) \geq 2$ for every vertex $x \in V(G)$.*

Proof. We first claim $\beta(x) \geq 1$. If not, there exists a vertex $x \in V(G)$, $\beta(x) = 0$, it means that all the edges in $E(x)$ are nontrivial. Let A be a S'_x-atom, by Corollary 1, $|A| \leq \frac{k-1}{2}$. Notice that $N(A)$ contains some element of $E(x)$, so if $|A| = 1$, then $E(x)$ contains a trivially edge, a contradiction. Hence $|A| \geq 2$ and there exists an edge xy in A. By the fact that G is K_4^--free and $|N(u) \cap N(v)| \leq 1$, we have $|N(u) \cup N(v)| \geq |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2k-1$. This implies $|A| \geq 2k-1-|N(A)| = k-1$. However, together with $|A| \leq \frac{k-1}{2}$ implies that $\frac{k-1}{2} \geq k-1$, a contradiction.

Now if $\beta(x) = 1$, assume xy is trivially, $z \in N(x) - \{y\}$. Take a k-cutset T containing $\{x,z\}$ and let F be a T- fragment, $\overline{F} = G - T - F$. Since $xy \in E(G)$, either $F \cap \{x,y\} = \emptyset$ or $\overline{F} \cap \{x,y\} = \emptyset$. Without loss of generality, we assume the former. Take an S'_x-end A such that $A \subseteq F$ and $N(A)$ contain an element of $E(x) \setminus E^*$, then $|A| \geq 2, |\overline{A}| \geq 2$. Clearly, both A and $\overline{A}$ contain an edge. Since G is K_4^--free, we have $|A| \geq k-1, |\overline{A}| \geq k-1$. As $A \cap N(x) \neq \emptyset$, we can take a vertex $w \in A \cap N(x)$. Take a k-cutset S containing $\{x,w\}$ and let B be an S- fragment, $\overline{B} = G - S - B$. Since $xw \in S$, now the fact $\beta(x) = 1$ show that $|B| \geq 2, |\overline{B}| \geq 2$. Then, again by the fact that G is K_4^--free, $|B| \geq k-1, |\overline{B}| \geq k-1$. Hence $|A| \geq k-1, |\overline{A}| \geq k-1, |B| \geq k-1, |\overline{B}| \geq k-1$. Similar to the proof of Assertion 4, we obtain a contradiction. This contradiction shows that $\beta(x) \geq 2$ for any $x \in V(G)$.

Assertion 6. *Let G be a K_4^--free contraction critical k-connected graph with $k \geq 4$ and k is even, $x \in V(G)$ such that $xy, xz \in E(x) \cap E^*, yz \in E(G)$. Then $\beta(x) \geq 3$.*

Proof. For otherwise, we may assume, by Assertion 5, $\beta(x) = 2$. That is to say, xy, xz are two trivially edges. Assume $w \in N(x) - \{y, z\}$, then xw is not trivially. Take a k-cutset T containing $\{x, w\}$ in G and F be a T- fragment, $\overline{F} = G - T - F$. Then either $E(F,T) \cap E^* \cap E(x) = \emptyset$ or $E(\overline{F},T) \cap E^* \cap E(x) = \emptyset$ since xyz is a triangle. Without loss of generality, we assume the former case. Now, as xw is not trivially, we have $|F| \geq 2$ and $|\overline{F}| \geq 2$. Further, both F and $\overline{F}$ contain some edges. So, similar to Assertion 5, $|F| \geq k-1, |\overline{F}| \geq k-1$. Take a $\mathcal{S}'_x$-end A in F. Similar to Assertion 5, we can get a contradiction.

We now complete the proof of Theorem 2, take a vertex $x \in V(G)$. If $\beta(x) = 2$, assume $xy, xz \in E^*$. By Assertion 6, $yz \notin E(G)$.

Notice that G doesn't contain K_4^-, so xy, xz is contained in two edge disjoint triangles xyu, xzv. Thus x is contained in a x-bowtie and $d(u) = d(v) = k$. If $\beta(x) \geq 3$, there are two edges xy, xz which are contained in two edge disjoint triangles. So we obtain the conclusion similarly.

5. Proof of Theorem 3

Let $k \geq 4$ be an even integer, and let G be K_4^--free contraction critical k-connected graph, E_t be the set of edges which is contained in a triangle. Denote $H = G[V_k], W = G[E(H) - E_t]$. For $e = xy$ in H, $e = xy$ is contained in at most one triangle since G doesn't contain K_4^-. If $e = xy \in E_t$, then let xyz be the triangle which contains xy, we define $v_e = z$. Let $p = \lfloor \frac{|H|}{2} \rfloor$ and let $U_i = \{v \in V | \ |E(G[N(v) \cap V_k])| = i\}$, for $i = 0, 1, \cdots, p$.

Let $E_i = \{e \in E(H) | \ v_e \in U_i\}$ if $i \geq 1$ and let $E_0 = E(W)$. Then $V(G) = \bigcup_{i=0}^{p} U_i, E(H) = \bigcup_{i=0}^{p} E_i$, and if $i \neq 0, |E_i| = i|U_i|$. Note that U_i, E_i maybe empty.

Let $\xi_1(v) = |\{e | e \in E(v, V_k) \cap E_t\}|$, $\xi_2(v) = |\{e | e \in E(v, V_k) \setminus E_t\}|$, $d_H(v) = |N(v) \cap V_k|$, then $d_H(v) = \xi_1(v) + \xi_2(v)$. By Assertion 6, if $v \in U_0$ then $\xi_1(v) \geq 2$; by Theorem 2, if $v \in U_1$ then $\xi_1(v) \geq 3$; if $v \in U_i (i \geq 2)$ then $\xi_1(v) \geq 2i \geq 2 + i$. This discussion implies

$$\begin{aligned} k|V_k| &= \textstyle\sum_{v \in V(G)} d_H(v) = \sum_{v \in V(G)} \xi_1(v) + \sum_{v \in V(G)} \xi_2(v) \\ &\geq \textstyle\sum_{i=0}^{p} \sum_{v \in U_i} \xi_1(v) + \sum_{v \in V(W)} \xi_2(v) \geq \sum_{i=0}^{p} \sum_{v \in U_i} (2+i) + 2|E_0| \\ &\geq 2 \textstyle\sum_{i=0}^{p} \sum_{v \in U_i} 1 + \sum_{i=1}^{p} \sum_{v \in U_i} i + 2|E_0| \\ &\geq 2 \textstyle\sum_{i=0}^{p} |U_i| + \sum_{i=1}^{p} i|U_i| + 2|E_0| \\ &\geq 2|V| + \textstyle\sum_{i=0}^{p} |E_i| \geq 2|V| + |E(H)|. \end{aligned}$$

By Theorem 2, $\delta(H) \geq 2$, then $2|E(H)| = \sum_{v \in V(H)} d_H(v) \geq \sum_{v \in V(H)} 2$ $\geq 2|H|$, so $|E(H)| \geq |H|$, thus $k|V_k| \geq 2|V| + |V_k|$, then we get $|V_k| \geq \frac{2}{k-1}|V| = \frac{2}{k-1}|G|$.

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