A sufficient condition for a graph to be a fractional (a, b, n)-critical deleted graph*

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Abstract: The toughness, as the parameter for measuring stability and vulnerability of networks, has been widely used in computer communication networks and ontology graph structure analysis. A graph G is called a fractional (a, b, n)-critical deleted graph if after deleting any n vertices from G, the resulting graph is still a fractional (a, b)-deleted graph. In this paper, we study the relationship between toughness and fractional (a, b, n)-critical deleted graph. A sufficient condition for a graph G to be a fractional (a, b, n)-critical deleted graph is determined.

Key words: toughness, fractional (g, f)-factor, fractional (g, f, n)-critical graph, fractional (a, b, n)-critical graph

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. The notations and terminologies used but undefined in this

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paper can be found in [1]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x \in V(G)$, we use $d_G(x)$ and $N_G(x)$ to denote the degree and the neighborhood of x in G, respectively. Let $\delta(G)$ denote the minimum degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$.

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional* (g,f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le \sum_{e \in E(x)} h(e) \le f(x)$. A graph G is called a *fractional* (g,f,n)*-critical graph* if after deleting any n vertices from G, the resulting graph still has a fractional (g,f)-factor. A graph G is called a *fractional* (g,f)*-deleted graph* if after deleting any edge e from G, the resulting graph still has a fractional (g,f)-factor. A graph G is called a *fractional* (g,f,n)*-critical deleted graph* if after deleting any n vertices from G, the resulting graph still is a fractional (g,f)-deleted graph. Furthermore, if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then fractional (g,f)-deleted graph, fractional (g,f,n)-critical graph, and fractional (g,f,n)-critical deleted graph are just fractional $[a,b]$-deleted graph, fractional (a,b,n)-critical graph, and fractional (a,b,n)-critical deleted graph, respectively. Several sufficient conditions for a graph to have fractional factor avoiding certain subgraphs can refer to [5], [6], [7], [8], [11], [12], [13], [14] and [15].

Let

$$\varepsilon(S,T) = \begin{cases} 2, & T \text{ is not independent set} \\ 1, & T \text{ is an independent set, and } e_G(T, V(G) \setminus (S \cup T)) \ge 1 \\ 0, & \text{Otherwise.} \end{cases}$$

The proof of our main result relies heavily on the following lemma.

Lemma 1 (Gao [3]) *Let G be a graph. Let a, b, n be non-negative integers such that $a \le b$. Then G is a fractional (a,b,n)-critical deleted graph if and* *only if*

$$b|S| - a|T| + d_{G-S}(T) \geq bn + \varepsilon(S,T) \tag{1}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n$.

The notion of *toughness* was first introduced by chvátal in [2]: if G is complete graph, $t(G) = \infty$; if G is not complete,

$$t(G) = \min\{\frac{|S|}{\omega(G-S)}\Big|\omega(G-S) \geq 2\}$$

and where $\omega(G-S)$ is the number of connected components of $G-S$.

Recently, Gao et al. [4] obtained a result that G is a fractional (a,b,n)-critical graph if $t(G) \geq \frac{ab-b+a-1}{b} + n$. It inspires us to think about the sufficient toughness condition for fractional (a,b,n)-critical deleted graphs. The contribution of our paper is to show that this bound of toughness is sufficient for a graph G to be a fractional (a,b,n)-critical deleted graph. Our main result to be proved in next section can be stated as follows.

Theorem 2 *Let G be a graph and let a, b be two nonnegative integers satisfying $2 \leq a \leq b$. Let n be a non-negative integer. $|V(G)| \geq n+a+2$ if G is complete. If $t(G) \geq \frac{ab-b+a-1}{b} + n$, then G is a fractional (a,b,n)-critical deleted graph.*

To prove Theorem 2, we need the following lemmas.

Lemma 3 (Chvátal [2]) *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

Lemma 4 (Liu and Zhang [9]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ *and* $i_j = |I \cap T_j|$ *for every* $j = 1, \ldots, k-1$.

The lemma below can be deduced from Lemma 2.2 in [9].

Lemma 5 (Liu and Zhang [9]) *Let* G *be a graph and let* $H = G[T]$ *such that* $d_G(x) = k-1$ *for every* $x \in V(H)$ *and no component of* H *is isomorphic to* K_k *where* $T \subseteq V(G)$ *and* $k \geq 2$. *Then there exist an independent set* I *and the covering set* $C = V(H) - I$ *of* H *satisfying*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2},$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$, $1 \leq i \leq k$ *and* $\sum_{i=1}^{k} |I^{(i)}| = |I|$.

2 Proof of Theorem 2

If G is complete, due to $|V(G)| \geq n+a+2$, clearly, G is a fractional (a,b,n)-critical deleted graph. In the following, we assume that G is not complete.

Suppose that G satisfies the conditions of Theorem 2, but is not a fractional (a,b,n)-critical deleted graph. According to Lemma 1 and $\varepsilon(S,T) \leq 2$, there exist disjoint subsets S and T of $V(G)$ such that

$$b|S| - a|T| + d_{G-S}(T) \leq bn+1 \tag{2}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$.

Claim 1 $d_{G-S}(x) \leq a-1$ *for any* $x \in T$.

Proof. If $d_{G-S}(x) \geq a$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T. □

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_a and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_a.

If $|V(H)| = 0$, then by (2), we deduce

$$b|S| \leq a|T_0| + al + bn + 1$$

or

$$|S| \leq \frac{a(|T_0| + l) + bn + 1}{b}.$$

Clearly, $\omega(G - S) = |T_0| + l \geq 1$. If $\omega(G - S) > 1$, then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{a(|T_0|+l)+bn+1}{b(|T_0|+l)} < \frac{a+bn+1}{b}$, which contradicts $t(G) \geq \frac{ab-b+a-1+bn}{b}$ and $b \geq a \geq 2$. If $\omega(G - S) = 1$, then $|T_0| + l = 1$. Hence $d_{G-S}(x) = a - 1$ or $d_{G-S}(x) = 0$ for $x \in V(G) \setminus S$. Since $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G)$, we have $2t(G) \leq a - 1 + |S| \leq a - 1 + \frac{a+bn+1}{b}$, which contradicts $t(G) \geq \frac{ab-b+a-1+bn}{b}$ and $b \geq a \geq 2$.

Now we consider that $|V(H)| > 0$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = a - 1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 5, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq \sum_{i=1}^{a}(a - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \tag{3}$$

and

$$|C_1| \leq \sum_{i=1}^{a}(a - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{4}$$

where $I^{(i)} = \{x \in I_1, d_{H_1}(x) = a - i\}$, $1 \leq i \leq a$ and $\sum\limits_{i=1}^{a} |I^{(i)}| = |I_1|$. Let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq a - 1$. Each component of H_2 has a vertex of degree at most $a - 2$ in $G - S$ by the definitions of H and H_2. According to Lemma 4, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-2)(a-j)i_j, \tag{5}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, a-1$. Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We infer

$$|U| \leq | S| + |C_1| + \sum_{j=1}^{a-1} j i_j + \sum_{i=1}^{a}(i-1)|I^{(i)}| \tag{6}$$

and

$$\omega(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{a-1} i_j, \tag{7}$$

where $t_0 = |T_0|$. Let $t(G) = t$. Then when $\omega(G-S) > 1$, we have

$$|U| \geq t\omega(G-S), \tag{8}$$

and it is also hold when $\omega(G-S) = 1$. In terms of (6), (7) and (8), we get

$$|S| + |C_1| \geq \sum_{j=1}^{a-1}(t-j)i_j + t(t_0+l) + t|I_1| - \sum_{i=1}^{a}(i-1)|I^{(i)}|. \tag{9}$$

In view of $a|T| - d_{G-S}(T) \geq b|S| - bn - 1$, we obtain

$$at_0 + al + |V(H_1)| + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j \geq b|S| - bn - 1$$

Combining with (9), we deduce

$$\begin{aligned} & at_0 + al + |V(H_1)| + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j + b|C_1| + bn + 1 \\ \geq \; & \sum_{j=1}^{a-1}(bt-bj)i_j + bt(t_0+l) + bt|I_1| - b\sum_{i=1}^{a}(i-1)|I^{(i)}|. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&|V(H_1)| + \sum_{j=1}^{a-1}(a-j)c_j + b|C_1| \\
\geq\ & \sum_{j=1}^{a-1}(bt - bj - a + j)i_j + (bt-a)(t_0+l) + bt|I_1| \\
& -b\sum_{i=1}^{a}(i-1)|I^{(i)}| - bn - 1. \qquad (10)
\end{aligned}
$$

By (3) and (4), we have

$$
|V(H_1)| + b|C_1| \leq \sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| - \frac{(b+1)|I^{(1)}|}{2}. \qquad (11)
$$

Using (5), (10) and (11), we get

$$
\begin{aligned}
&\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| \\
\geq\ & \sum_{j=1}^{a-1}(bt - bj - a + j)i_j + bt|I_1| + \frac{(b+1)|I^{(1)}|}{2} \\
& -b\sum_{i=1}^{a}(i-1)|I^{(i)}| + (bt-a)(t_0+l) - bn - 1. \qquad (12)
\end{aligned}
$$

The following proof splits into two cases by the value of $t_0 + l$.

Case 1. $t_0 + l \geq 1$. By $bt \geq ab - b + a - 1 + bn$, we have $(bt-a)(t_0 + l) - bn - 1 \geq ab - b - 2 = b(a-1) - 2 \geq 0$. Thus, (12) becomes

$$
\begin{aligned}
&\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| \\
\geq\ & \sum_{j=1}^{a-1}(bt - bj - a + j)i_j + bt|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}|.
\end{aligned}
$$

And then, at least one of the following two cases must hold.

Subcase 1.1. $\sum_{j=1}^{a-1}(a-2)(a-j)i_j \geq \sum_{j=1}^{a-1}(bt - bj - a + j)i_j$.

Then, there is at least one j such that

$$
(a-2)(a-j) \geq bt - bj - a + j,
$$

which implies

$$\begin{aligned} & ab - b + a - 1 + bn \le bt \le (a-2)(a-j) + bj + a - j \\ = \quad & a(a-2) + (b-a+1)j + a \le ab - b + a - 1. \end{aligned}$$

Hence, $\sum_{j=1}^{a-2} i_j = 0$, which contradicts the definition of H_2 and the choice of I_2 (see the proof of Lemma [9] such that $\sum_{j=1}^{a-2} i_j \neq 0$).

Subcase 1.2.

$$\begin{aligned} & \sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| \\ \ge \quad & bt|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}| \\ \ge \quad & (ab - b + a - 1 + bn)|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}| \\ \ge \quad & (ab - b + a - 1)|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}|. \end{aligned}$$

This implies

$$\sum_{i=2}^{a}(-i+2)|I^{(i)}| + (-\frac{b}{2} + \frac{1}{2})|I^{(1)}| \ge 0.$$

If $t_0 + l \ge 2$ or $(a, b) \neq (2, 2)$, then by $(bt - a)(t_0 + l) - bn - 1 \ge 1$ we get

$$\sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| \ge bt|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}| + 1,$$

and

$$\sum_{i=2}^{a}(-i+2)|I^{(i)}| + (-\frac{b}{2} + \frac{1}{2})|I^{(1)}| \ge 1,$$

a contradiction.

If $n \ge 1$, we obtain

$$\begin{aligned} & \sum_{i=1}^{a}(ab - bi + a - i + 1)|I^{(i)}| \\ \ge \quad & (ab - b + a - 1)|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}| + 2. \end{aligned}$$

Hence, we infer

$$\sum_{i=2}^{a}(-i+2)|I^{(i)}|+(-\frac{b}{2}+\frac{1}{2})|I^{(1)}|\geq 2,$$

a contradiction.

In conclusion, we have $n=0$ and $(a,b)=(2,2)$. Then the result follows from the main result in [10] which determined that G is fractional 2-deleted graph if $t(G)\geq\frac{3}{2}$.

Case 2. $t_0+l=0$. In this case, (12) becomes

$$\begin{aligned}&\sum_{j=1}^{a-1}(a-2)(a-j)i_j+\sum_{i=1}^{a}(ab-bi+a-i+1)|I^{(i)}|\\ \geq\ &\sum_{j=1}^{a-1}(bt-bj-a+j)i_j+bt|I_1|+\frac{(b+1)|I^{(1)}|}{2}\\ &-b\sum_{i=1}^{a}(i-1)|I^{(i)}|-bn-1.\end{aligned}\tag{13}$$

Subcase 2.1. $|I_1|=0$. In this subcase, (13) becomes

$$\sum_{j=1}^{a-1}((a-2)(a-j)-(bt-bj-a+j))i_j+bn+1\geq 0.\tag{14}$$

Let

$$\begin{aligned}h_j &= (a-2)(a-j)-(bt-bj-a+j)\\ &= a^2+(b-a+1)j-a-bt\\ &\leq a^2+(b-a+1)j-a-b\cdot\frac{ab-b+a-1+bn}{b}\\ &= a^2+(b-a+1)j-ab-2a+b+1-bn.\end{aligned}$$

Then $\max\{h_j\}=h_{a-1}=-bn$ and the second largest value of h_j is $h_{a-2}=-bn-b+a-1$. Analysis the proof of Lemma 4 in [9], for each connected component of H_2, choose a vertex with the smallest degree and add it to I_2. Hence, by the definition of H_2, we confirm that H_2 is connected (only one connected component), each vertex in I_2 has degree $a-1$ in $G-S$ except one vertex has degree $a-2$ in $G-S$, and $b=a$. This fact implies

$$|C_2| \leq (a-2)+(|I_2|-1)(a-1-1)=|I_2|(a-2),$$

$$|T| \leq| I_2|(a-1),$$

and

$$|S| \leq \frac{|T|+1+bn}{a} \leq |I_2|+\frac{1-|I_2|}{a}+n.$$

If $|I_2|=1$, then $|S|=1+n$, $\delta(G) \leq |S|+(a-1)=a+n$, which contradicts $\delta(G) \geq 2t(G) > a+n$. Hence, $|I_2| \geq 2$ and

$$\begin{aligned} a-\frac{1}{a}+n \leq t(G) &\leq \frac{|U|}{\omega(G-U)} \leq \frac{\frac{1-|I_2|}{a}+|I_2|+|I_2|(a-2)+n}{|I_2|} \\ &= (a-1-\frac{1}{a})+\frac{1}{a|I_2|}+\frac{n}{|I_2|}, \end{aligned}$$

where $U=S \cup C_2 \cup (N_G(I_2) \cap W)$. This reveals $n(1-\frac{1}{|I_2|}) \leq \frac{1}{a|I_2|}-1$, which contradicts $a \geq 2$ and $|I_2| \geq 2$.

Subcase 2.2. $|I_2|=0$. In this subcase, (13) becomes

$$\sum_{i=1}^{a}(ab-bi+a-i+1)|I^{(i)}|-bt|I_1|-\frac{(b+1)|I^{(1)}|}{2}+b\sum_{i=1}^{a}(i-1)|I^{(i)}|+bn+1 \geq 0.$$

This implies

$$\sum_{i=2}^{a}(-i+2)|I^{(i)}|+(-\frac{b}{2}+\frac{1}{2})|I^{(1)}|+1 \geq 0.$$

Then we get $\sum_{i=4}^{a}|I^{(i)}|=0$, $|I^{(3)}| \leq 1$ and $|I^{(1)}| \leq 2$. Now, we consider following three subcases:

Subcase 2.2.1. $|I^{(1)}|=1$. In this subcase, we have $\sum_{i=3}^{a}|I^{(i)}|=0$. By analyzing proof process of Lemma 2.2 in [9]: "for each vertex $x \in I_n$ and $d_{H_n}(x)=k-1$, there exists a vertex $y \in I_n$ such that $N_{H_n}(x) \cap N_{H_n}(y) \neq \emptyset$", we obtain $|I_1| \geq 2$,

$$|T| \leq (a-1)+(|I_1|-1)(a-1)=|I_1|(a-1),$$

$$|S| \leq \frac{|T|+1+bn}{b} \leq \frac{|I_1|(a-1)+1+bn}{b},$$

and

$$\begin{aligned} |U| &\leq |S|+|C_1|+\sum_{i=1}^{a}(i-1)|I^{(i)}| \\ &\leq \frac{|I_1|(a-1)+1+bn}{b}+|I_1|(a-1)-|I_1|+(|I_1|-1) \\ &= \frac{|I_1|(a-1)+1+bn}{b}+|I_1|(a-1)-1. \end{aligned}$$

Thus,

$$\frac{ab-b+a-1+bn}{b} \leq t(G) \leq \frac{|U|}{\omega(G-U)} \leq \frac{\frac{|I_1|(a-1)+1+bn}{b}+|I_1|(a-1)-1}{|I_1|}.$$

This implies $bn(|I_1|-1) \leq 1-b$, a contradiction.

Subcase 2.2.2. $|I^{(1)}|=2$. In this subcase, $\sum\limits_{i=3}^{a}|I^{(i)}|=0$. We can get a contradiction via the discussion similar as Subcase 2.2.1.

Subcase 2.2.3. $|I^{(1)}|=0$. In this subcase, we have $\sum_{i=4}^{a}|I^{(i)}|=0$ and $|I^{(3)}| \leq 1$. If $|I_1|=1$, then $|S| \leq \frac{(a-1)+bn+1}{b}$. Thus, we infer

$$\frac{(a-1)+bn+1}{b}+a-1 \geq a-1+|S| \geq \delta(G) \geq 2t(G) \geq \frac{2(ab-b+a-1+bn)}{b}.$$

A contradiction. Hence, $|I_1| \geq 2$. Let $Y = N_G(I_1) \cap W$.

If there is a vertex $y \in Y$ such that y only adjacent to one vertex in I_1. Reset

$$U = S \cup C_1 \cup (N_G(I_1) \cap (W-\{y\})).$$

Then, we have

$$|U| \leq |S|+|I_1|(a-1)-1 \leq \frac{|I_1|(a-1)+1+bn}{b}+|I_1|(a-1)-1,$$

and by $|I_1| \geq 2$,

$$\frac{ab-b+a-1+bn}{b} \leq t(G) \leq \frac{|U|}{\omega(G-U)} \leq \frac{\frac{|I_1|(a-1)+1+bn}{b}+|I_1|(a-1)-1}{|I_1|}.$$

This implies $bn(|I_1|-1) \leq 1-b$, a contradiction.

If each vertex in Y adjacent to at least two vertices in I_1. Let $U = S \cup C_1 \cup (N_G(I_1) \cap W)$, we get,

$$|U| \leq |S| + |I_1|(a-2) + \frac{|I_1|}{2} \leq \frac{|I_1|(a-1)+1+bn}{b} + |I_1|(a-2) + \frac{|I_1|}{2},$$

and by $|I_1| \geq 2$,

$$\begin{aligned} \frac{ab-b+a-1+bn}{b} &\leq t(G) \leq \frac{|U|}{\omega(G-U)} \\ &\leq \frac{\frac{|I_1|(a-1)+1+bn}{b} + |I_1|(a-2) + \frac{|I_1|}{2}}{|I_1|}. \end{aligned}$$

That is to say, $bn(|I_1|-1) \leq 1 - \frac{b|I_1|}{2}$, which contradicts $b \geq 2$ and $|I_1| \geq 2$.

Subcase 2.3. $|I_1| \neq 0$ **and** $|I_2| \neq 0$. From what we have discussed in Subcase 2.1, we get $\sum_{j=1}^{a-1}(a-2)(a-j)i_j \leq \sum_{j=1}^{a-1}(bt-bj-a+j)i_j + bn + 1$. Then, we deduce

$$\sum_{i=1}^{a}(ab-bi+a-i+1)|I^{(i)}| \geq bt|I_1| + \frac{(b+1)|I^{(1)}|}{2} - b\sum_{i=1}^{a}(i-1)|I^{(i)}|.$$

This implies

$$\sum_{i=2}^{a}(-i+2)|I^{(i)}| + (-\frac{b}{2}+\frac{1}{2})|I^{(1)}| \geq 0.$$

Thus, we have $\sum_{i=4}^{a}|I^{(i)}| = 0$, $|I^{(3)}| \leq 1$, $|I^{(1)}| \leq 2$ and $n = 0$ by what we have discussed in Subcase 1.2. We only to discuss the situation of $|I^{(1)}| = 0$, other two situations for $|I^{(1)}| = 1$ and $|I^{(1)}| = 2$ can be considered in a similar way.

Under the condition of $|I^{(1)}| = 0$, we are sure that $\sum_{i=4}^{a}|I^{(i)}| = 0$ and $|I^{(3)}| \leq 1$. We infer that

$$|T| \leq |I_1|(a-1) + |I_2|(a-1) = (a-1)(|I_1|+|I_2|),$$

$$|S| \leq \frac{|T|+1}{b} \leq \frac{(|I_1|+|I_2|)(a-1)+1}{b}.$$

Since $|I_1| + |I_2| \geq 2$, we get

$$\frac{ab-b+a-1}{b} \leq t(G) \leq \frac{|U|}{\omega(G-U)} \leq \frac{|S| + |I_2|(a-2) + |I_1|(a-1)}{|I_1|+|I_2|},$$

where $U = S \cup C_1 \cup (N_G(I_1) \cap W) \cup C_2 \cup (N_G(I_2) \cap W)$. Then, we get

$$(ab-b+a-1)(|I_1|+|I_2|) \leq (|I_1|+|I_2|)(a-1)+1+(ab-2b)(|I_1|+|I_2|)+|I_1|b.$$

This implies $\frac{1}{b} \geq |I_2|$, a contradiction.

We complete the proof of the theorem. □

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