The (-1)-critically duo-free tournaments

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Abstract

Given a tournament $T = (V, A)$, a subset X of V is an interval of T provided that for any $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\emptyset$, $\{x\}$ $(x \in V)$ and V are intervals of T, called trivial intervals. A two-element interval of T is called a duo of T. The tournaments which do not admit any duo, are called duo-free tournaments. A vertex x of a duo-free tournament is d-critical if $T - x$ has at least one duo. In 2005, J.F. Culus and B. Jouve [5] characterized the duo-free tournaments, all the vertices of which are d-critical, called tournaments without acyclic interval. In this paper, we characterize the duo-free tournaments which admit exactly one non-d-critical vertex, called (-1)-critically duo-free tournaments.
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1 Basic definitions

A *graph* (or *digraph*) $G := (V(G), A(G))$, more simply (V, A), consists of a finite *vertex* set V with an *arc* set A of ordered pairs of distinct vertices of G. *The order* (*or the cardinality*) of G, denoted by $|V(G)|$, is that of $V(G)$.

The notion of *isomorphism*, *subgraph* and *embedding* are defined in the following manner. First, let $G := (V, A)$ and $G' := (V', A')$ be two graphs. A one-to-one correspondence f from V onto V' is an *isomorphism from* G *onto* G' provided that for $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The graphs G and G' are then said to be *isomorphic*, which is denoted by $G \simeq G'$. Second, given a graph $G := (V, A)$, with each subset X of V is associated the *subgraph* $G[X] := (X, A \cap (X \times X))$ of G *induced* by X. For each subset X of V (resp. $x \in V$), the subgraph $G[V \setminus X]$ (resp. $G[V \setminus \{x\}]$) is denoted by $G - X$ (resp. $G - x$). For two graphs G and G', if G' is isomorphic to a subgraph of G, then we say that G' *embeds into* G.

A *symmetric graph* is a graph G satisfying: for $x \neq y$ in $V(G)$, if $(x, y) \in A(G)$, then $(y, x) \in A(G)$. For example for every integer $n \geq 2$, the *path* P_n is the symmetric graph defined on $\{0, 1, \ldots, n-1\}$ as follows: for all $i, j \in \{0, 1, \ldots, n-1\}$, (i, j) is an arc of P_n if and only if $|i - j| = 1$. Besides, for each integer $n \geq 3$ the *cycle* C_n of length n is the symmetric graph obtained from the path P_n by adding arcs $(0, n-1)$ and $(n-1, 0)$. A *cycle* is a graph which is isomorphic to C_n, for some $n \geq 3$. Given a symmetric graph G and two vertices $x, y \in V(G)$, the vertex y is said to be a *neighbor* of x (in G) if $(x, y) \in A(G)$. We denote by $N_G(x)$ the set of neighbors of x in G. The *degree* of x is $d_G(x) := |N_G(x)|$. When $N_G(x) = \emptyset$, the vertex x is said to be an *isolated* vertex of G. An *empty graph* is a symmetric graph which all the vertices are isolated.

Let G be a symmetric graph, an equivalence relation $\mathcal{R}$ is defined on $V(G)$ as follows: for all $x \neq y$ in $V(G)$, $x\mathcal{R}y$ if there is a sequence $x_0 := x, \cdots, x_n := y$ of vertices of G such that for all $i \in \{0, \cdots, n-1\}$, $(x_i, x_{i+1}) \in A(G)$. The equivalence classes of $\mathcal{R}$ are called *connected components* of G. A *non trivial connected component of* G is a connected component of G which is not reduced to a singleton. We say that G is a *connected graph* if it admits a unique connected component.

A graph $T := (V, A)$ is said to be a *tournament* if for $x \neq y$ in V, $(x, y) \in A$ if and only if $(y, x) \notin A$. For two distinct vertices x and y of a tournament T, $x \longrightarrow y$ means that $(x, y) \in A(T)$. A *transitive* tournament or a *total order* is a tournament T such that for $x, y, z \in V(T)$, if $x \longrightarrow y$ and $y \longrightarrow z$, then $x \longrightarrow z$. If x and y are two distinct vertices of a total order, the notation $x < y$ means that $x \longrightarrow y$. Let T be a tournament. For all $x, y, z \in V(T)$ we say that x *separates* y and z if $y \longrightarrow x \longrightarrow z$ or $z \longrightarrow x \longrightarrow y$. For $x \in V(T)$ and $Y \subset V(T)$, $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) means that for any $y \in Y$, $x \longrightarrow y$ (resp. $y \longrightarrow x$). Let X and Y be two disjoint subsets of $V(T)$, $X \longrightarrow Y$ means that for any $(x, y) \in X \times Y$, $x \longrightarrow y$. The *dual* of a tournament $T = (V, A)$ is the tournament obtained from T by reversing all its arcs. This tournament is denoted by $T^* := (V, A^*)$, where $A^* := \{(x, y) : (y, x) \in A\}$.

2 Indecomposable tournaments

Let T be a tournament. We introduce an equivalence relation on the ordered pairs of distinct vertices of T, denoted by $\equiv_T$ (or $\equiv$) and defined as follows: for $x \neq y$ in $V(T)$ and $u \neq v$ in $V(T)$, $(x, y) \equiv_T (u, v)$ (or $(x, y) \equiv (u, v)$) if $(x, y) = (u, v)$ or $|\{(x, y), (u, v)\} \cap A| \neq 1$. Given X a subset of $V(T)$ and $x \in V(T) \setminus X$, the notation $x \sim X$ means that for all $y, z \in X$, $(x, y) \equiv (x, z)$, otherwise we notate $x \not\sim X$. The notation $X \sim Y$ means that for all $x, x' \in X$ and $y, y' \in Y$, $(x, y) \equiv (x', y')$, otherwise we notate $X \not\sim Y$. A subset I of $V(T)$ is an *interval* [7, 9, 10] (or a *clan* [6], or an *homogenous subset* [8]) of T provided that for all $a, b \in I$ and $x \in V \setminus I$, $(a, x) \equiv (b, x)$. For example $\emptyset$, $\{x\}$ where $x \in V(T)$, and $V(T)$ are intervals of T, called *trivial intervals*. A tournament is then said to be *indecomposable* ([9, 10]) (or *primitive* [6]) if all its intervals are trivial, otherwise it is said to be *decomposable*. Given a tournament T, a partition of $V(T)$, all the elements of which are intervals of T, is called an *interval partition* of T.

Now we introduce some notations and recall some results on indecomposable tournaments.

Definition 2.1 *([6]) Given a tournament $T := (V, A)$, with each subset X of V, such that $|X| \geq 3$ and $T[X]$ is indecomposable, are associated the following subsets of $V \setminus X$:*

- *$[X] := \{x \in V \setminus X : x \sim X\}$.*
- *$X(u) := \{x \in V \setminus X : \{u, x\}$ is an interval of $T[X \cup \{x\}]\}$ for every $u \in X$.*
- *$Ext(X) := \{x \in V \setminus X : T[X \cup \{x\}]$ is indecomposable$\}$.*

Lemma 2.2 *([6]) Let $T := (V, A)$ be a tournament and X be a subset of V such that $|X| \geq 3$ and $T[X]$ is indecomposable.*

- *The family $\{X(u) : u \in X\} \cup \{Ext(X), [X]\}$ constitutes a partition of $V \setminus X$.*
- *Given $u \in X$, for all $x \in X(u)$ and for all $y \in V \setminus (X \cup X(u))$, if $T[X \cup \{x, y\}]$ is decomposable, then $\{u, x\}$ is an interval of $T[X \cup \{x, y\}]$.*
- *For every $x \in [X]$ and for every $y \in V \setminus (X \cup [X])$, if $T[X \cup \{x, y\}]$ is decomposable, then $X \cup \{y\}$ is an interval of $T[X \cup \{x, y\}]$.*
- *Given $x, y \in Ext(X)$, with $x \neq y$, if $T[X \cup \{x, y\}]$ is decomposable, then $\{x, y\}$ is an interval of $T[X \cup \{x, y\}]$.*

Lemma 2.3 *([10]) If $T := (V, A)$ is an indecomposable tournament such that $|V| \geq 7$, then there are distinct $x, y \in V$ such that $T - \{x, y\}$ is indecomposable.*

Given a tournament T, a *duo* [1] of T is a two-element interval of T. The tournaments which do not admit any duo are called *duo-free tournaments.*

3 The critical vertices of an indecomposable tournament

Let T be an indecomposable tournament. A vertex x of T is *critical* if $T - x$ is decomposable. The tournament T is said to be *critical* if all its vertices are critical. In order to present our main result and the characterization of the critical tournaments due to J.H. Schmerl and W.T. Trotter [10], we introduce the tournaments T_{2h+1}, U_{2h+1} and V_{2h+1} defined on $\{0, 1, \ldots, 2h\}$, with $h \geq 1$, as follows :

- $T_{2h+1}[\{0, 1, \ldots, h\}] = U_{2h+1}[\{0, 1, \ldots, h\}] = 0 < 1 < \cdots < h$, $T_{2h+1}[\{h+1, \ldots, 2h\}] = (U_{2h+1})^\star[\{h+1, \ldots, 2h\}] = h+1 < \cdots < 2h$. For any $i \in \{0, 1, \ldots, h-1\}$, if $j \in \{i+1, \ldots, h\}$ and $k \in \{0, 1, \ldots, i\}$, then $(j, i+h+1)$ and $(i+h+1, k)$ belong to $A(T_{2h+1})$ and $A(U_{2h+1})$.
- $V_{2h+1}[\{0, \ldots, 2h-1\}] = 0 < \cdots < 2h-1$ and for any $i \in \{0, \ldots, h-1\}$, $(2i+1, 2h)$ and $(2h, 2i)$ belong to $A(V_{2h+1})$.

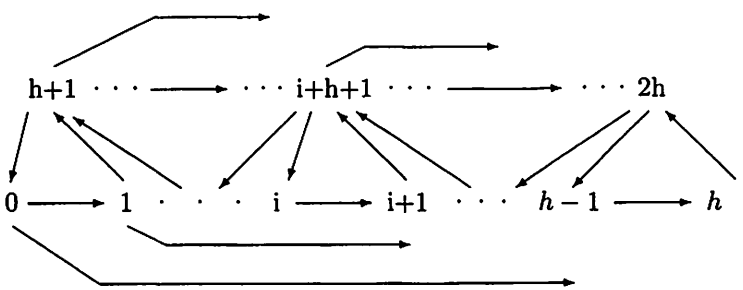

Figure 1: T_{2h+1}.

Theorem 3.1 *([10]) Up to isomorphism, the critical tournaments of cardinality* ≥ 5 *are the tournaments* T_{2h+1}, U_{2h+1} *and* V_{2h+1}, *where* $h \geq 2$.

A tournament T of order ≥ 5 is said to be (-1)-*critical* when it admits exactly one non-critical vertex [2]. The (-1)-critical tournaments were characterized by H. Belkhechine, I. Boudabbous and J. Dammak [2]. In particular, we have the following lemma.

Lemma 3.2 *([2]) The order of any* (-1)-*critical tournament is odd and* ≥ 7.

4 The d-critical vertices of a duo-free tournament

A vertex x of a duo-free tournament T is *d-critical* if $T - x$ admits at least one duo. A duo-free tournament, of order ≥ 3, is *critically duo-free* (*or critically without duo*) if all its vertices are d-critical. We generalize this definition by saying that a duo-free tournament of order ≥ 4 is $(-k)$-*critically duo-free*, or $(-k)$-*critically without duo*, when it admits exactly k non-d-critical vertices.

The following theorem due to J. F. Culus and B. Jouve [5] characterizes the critically duo-free tournaments.

Theorem 4.1 *([5]) A tournament* T *is critically duo-free if and only if* T *admits an interval partition* P *such that for every* $X \in P$, $T[X]$ *is isomorphic to some* T_{2h+1}.

A *diamond* is a tournament on 4 vertices admitting only one interval of cardinality 3. The *center* of a diamond δ is the unique vertex $a \in V(\delta)$ satisfying $a \sim (V(\delta) - \{a\})$. Up to isomorphism, there are exactly two diamonds δ^+ and $\delta^- = (\delta^+)^*$, where δ^+ is the tournament defined on $\{0,1,2,3\}$ by $\delta^+(\{0,1,2\}) = T_3$ and $\{0,1,2\} \rightarrow 3$. A tournament isomorphic to δ^+ (resp. isomorphic to δ^-) is said to be a *positive diamond* (resp. *negative diamond*).

A *double-diamond* is a tournament Δ on 7 vertices admitting an interval partition $P = \{X, Y\}$ such that $\Delta(X) \simeq T_3$, $\Delta(Y)$ is a positive diamond and $Y \rightarrow X$. The *center* of a double-diamond Δ is the unique vertex $a \in V(\Delta)$ satisfying: for all $x \neq y$ in $V(\Delta) \setminus \{a\}$, $\Delta[\{a, x, y\}]$ is a total order.

In the present paper, we characterize the (-1)-critically duo-free tournaments. When a vertex a is the only non d-critical vertex of a (-1)-critically duo-free tournament, we say that this tournament is (-1)-*critically duo-free at* a, or (-1)-*critically without duo at* a. The following theorem is the main result.

Theorem 4.2 *A tournament T, of order ≥ 4, is (-1)-critically duo-free at a if and only if T admits an interval partition P such that there is a unique $X \in P$ satisfying : $X = \{a\}$, or $T[X]$ is a diamond of center a, or $T[X]$ is a double-diamond of center a and for any $Y \in P\setminus\{X\}$, $T[Y]$ is isomorphic to some T_{2h+1}.*

5 The d-indecomposability graph of a duo-free tournament

By analogy with the indecomposability graph of an indecomposable graph [3], the notion of *d-indecomposability graph* was introduced by Y. Boudabbous and A. Salhi [4] in the following manner. With each duo-free tournament $T := (V, A)$ is associated its *d-indecomposability graph* $D(T)$ defined on V as follows: For all $x \neq y$ in V, (x, y) is an arc of $D(T)$ if $T - \{x, y\}$ has no duo. Notice that $D(T)$ is a symmetric graph and $D(T) = D(T^*)$. This graph is an important tool in the present study concerning the (-1)-critically duo-free tournaments. We recall the following result.

Lemma 5.1 *([4]) Let T be a critically duo-free tournament and K be a connected component of $D(T)$, then $D(T)[K]$ is a cycle of odd length. Moreover K is an interval of T and $T[K]$ is isomorphic to some T_{2h+1}.*

We begin with establishing the following Lemma.

Lemma 5.2 *Let $T := (V, A)$ be a duo-free tournament of order ≥ 4 and x be a d-critical vertex of T. If x is non-isolated in $D(T)$, then $d_{D(T)}(x) = 2$ and $N_{D(T)}(x)$ is the unique duo of $T - x$.*

Proof. Let $x \in V$ be a d-critical vertex of T. Assume that x is non-isolated in $D(T)$. Let $I_x := \{i_x, j_x\}$ be a duo of $T - x$. For all $t \in V\setminus(I_x \cup \{x\})$, I_x is a duo of $T - \{x, t\}$, then $t \notin N_{D(T)}(x)$. So, $N_{D(T)}(x) \subseteq I_x$. As $T - \{x, i_x\} \simeq T - \{x, j_x\}$ and $d_{D(T)}(x) \neq 0$, then $N_{D(T)}(x) = I_x$. In particular, $N_{D(T)}(x)$ is the unique duo of $T - x$. □

Remark 5.3 *Let $T := (V, A)$ be a duo-free tournament of order ≥ 4 and x be a d-critical vertex of T. The vertex x is isolated in $D(T)$ if and only if $T - x$ admits at least two duos.*

Indeed, let x be a d-critical vertex of T. If x is non-isolated in $D(T)$ then, by Lemma 5.2, $N_{D(T)}(x)$ is the unique duo of $T - x$. Conversely, assume that $T - x$ admits a unique duo $I_x := \{i_x, j_x\}$, then $T - \{x, i_x\}$ and $T - \{x, j_x\}$ are isomorphic and duo-free tournaments, so x is non-isolated in $D(T)$.

From the previous Lemma we obtain:

Corollary 5.4 *Let T be a (-1)-critically duo-free tournament at a. If K is a non trivial connected component of $D(T)$ not containing a, then $D(T)[K]$ is a cycle.*

Lemma 5.5 *Let T be a (-1)-critically duo-free tournament. If a cycle C_l embeds into $D(T)$, then this cycle is of odd length. Moreover, if X is a subset of $V(T)$ satisfying: $D(T)[X] \simeq C_l$, then X is an interval of T.*

Proof. We suppose that a cycle C_l embeds into $D(T)$, $l \geq 3$. Up to isomorphism, we can assume that $C_l = D(T)[\{0, 1, \cdots, l-1\}]$ and all elements of $\{1, \cdots, l-1\}$ are d-critical vertices of T. Since $N_{D(T)}(l-1) = \{l-2, 0\}$, then according to Lemma 5.2, $\{0, l-2\}$ is a duo of $T - \{l-1\}$. It follows that l is odd, otherwise, again by Lemma 5.2, $(l-1, 0) \equiv (l-1, 2i)$, for $i \in \{0, ..., \frac{l-2}{2}\}$. In particular, $(l-1, 0) \equiv (l-1, l-2)$ so that $\{l-2, 0\}$ is a duo of T, contradiction. Let $l = 2n+1$, $n \geq 1$. If $V(T)\backslash\{0, ..., 2n\} \neq \emptyset$, then $V(C_{2n+1}) = \{0, ..., 2n\}$ is a non-trivial interval of T. Indeed, let $x \in V(T)\backslash\{0, ..., 2n\}$. Again by Lemma 5.2, $(x, 1) \equiv (x, 2i+1)$ for $i \in \{0, ..., n-1\}$ and $(x, 0) \equiv (x, 2i)$ for $i \in \{0, ..., n\}$. Moreover, since $N_{D(T)}(2n) = \{2n-1, 0\}$, then according to Lemma 5.2, $(x, 0) \equiv (x, 2n-1)$. Then, $V(C_{2n+1}) = \{0, ..., 2n\}$ is an interval of T. □

Lemma 5.6 *Let T be a (-1)-critically duo-free tournament at a, then $d_{D(T)}(a) = 0$.*

Proof. Assume by contradiction that $d_{D(T)}(a) \geq 1$. Let K be the connected component of $D(T)$ containing a. From Lemma 5.2, for all $x \in K\backslash\{a\}$, $d_{D(T)}(x) = 2$. So, $d_{D(T)}(a)$ is even, then for all $x \in K$, $d_{D(T)}(x)$ is even and non-zero, then a cycle C_m embeds into $D(T)[K]$. Set X a subset of $V(T)$ such that $D(T)[X] \simeq C_m$. If $a \notin X$, since $D(T)[K]$ is connected, there is $x \in C_m$ and $y \in K\backslash C_m$ such that $y \in N_{D(T)}(x)$, this contradicts the fact that $d_{D(T)}(x) = 2$, so $a \in X$. From Lemma 5.5, the cycle C_m is of odd length and X is an interval of T. Let $n \geq 1$ such that $D(T)[X] \simeq C_{2n+1}$. Up to isomorphism, we can assume that $X = \{0, 1, \cdots, 2n\}$, $D(T)[X] = C_{2n+1}$ and $a = 0$. Up to duality, we can assume that $0 \longrightarrow 1$ in T. Since for $i \in \{1, ..., 2n-1\}$, $\{i-1, i+1\}$ is a duo of $T - i$ and not a duo of T, then $0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow 2n$ in T. Moreover, since $\{2n-1, 0\}$ is a duo of $T - 2n$ and not a duo of T, then $2n \longrightarrow 0$ in T. Necessarily $n \geq 2$, otherwise, since $\{0, 1, 2\}$ is an interval of T, then $\{1, 2\}$ is a duo of $T - 0$ and this contradicts the fact that 0 is a non d-critical vertex of T.
For all $i \in \{0, \cdots, n-1\}$, $\{2i, 2i+2\}$ is a duo of $T - (2i+1)$. Since $2n \longrightarrow 0$ then $2n \longrightarrow 2i$ for all $i \in \{0, \cdots, n-1\}$. On the other hand, since $1 \longrightarrow 2$ then $1 \longrightarrow 2i$ for all $i \in \{1, \cdots, n\}$. So $\{1, 2n\} \longrightarrow 2i$ for all

$i \in \{1, \cdots, n-1\}$.
For all $i \in \{0, \cdots, n-2\}$, $\{2i+1, 2i+3\}$ is a duo of $T - (2i+2)$. Since $2n-1 \longrightarrow 2n$, then $2i+1 \longrightarrow 2n$ for all $i \in \{0, \cdots, n-1\}$. On the other hand, $N_{D(T)}(2n) = \{2n-1, 0\}$ and $0 \longrightarrow 1$ then, from Lemma 5.2, $\{2n-1, 0\} \longrightarrow 1$; consequently $2i+1 \longrightarrow 1$ for all $i \in \{1, \cdots, n-1\}$. So $2i+1 \longrightarrow \{1, 2n\}$ for all $i \in \{1, \cdots, n-1\}$.
We deduce that $\{1, 2n\}$ is a duo of $T[X] - 0$. Since X is an interval of T, then $\{1, 2n\}$ is a duo of $T - 0$. This contradicts the fact that 0 is a non d-critical vertex of T. □

The result bellow follows from Corollary 5.4, Lemmas 5.5 and 5.6.

Corollary 5.7 *Let T be a (-1)-critically duo-free tournament and K be a non-trivial connected component of $D(T)$, then $D(T)[K]$ is a cycle of odd length and K is an interval of T.*

Remark 5.8 *Every (-1)-critically duo-free tournament is decomposable.*

Proof. Let T be a (-1)-critically duo-free tournament at a. Assume by contradiction that T is indecomposable. The tournaments with 4 vertices are all decomposable. Moreover T_5, U_5 and V_5 are the only indecomposable tournaments of order 5 and are not (-1)-critically duo-free. So, T has order ≥ 6. Furthermore, T is critical or (-1)-critical depending on whether $T-a$ is decomposable or not. Then, from Theorem 3.1 and Lemma 3.2, T has odd order. It follows that $|V(T)| \geq 7$. By Lemma 5.6, a is an isolated vertex of $D(T)$. If $D(T)$ is not an empty graph then, from Corollary 5.4, a cycle embeds into $D(T)$. From Lemma 5.5, this cycle is a non-trivial interval of T, this contradicts the fact that T is indecomposable. Then $D(T)$ is an empty graph. It follows that for all $x \neq y$ in $V(T)$, $T - \{x, y\}$ admits a duo, this contradicts Lemma 2.3. □

6 Proof of Theorem 4.2

The sufficient condition of Theorem 4.2 follows from the following two remarks and Lemma:

Remark 6.1 *A diamond (resp. a double-diamond) is (-1)-critically duo-free tournaments at its center.*

Remark 6.2 *For any integer $h \geq 1$, the tournament T_{2h+1} is both indecomposable and critically duo-free.*

Lemma 6.3 *Let T be a tournament, of order ≥ 4, having an interval partition P such that there is a unique $X \in P$ satisfying: $X = \{a\}$, or $T[X]$*

is a diamond of center a, or $T[X]$ is a double-diamond of center a and for all $Y \in P\backslash\{X\}$, $T[Y]$ is isomorphic to some T_{2h+1}. Then $T - a$ and T are duo-free.

Proof. First, we verify that $T - a$ is duo-free. If $X = \{a\}$, we notice that $P\backslash\{X\}$ is an interval partition of $T - a$ and we deduce, by Theorem 4.1, that $T - a$ is critically duo-free. Now we assume that $|X| > 1$, then either $T[X \setminus \{a\}] \simeq T_3$ or $T[X \setminus \{a\}]$ admits an interval partition $\{Y_1, Y_2\}$ such that $T[Y_1] \simeq T[Y_2] \simeq T_3$, depending on whether $T[X]$ is a diamond or $T[X]$ is a double-diamond. Putting $P' := (P\backslash\{X\}) \cup \{X\backslash\{a\}\}$ in the first case and $P' := (P\backslash\{X\}) \cup \{Y_1, Y_2\}$ in the second case, we see that P' is an interval partition of $T-a$ such that for all $Z \in P'$, $(T-a)[Z]$ is isomorphic to some T_{2h+1} with $h \geq 1$. It follows, from Theorem 4.1, that $T - a$ is critically duo-free. In particular $T - a$ is duo-free.

At present we show that T is duo-free. Assume by contradiction that T admits a duo $\{\alpha, \beta\}$. Since $T - a$ is duo-free, then necessarily $a \in \{\alpha, \beta\}$. We suppose that $a = \beta$. Since, by Remark 6.1, $T[X]$ is either of order 1 or (-1)-critically duo-free at a, we deduce that there is $Y \in P\backslash\{X\}$ such that $\alpha \in Y$. So, $a \in Y(\alpha)$. Moreover, as $X \sim Y$, $a \in [Y]$ which contradicts Lemma 2.2. □

To prove the converse, we first prove the following results.

Lemma 6.4 *Let T be a (-1)-critically duo-free tournament at a. Then $T - a$ is a critically duo-free tournament.*

Proof. Since a is a non d-critical vertex of T, then $T - a$ is a duo-free tournament. From Lemma 5.6, $d_{D(T)}(a) = 0$, then for all $x \in V(T)\backslash\{a\}$, $T - \{a, x\}$ admits at least one duo. So, for all $x \in V(T)\backslash\{a\}$, x is a d-critical vertex of $T - a$. Consequently $T - a$ is a critically duo-free tournament. □

Proposition 6.5 *Let T be a (-1)-critically duo-free tournament at a and K be a non-trivial connected component of $D(T)$, then K is a connected component of $D(T - a)$. Moreover $T[K]$ is isomorphic to some T_{2h+1}.*

Proof. It is sufficient to show that for all non isolated vertex x of $D(T)$, $N_{D(T-a)}(x) = N_{D(T)}(x)$. Let $x \in V(T)$ be a non-isolated vertex of $D(T)$, then according to Lemma 5.6, x is a d-critical vertex of T. From Lemma 5.2, $N_{D(T)}(x)$ is the unique duo of $T - x$. Since $a \notin N_{D(T)}(x)$, then $N_{D(T)}(x)$ is a duo of $T - \{a, x\}$. Furthermore, from Lemma 6.4, $T - a$ is critically duo-free then, by Lemma 5.1, all vertices of $T - a$ are non isolated in $D(T - a)$. Finally, by Lemma 5.2, $N_{D(T)}(x)$ is the unique duo of $T - \{a, x\}$ and $N_{D(T-a)}(x) = N_{D(T)}(x)$.

From what precedes, K is a connected component of $D(T-a)$. Moreover, since $T-a$ is critically duo-free, we deduce by Lemma 5.1 that $T[K]$ is isomorphic to some T_{2h+1}. □

Lemma 6.6 *Let T be a (-1)-critically duo-free tournament at a and K be a connected component of $D(T-a)$ formed by isolated vertices of $D(T)$. Then for every $x \in K$, there is $\alpha \in K$ such that $\{a, \alpha\}$ is a duo of $T-x$.*

Proof. Let $x \in K$, there exists $\alpha \in V(T) \setminus \{a\}$ such that $\{a, \alpha\}$ is a duo of $T-x$, otherwise, set J_x a duo of $T-x$, then $a \notin J_x$, so J_x is a duo of $T-\{a,x\}$. Furthermore, as $T-a$ is a critically duo-free tournament, according to Lemma 5.1, $d_{D(T-a)}(x) = 2$. Moreover from Lemma 5.2, J_x is the unique duo of $T-\{a,x\}$ and $J_x = N_{D(T-a)}(x)$. It follows that $N_{D(T-a)}(x)$ is the unique duo of $T-x$. Consequently, by Remark 5.3, x is a non-isolated vertex of $D(T)$, contradiction. We prove now that $\alpha \in K$. Suppose the contrary. Let $y \in N_{D(T-a)}(x)$. From what precedes, there exists $\beta \in V(T) \setminus \{a\}$ such that $\{a, \beta\}$ is a duo of $T-y$. Necessarily, $\alpha \neq \beta$ because x separates a and α. Set K' be the connected component of $D(T-a)$ containing α. We have $\beta \notin K'$, otherwise $\{\alpha, \beta\}$ will be a duo of $T[K']$ which is indecomposable. Furthermore, since $\{a, \beta\}$ is a duo of $T-y$ and $\{a, \alpha\}$ is a duo of $T-x$, for all $z \in K' \setminus \{\alpha\}$, $(z,\beta) \equiv (z,a) \equiv (z,\alpha)$. It follows that $\beta \in K'(\alpha)$, impossible because $\beta \in [K']$. □

Proposition 6.7 *Let T be a (-1)-critically duo-free tournament at a and K be a connected component of $D(T-a)$ formed by isolated vertices of $D(T)$. Then the following assertions are verified.*

1. *$T[K]$ is isomorphic to T_3.*

2. *$T[K \cup \{a\}]$ is a diamond of center a.*

3. *$K \cup \{a\}$ is an interval of T.*

Proof. Since $T-a$ is critically duo-free then, from Lemma 5.1, there exists $h \geq 1$ such that $T[K] \simeq T_{2h+1}$. Moreover, up to isomorphism, we may assume that $T[K] = T_{2h+1}$. Assume by contradiction that $h \geq 2$. Since 0 is an isolated vertex of $D(T)$, then $T - \{0, h+1\}$ admits at least a duo.

First, we show that a duo of $T-\{0,h+1\}$ is a duo of $T[K\cup\{a\}]-\{0,h+1\}$ containing a. Since $T[K] - \{0, h+1\} \simeq T_{2h-1}$, then by Theorem 4.1, $(T-a) - \{0, h+1\}$ is a critically duo free tournament. In particular, $(T-a) - \{0, h+1\}$ does not admit any duo. It follows that every duo of $T - \{0, h+1\}$ contains a. Let $\beta \in V(T) \setminus \{0, a\}$ such that $\{a, \beta\}$ is a duo of $T - \{0, h+1\}$. We have to prove that $\beta \in K$. Suppose the contrary. Since K is an interval of $T-a$, then $\beta \sim K$. It follows that $a \sim K \setminus \{0, h+1\}$. From Lemma 6.6, there exists $\alpha \in K$ such that $\{a, \alpha\}$ is a duo of $T-0$. So,

$\alpha = h+1$, otherwise $\beta \in (K \setminus \{0, h+1\})(\alpha)$ and this contradicts Lemma 2.2. We obtain $h+1 \in [K \setminus \{0, h+1\}]$, which is impossible because $h \geq 2$. Throughout the following $\beta \in K$ such that $\{a, \beta\}$ is a duo of $T - \{0, h+1\}$.

Second, we show that $T[K \cup \{a\}]$ is indecomposable. Since $a \in (K \setminus \{0, h+1\})(\beta)$ then, by lemma 2.2, $a \notin [K]$. So, it is enough to show that for all $i \in K$, $a \notin K(i)$.

Assume that there is $i \in K$ such that $a \in K(i)$, then $\{i, a\}$ is a duo of $T[K \cup \{a\}]$. Without loss of generality, we can assume that $i = h$. Since $a \in K(h)$ and $\{a, \beta\}$ is a duo of $T - \{0, h+1\}$, then $a \in (K \setminus \{0, h+1\})(h)$ and $a \in (K \setminus \{0, h+1\})(\beta)$. It follows by Lemma 2.2 that $\beta = h$. So, $\{a, h\}$ is both a duo of $T - \{0, h+1\}$ and a duo of $T[K \cup \{a\}]$. Then $\{a, h\}$ is a duo of T, which is impossible because T is duo-free.

Moreover, from Lemma 6.6, for all $i \in K$, there is a duo of $T - i$ included in $K \cup \{a\}$. It follows that $T[K \cup \{a\}]$ is a (-1)-critical tournament, but this contradicts lemma 3.2 because $|K \cup \{a\}|$ is even.

In conclusion $h = 1$ and $T[K]$ is isomorphic to T_3.

We will show now that $T[K \cup \{a\}]$ is a diamond. By contradiction, we assume that $a \in K(i)$, where $i \in K$. Then $T[(K \cup \{a\}) \setminus \{i\}] \simeq T_3$. It follows that for all $j \in K \setminus \{i\}$, $\{a, j\}$ is not a duo of $T - i$, but this contradicts Lemma 6.6. Finally, since all tournaments on 4 vertices are decomposables, we deduce by Lemma 2.2 that $a \in [K]$, thus $T[K \cup \{a\}]$ is a diamond.

To finish the proof, we will show that $K \cup \{a\}$ is an interval of T. Since K is an interval of $T - a$, it is sufficient to verify that for all $x \in V(T) \setminus (K \cup \{a\})$ and $y \in K$, $(x, y) \equiv (x, a)$. Let $y \in K$ and $z \in K$ such that $\{a, z\}$ is a duo of $T - y$. For all $x \in V(T) \setminus (K \cup \{a\})$, $(x, a) \equiv (x, z) \equiv (x, y)$. □

Proposition 6.8 *Let T be a (-1)-critically duo-free tournament at a. Set $W(T)$ the set of isolated vertices of $D(T)$. If $W(T) \neq \{a\}$, then $T[W(T)]$ is a diamond of center a, or a double-diamond of center a.*

Proof. From Lemma 5.6, $d_{D(T)}(a) = 0$, then $a \in W(T)$. From Proposition 6.7, $T[W(T)]$ is either of order 1 or admits an interval partition $\{\{a\}, K_1, K_2, \ldots, K_n\}$, where $n \geq 1$ such that for all $i \in \{1, 2, \ldots, n\}$, $T[K_i] \simeq T_3$ and both K_i and $K_i \cup \{a\}$ are intervals of T. In particular $|W(T)|$ is of the form $1 + 3p$, where p is a positive integer.

Assume that $|W(T)| > 7$, then $n \geq 3$. We have $T[K_i \cup \{a\}]$ is a diamond for $i \in \{1, 2, 3\}$. Up to isomorphism, we can assume that $a \longrightarrow K_1$ and $a \longrightarrow K_2$. Since $K_1 \cup \{a\}$ is an interval of T and $a \longrightarrow K_2$, then $K_1 \longrightarrow K_2$. Besides, $K_2 \cup \{a\}$ is an interval of T and $a \longrightarrow K_1$, then $K_2 \longrightarrow K_1$, absurd. The form of $T[W(T)]$ follows immediately using Proposition 6.7. □

Now we complete the proof of Theorem 4.2. Given a (-1)-critically duo-free tournament T at a. Let K be a non-trivial connected component of $D(T)$ then K does not contain a. From Corollary 5.7, $D(T)[K]$ is a cycle and K is an interval of T. Proposition 6.5 states that $T[K]$ is isomorphic to some T_{2h+1}.

Proposition 6.8 ensures that $W(T) = \{a\}$, or $T[W(T)]$ is a diamond of center a, or $T[W(T)]$ is a double-diamond of center a. Assume that $W(T) \neq \{a\}$. It follows that $T[W(T)]$ admits an interval partition $\{\{a\}, K_1, \dots, K_n\}$ with $n = 1$ or 2 and $T[K_1] \simeq T[K_n] \simeq T_3$. Since K_1 and K_n are connected components of $D(T - a)$ formed by isolated vertices of $D(T)$ then, from Proposition 6.7, $K_1 \cup \{a\}$ and $K_n \cup \{a\}$ are intervals of T, thus their union $W(T)$ is an interval of T.

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