Upper Bounds on the D(β)-Vertex-Distinguishing Total-Chromatic Number of Graphs

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Abstract: In this paper, we study the upper bounds for the D(β)-vertex-distinguishing total-

chromatic numbers by probability method, and obtain: Let Δ be the maximum degree of G, then

$$\chi_{\beta vt}(G) \leq \begin{cases} 16\Delta^{(\beta+1)/(2\Delta+2)}, & \Delta \geq 3, \beta \geq 4\Delta+3; \\ 13\Delta^{(\beta+4)/4}, & \Delta \geq 4, \beta \geq 5; \\ 10\Delta^2, & \Delta \geq 3, 2 \leq \beta \leq 4. \end{cases}.$$

Key words: random coloring;probability method;positive probability; D(β)-vertex-

distinguishing total-chromatic number

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Graph theory is a sort of models which can be applied in various science fields such as biology, computer science, combinatorial optimization,

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etc. It is well known to compute the chromatic number of a graph is NP-hard problem. In [1][2], the people have some results about it by combined methods. At ICM2002, Noga Alon advanced a new theory that graph coloring could studied by probability methods. For instance, some conclusions have been gotten by probability methods, see [3-8]. In 2006, Zhang Zhongfu present a new concept of the k-D(β)-vertex-distinguishing total-coloring and conjecture, see [9]. In this paper, we study the upper bounds for the D(β)-vertex-distinguishing total-chromatic numbers by probability method.

All the graphs $G = G(V, E)$ discussed in this paper are finite, undirected, simple and connected. Let Δ(or $\Delta(G)$) be the maximum degree of G, δ(or $\delta(G)$) be the minimum degree of G, and if $u \in V(G)$, then $d(u)$ is the degree of u in G and $C(u) = \{f(u)\} \cup \{f(uu')|uu' \in E(G)\}$.

Definition 1[9]**.** Let $G(V, E)$ be a connected graph with order at least 2. Suppose k, β are both positive integers and f is a mapping from $V(G) \bigcup E(G)$ to $C = \{1, 2, \cdots, k\}$. For all $u \in V(G)$, the set $C - C(u)$ is denoted by $\overline{C}(u)$. If

(i) for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$;

(ii) for any $uv \in E(G), u \neq v$, we have $f(u) \neq f(v), f(u) \neq f(uv), f(v) \neq f(uv)$, then f is called a k-proper-total-coloring. If f is a k-proper-total-coloring, and

(iii) for any $u, v \in V(G), u \neq v, d(u, v) \leq \beta$, where $d(u, v)$ denotes the distance between u and v, we have $C(u) \neq C(v)$, i.e. $\overline{C}(u) \neq \overline{C}(v)$, then f is called a k-$D(\beta)$-vertex-distinguishing total-coloring of the graph G(k-$D(\beta)$-VDTC of G in brief) and the number $\chi_{\beta vd}(G)$=min{k|G has a k-$D(\beta)-$VDTC} is called a D(β)-vertex-distinguishing total-chromatic number of G.

For other terminologies and notations, the interested reader may refer to [11].

Lemma 2 (The General Local Lemma)[10]**.** Consider a set $\xi = \{A_1, A_2, \cdots, A_n\}$ of (typically bad) events such that each A_i is mutually independent of $\xi - (D_i \bigcup \{A_i\})$, for some $D_i \subseteq \xi$. If we have reals $x_1, x_2, \cdots, x_n \in [0,1)$ such that for each $1 \le i \le n$, $Pr(A_i) \le x_i \prod_{A_j \in D_i} (1 - x_j)$, then the probability that none of the events in ξ occur is at least $\prod_{i=1}^{n}(1 - x_i) > 0$.

Theorem 1. Let Δ be the maximum degree of G, then

$$\chi_{\beta vt}(G) \le \begin{cases} 16\Delta^{(\beta+1)/(2\Delta+2)}, & \Delta \ge 3, \beta \ge 4\Delta + 3; \\ 13\Delta^{(\beta+4)/4}, & \Delta \ge 4, \beta \ge 5; \\ 10\Delta^2, & \Delta \ge 3, 2 \le \beta \le 4. \end{cases}$$

Proof. We only give the proof for the case that $\Delta \ge 3, \beta \ge 4\Delta + 3$, the other cases can be proved similarly. The proof consists of five steps. We assign to each edge and vertex of G a uniformly random coloring from $\{1, 2, \cdots, 16\Delta^{(\beta+1)/(2\Delta+2)}\}$, named this new coloring f. We will use lemma 2 to show that the probability that f is a D(β)-vertex-distinguishing total-coloring is positive.

Step 1. The following bad events are defined:

Type 1. For each pair of adjacent edges e, f, let $A_{e,f}$ be the event that both e and f are colored with the same color;

Type 2. For each pair of adjacent vertices u, v, let $B_{u,v}$ be the event that both u and v are colored with the same color;

Type 3. For each pair of incident vertex w and edge e, let $E_{w,e}$ be the event that both w and e are colored with the same color;

Type 4. For each edge $e = uv_1$, such that $d(u) = d(v_1) \ge \delta(G)$, let C_e be the color set of vertices u, v_1 and all edges which are incident u or v_1, then E_{C_e} be the event that the coloring of u, v_1 and all edges which are incident to u or v_1 satisfy proper total coloring, and $C(u) = C(v_1)$;

Type 5. For each path whose length is 2, $P_{uv_2} = uev_1fv_2$, such that $d(u) = d(v_2) \ge \delta(G)$, let P_{uv_2} be the set of vertices u, v_2 and all edges which are incident u or v_2, then $E_{P_{uv_2}}$ be the event that the coloring of u, v_2 and all edges which are incident to u or v_2 satisfy proper total coloring, and $C(u) = C(v_2)$;

... ...

Type $(\beta+3)$. For each path whose length is β, $P_{uv_\beta} = uev_1fv_2\cdots v_\beta$, such that $d(u) = d(v_\beta) \geq \delta(G)$, let P_{uv_β} be the set of vertices u, v_β and all edges which are incident u or v_β, then $E_{P_{uv_\beta}}$ be the event that the coloring of u, v_β and all edges which are incident to u or v_β satisfy proper total coloring, and $C(u) = C(v_\beta)$.

It remains to show that with positive probability none of these events happen, then f is a $D(\beta)-$VDTC of G. Let us construct a dependency graph H whose nodes are all the events of these $\beta+3$ types, in which two nodes E_X and E_Y(where each of X and Y is either a pair of adjacent edges, a pair of adjacent vertices, a pair of incident vertex and edge, or a set of two vertices and all edges which are incident to any vertex of these two vertices) are adjacent if and only if X and Y contain one common edge or vertex. Since the occurrence of each event E_X(or E_Y) depends only on the edges and vertices of X(or Y), H is dependency graph for our events. In order to apply The General Local Lemma, we need to estimate for the probability of each event and the number of nodes of each type in H which are adjacent to any given node. These estimates are given in the two steps below.

Step 2. Estimates the probability of each event:

$$Pr(A_{e,f}) = Pr(B_{u,v}) = Pr(E_{w,e}) = \frac{1}{16\Delta^{(\beta+1)/(2\Delta+2)}};$$

$$Pr(E_{C_e}) = \frac{1}{\binom{16\Delta^{(\beta+1)/(2\Delta+2)}}{1}+\binom{16\Delta^{(\beta+1)/(2\Delta+2)}}{2}+\ \cdots\ +\binom{16\Delta^{(\beta+1)/(2\Delta+2)}}{\Delta}}$$
$$\leq \frac{1}{\Delta\cdot 16\Delta^{(\beta+1)/(2\Delta+2)}};$$

$$Pr(E_{P_{uv_2}}) = Pr(E_{P_{uv_3}}) = \cdots = Pr(E_{P_{u(v_{\beta-1})}}) = Pr(E_{P_{uv_\beta}})$$
$$= \frac{1}{(16\Delta^{(\beta+1)/(2\Delta+2)})^{\Delta+1}} \cdot \frac{1}{(16\Delta^{(\beta+1)/(2\Delta+2)})^{\Delta+1}} = \frac{1}{(16\Delta^{(\beta+1)/(2\Delta+2)})^{2\Delta+2}}.$$

Step 3. Estimate the dependency events number :

In Type 1, Type 2, $\cdots$, Type $(\beta+3)$, the most dependency events number with event $A_{e,f}$ of Type 1 are $4\Delta-5$ which constitute a set D_{11}, 0 which constitute a set D_{12}, 4 which constitute a set D_{13}, $2\Delta-2$ which constitute a set D_{14}, $2\Delta(\Delta-1)-1$ which constitute a set D_{15}, $2\Delta(\Delta-1)^2$ which constitute a set D_{16}, $\cdots$, $2\Delta(\Delta-1)^{\beta-1}$ which constitute a set $D_{1(\beta+3)}$; the most dependency events number with event $B_{u,v}$ of Type 2 are 0 which constitute a set D_{21}, $2\Delta - 1$ which constitute a set D_{21}, 2Δ which constitute a set D_{22}, $2\Delta - 2$ which constitute a set D_{23}, $2\Delta(\Delta - 1)$ which constitute a set D_{24}, $2\Delta(\Delta-1)^2$ which constitute a set D_{25}, $\cdots$, $2\Delta(\Delta-1)^{\beta-1}$ which constitute a set $D_{2(\beta+3)}$; the most dependency events number with event $E_{w,e}$ of Type 3 are $2(\Delta-1)$ which constitute a set D_{31}, Δ which constitute a set D_{32}, $\Delta + 1$ which constitute a set D_{33}, $\Delta - 1$ which constitute a set D_{34}, $\Delta(\Delta - 1)$ which constitute a set D_{35}, $\Delta(\Delta - 1)^2$ which constitute a set D_{36}, $\cdots$, $\Delta(\Delta - 1)^{\beta-1}$ which constitute a set $D_{3(\beta+3)}$; the most dependency events number with event E_{C_e} of Type 4 are$2(\Delta - 1)^2$ which constitute a set D_{41}, $2(\Delta - 1) + 1$ which constitute a set D_{42}, $2(\Delta - 1)$ which constitute a set D_{43}, $2(\Delta - 1)^2 + 2(\Delta - 1) + 1$ which constitute a set D_{44}, $2\Delta(\Delta-1)^2+\Delta(\Delta-1)$ which constitute a set D_{45}, $2\Delta^2(\Delta - 1)^2$ which constitute a set D_{46}, $\cdots$,$2\Delta^2(\Delta - 1)^{\beta-1}$ which constitute a set $D_{4(\beta+3)}$; the most dependency events number with event $E_{P_{uv_2}}$ of Type 5 are $2(\Delta - 1)^2 + 2(\Delta - 2)$ which constitute a set D_{51}, 2Δ which constitute a set D_{52}, 2Δ which constitute a set D_{53}, $2\Delta(\Delta - 1) + 2\Delta$ which constitute a set D_{54}, $2\Delta(\Delta - 1)^2 + 2\Delta(\Delta - 1) - 2$ which constitute a set D_{55}, $2\Delta^2(\Delta - 1)^2 + 2(\Delta - 1)^2$ which constitute a set D_{56}, $\cdots$, $2\Delta^2(\Delta - 1)^{\beta-1} + 2(\Delta - 1)^{\beta-1}$ which constitute a set $D_{5(\beta+3)}$; the most dependency events number with each event of $E_{P_{uv_{k-3}}}$ of Type k are $2\Delta(\Delta - 1)$which constitute a set D_{k1}, 2Δ which constitute a set D_{k2}, 2Δ which constitute a set D_{k3},$2\Delta(\Delta-1)+2\Delta$ which constitute a set D_{k4}, $2\Delta(\Delta-1)^2+(3\Delta+2)(\Delta - 1)$ which constitute a set D_{k5}, $2\Delta(\Delta - 1)^3 + 4\Delta(\Delta - 1)^2$ which constitute a set D_{k6}, $\cdots$, $2\Delta(\Delta - 1)^{\beta} + 4\Delta(\Delta - 1)^{\beta-1}$ which constitute a set $D_{k(\beta+3)} (k = 6, 7, \cdots , \beta + 3)$.

We only give the statement for one of these, and the other can be stated similarly. For each event $A_{e,f}$ of Type 1, let $e = u_1u_2, f = u_2u_3 (u_1 \neq u_3)$, at most $2\Delta - 2$ edges are adjacent to e(or f), so each event of Type 1 is incident to at most $(2\Delta-2)+(2\Delta-2)-1 = 4\Delta-5$ events of Type 1. We can easily see that each event of Type 1 is incident to 0 event of Type 2. At most 2 vertices are incident to e(or f), so each event of Type 1 is incident to at most 4 events of Type 3. At most $\Delta-1$ vertices are adjacent to u_1(or u_3), so each event of Type 1 is incident to at most $2\Delta-2$ events of Type 4. At most $\Delta(\Delta-1)$ vertices which the distance of $u_1(or u_3)$ and these vertices is 2, and Type 5 don't include the event $E_{P_{u_1u_3}}$, so each event of Type 1 is incident to at most $2\Delta(\Delta-1)-1$ events of Type 5. At most $\Delta(\Delta-1)^k$ vertices which the distance of $u_1(or u_3)$ and these vertices is k+1, so each event of Type 1 is incident to at most $2\Delta(\Delta-1)^k$ events of Type (k+4) (k=2,3,$\cdots$,$\beta-1$).

Step 4. Find the real constant $x_i(0 \leq x_i < 1)$ for applying lemma 2.

Let $\frac{2}{16\Delta^{(\beta+1)/(2\Delta+2)}}$, $\frac{2}{16\Delta^{(\beta+1)/(2\Delta+2)}}$, $\frac{2}{16\Delta^{(\beta+1)/(2\Delta+2)}}$, $\frac{2}{\Delta\cdot 16\Delta^{(\beta+1)/(2\Delta+2)}}$, $\frac{2}{(16\Delta^{(\beta+1)/(2\Delta+2)})^{2\Delta+2}}$, $\cdots$, $\frac{2}{(16\Delta^{(\beta+1)/(2\Delta+2)})^{2\Delta+2}}$ be the constants associated with events of Type 1, Type 2,$\cdots$, Type $(\beta+3)$.

Step 5. Conclude that with positive probability no events of Type 1, Type 2,$\cdots$, Type $(\beta+3)$ provided that :

Let $m = 16\Delta^{(\beta+1)/(2\Delta+2)}$.

$$\begin{aligned}\frac{1}{m} \leq & \frac{2}{m}(1-\frac{2}{m})^{4\Delta-1}\cdot(1-\frac{2}{\Delta m})^{2\Delta-2}\cdot(1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)-1}\cdot \\ & (1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^2}\cdot\ \cdots\ \cdot(1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^{\beta-1}} \\ \leq & \frac{2}{m}\cdot(\prod_{A_{e,f}\in D_{11}}(1-\frac{2}{m}))\cdot(\prod_{B_{u,v}\in D_{12}}(1-\frac{2}{m}))\cdot(\prod_{E_{w,e}\in D_{13}}(1-\frac{2}{m}))\cdot \\ & (\prod_{E_{C_e}\in D_{14}}(1-\frac{2}{\Delta m}))\cdot(\prod_{E_{P_{uv_2}}\in D_{15}}(1-\frac{2}{m^{2\Delta+2}}))\cdot\ \cdots\ \cdot \\ & (\prod_{E_{P_{uv_2}}\in D_{1(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})); \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{1}{m} \leq & \frac{2}{m}(1-\frac{2}{m})^{4\Delta-1}\cdot(1-\frac{2}{\Delta m})^{2\Delta-2}\cdot(1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)}\cdot \\ & (1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^2}\cdot\ \cdots\ \cdot(1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^{\beta-1}} \\ \leq & \frac{2}{m}\cdot(\prod_{A_{e,f}\in D_{21}}(1-\frac{2}{m}))\cdot(\prod_{B_{u,v}\in D_{22}}(1-\frac{2}{m}))\cdot(\prod_{E_{w,e}\in D_{23}}(1-\frac{2}{m}))\cdot \\ & (\prod_{E_{C_e}\in D_{24}}(1-\frac{2}{\Delta m}))\cdot(\prod_{E_{P_{uv_2}}\in D_{25}}(1-\frac{2}{m^{2\Delta+2}}))\cdot\ \cdots\ \cdot \\ & (\prod_{E_{P_{uv_2}}\in D_{2(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})); \qquad (2)\end{aligned}$$

$$\begin{aligned}\frac{1}{m} \leq & \frac{2}{m}(1-\frac{2}{m})^{4\Delta-1}\cdot(1-\frac{2}{\Delta m})^{\Delta-1}\cdot(1-\frac{2}{m^{2\Delta+2}})^{\Delta(\Delta-1)}\cdot \\ & (1-\frac{2}{m^{2\Delta+2}})^{\Delta(\Delta-1)^2}\cdot\ \cdots\ \cdot(1-\frac{2}{m^{2\Delta+2}})^{\Delta(\Delta-1)^{\beta-1}} \\ \leq & \frac{2}{m}\cdot(\prod_{A_{e,f}\in D_{31}}(1-\frac{2}{m}))\cdot(\prod_{B_{u,v}\in D_{32}}(1-\frac{2}{m}))\cdot(\prod_{E_{w,e}\in D_{33}}(1-\frac{2}{m}))\cdot \\ & (\prod_{E_{C_e}\in D_{34}}(1-\frac{2}{\Delta m}))\cdot(\prod_{E_{P_{uv_2}}\in D_{35}}(1-\frac{2}{m^{2\Delta+2}}))\cdot\ \cdots\ \cdot \\ & (\prod_{E_{P_{uv_2}}\in D_{3(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})); \qquad (3)\end{aligned}$$

$$\frac{1}{\Delta m} \leq \frac{2}{\Delta m}(1-\frac{2}{m})^{2(\Delta-1)^2} \cdot (1-\frac{2}{m})^{2(\Delta-1)+1} \cdot (1-\frac{2}{m})^{2(\Delta-1)} \cdot$$
$$(1-\frac{2}{\Delta m})^{2(\Delta-1)^2+2(\Delta-1)+1} \cdot (1-\frac{2}{m^{2\Delta+2}})^{2(\Delta-1)^2+\Delta(\Delta-1)} \cdot$$
$$(1-\frac{2}{m^{2\Delta+2}})^{2\Delta^2(\Delta-1)^2} \cdot \ \cdots \ \cdot (1-\frac{2}{m^{2\Delta+2}})^{2\Delta^2(\Delta-1)^{\beta-1}}$$
$$\leq \frac{2}{\Delta m} \cdot (\prod_{A_{e,f} \in D_{41}}(1-\frac{2}{m})) \cdot (\prod_{B_{u,v} \in D_{42}}(1-\frac{2}{m})) \cdot (\prod_{E_{w,e} \in D_{43}}(1-\frac{2}{m})) \cdot$$
$$(\prod_{E_{C_e} \in D_{44}}(1-\frac{2}{\Delta m})) \cdot (\prod_{E_{P_{uv_2}} \in D_{45}}(1-\frac{2}{m^{2\Delta+2}})) \cdot \ \cdots \ \cdot$$
$$(\prod_{E_{P_{uv_2}} \in D_{4(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})); \qquad (4)$$

$$\frac{1}{m^{2\Delta+2}} \leq \frac{2}{m^{2\Delta+2}}(1-\frac{2}{m})^{2(\Delta-1)^2+2(\Delta-2)} \cdot (1-\frac{2}{m})^{2\Delta} \cdot (1-\frac{2}{m})^{2\Delta} \cdot$$
$$(1-\frac{2}{\Delta m})^{2\Delta(\Delta-1)+2\Delta} \cdot (1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^2+2\Delta(\Delta-1)-2} \cdot$$
$$(1-\frac{2}{m^{2\Delta+2}})^{2\Delta^2(\Delta-1)^2+2(\Delta-1)^2} \cdot \ \cdots \ \cdot (1-\frac{2}{m^{2\Delta+2}})^{2\Delta^2(\Delta-1)^{\beta-1}+2(\Delta-1)^{\beta-1}}$$
$$\leq \frac{2}{m^{2\Delta+2}} \cdot (\prod_{A_{e,f} \in D_{51}}(1-\frac{2}{m})) \cdot (\prod_{B_{u,v} \in D_{52}}(1-\frac{2}{m})) \cdot$$
$$(\prod_{E_{w,e} \in D_{53}}(1-\frac{2}{m})) \cdot (\prod_{E_{C_e} \in D_{54}}(1-\frac{2}{\Delta m})) \cdot$$
$$(\prod_{E_{P_{uv_2}} \in D_{55}}(1-\frac{2}{m^{2\Delta+2}})) \cdot \ \cdots \ \cdot (\prod_{E_{P_{uv_2}} \in D_{5(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})); \ (5)$$

$$\frac{1}{m^{2\Delta+2}} \leq \frac{2}{m^{2\Delta+2}}(1-\frac{2}{m})^{2\Delta(\Delta-1)} \cdot (1-\frac{2}{m})^{2\Delta} \cdot (1-\frac{2}{m})^{2\Delta} \cdot$$
$$(1-\frac{2}{\Delta m})^{2\Delta(\Delta-1)+2\Delta} \cdot (1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^2+(3\Delta+2)(\Delta-1)} \cdot$$
$$(1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^3+4\Delta(\Delta-1)^2} \cdot \ \cdots \ \cdot (1-\frac{2}{m^{2\Delta+2}})^{2\Delta(\Delta-1)^{\beta}+4\Delta(\Delta-1)^{\beta-1}}$$
$$\leq \frac{2}{m^{2\Delta+2}} \cdot (\prod_{A_{e,f} \in D_{(\beta+3)1}}(1-\frac{2}{m})) \cdot (\prod_{B_{u,v} \in D_{(\beta+3)2}}(1-\frac{2}{m})) \cdot$$
$$(\prod_{E_{w,e} \in D_{(\beta+3)3}}(1-\frac{2}{m})) \cdot (\prod_{E_{C_e} \in D_{(\beta+3)4}}(1-\frac{2}{\Delta m})) \cdot$$
$$(\prod_{E_{P_{uv_2}} \in D_{(\beta+3)5}}(1-\frac{2}{m^{2\Delta+2}})) \cdot \cdots \cdot (\prod_{E_{P_{uv_2}} \in D_{(\beta+3)(\beta+3)}}(1-\frac{2}{m^{2\Delta+2}})).$$
$$(6), \ \ldots \ ,(\beta+3)$$

Now, since $(1-\frac{1}{z})^z \geq \frac{1}{4}$ for all real $z \geq 2$, we can prove that the inequalities $(1),(2),\cdots,(\beta+3)$ are true, when $\Delta \geq 3, \beta \geq 4\Delta+3$. Using lemma 2, G has a $(16\Delta^{(\beta+1)/(2\Delta+2)}) - D(\beta)-$VDTC, when $\Delta \geq 3, \beta \geq 4\Delta+3$.

This completes the proof.

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