On E_4-Cordial Graphs

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Abstract

In this paper we prove the graphs P_n $(n \geq 3)$, C_n $(n \geq 3)$ where $n \not\equiv 4 \pmod 8$ and K_n $(n \geq 3)$ are E_4-cordial graphs. Also we prove that every graph of order ≥ 3 is a subgraph of an E_4-Cordial Graph.

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1 Introduction

Cahit and Yilmaz [1] have defined a new graph labeling by combining k-equitable labeling and edge graceful labeling of graphs, called E_k-cordial labeling. In this paper we prove that the graphs P_n $(n \geq 3)$, C_n $(n \geq 3)$ where $n \not\equiv 4 \pmod 8$ and K_n $(n \geq 3)$ are E_4-cordial graphs. Also we prove that every graph of order ≥ 3 is a subgraph of an E_4-Cordial Graph.

All graphs considered in this paper are finite, undirected simple graphs. For all notations in Graph theory we follow [3] and all terminology regarding labeling we follow [2].

Definition 1. A (p, q) graph G is E_k-cordial, if it is possible to label the edges of G with the numbers from the set $\{0, 1, 2, \ldots, k-1\}$ in such a way that at each vertex v, the sum modulo k of the labels on the edges incident with v satisfies the inequalities

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1 \quad \forall i, j,$$

where $v_f(i)$ and $e_f(j)$ are respectively the number of vertices labeled with i and the number of edges labeled with j.

2 Main Results

Theorem 2. The graph P_n for $n \geq 3$ is E_4-cordial.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n and $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ be the edges.
Define $f : E(P_n) \to \{0, 1, 2, 3\}$ as follows:
Case(i). For $n \equiv 0, 1, 4 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 0, 1 \pmod 8 \\ 1, & \text{if } i \equiv 2 \pmod 4 \\ 2, & \text{if } i \equiv 3 \pmod 4 \\ 3, & \text{if } i \equiv 4, 5 \pmod 8 \end{cases}$$

From the above labeling we obtain the following:

(a) If $n = 8r, r \geq 1$, then $v_f(0) = v_f(1) = v_f(2) = v_f(3) = 2r$ and $e_f(0) = 2r - 1, e_f(1) = e_f(2) = e_f(3) = 2r$.

(b) If $n = 8r + 1, r \geq 1$, then $v_f(0) = 2r + 1, v_f(1) = v_f(2) = v_f(3) = 2r$ and $e_f(0) = 2r - 1, e_f(1) = e_f(2) = e_f(3) = 2r$.

(c) If $n = 8r + 4, r \geq 0$, then $v_f(0) = v_f(1) = v_f(2) = v_f(3) = 2r + 1$ and $e_f(0) = e_f(1) = e_f(2) = 2r + 1, e_f(3) = 2r$.

Case(ii). For $n \equiv 2 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 1,2 \pmod 8 \text{ where } i > 8 \text{ and i=3,6} \\ 1, & \text{if } i \equiv 3,7 \pmod 8 \text{ where } i > 8 \text{ and i=1,5} \\ 2, & \text{if } i \equiv 0 \pmod 4 \text{ where } i > 8 \text{ and i=2,7} \\ 3, & \text{if } i \equiv 5,6 \pmod 8 \text{ where } i > 8 \text{ and i=4,8} \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r + 2, r \geq 1$, then $v_f(0) = v_f(2) = 2r, v_f(1) = v_f(3) = 2r + 1$ and $e_f(0) = 2r + 1, e_f(1) = e_f(2) = e_f(3) = 2r$.

Case(iii). For $n \equiv 3 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 2,3 \pmod 8 \text{ where } i > 8 \text{ and } i = 3 \\ 1, & \text{if } i \equiv 0 \pmod 4 \text{ and } i = 1 \\ 2, & \text{if } i \equiv 1,5 \pmod 8 \text{ where } i > 1 \text{ and } i = 2 \\ 3, & \text{if } i \equiv 6,7 \pmod 8 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r + 3, r \geq 0$, then $v_f(0) = 2r, v_f(1) = v_f(2) = v_f(3) = 2r + 1$ and $e_f(0) = e_f(3) = 2r, e_f(1) = e_f(2) = 2r + 1$.

Case(iv). For $n \equiv 5 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 4,5 \pmod 8 \text{ where } i > 4 \text{ and } i = 3 \\ 1, & \text{if } i \equiv 2 \pmod 4 \text{ where } i > 4 \text{ and } i = 1 \\ 2, & \text{if } i \equiv 3,7 \pmod 4 \text{ where } i > 4 \text{ and } i = 4 \\ 3, & \text{if } i \equiv 0,1 \pmod 8 \text{ where } i > 4 \text{ and } i = 2 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r+5, r \geq 0$, then $v_f(0) = v_f(1) = v_f(3) = 2r+1, v_f(2) = 2r+2$ and $e_f(0) = e_f(1) = e_f(2) = e_f(3) = 2r + 1$.

Case(v). For $n \equiv 6 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 6,5 \pmod 8 \text{ where } i > 5 \text{ and } i = 1 \\ 1, & \text{if } i \equiv 3 \pmod 4 \text{ where } i > 5 \text{ and } i = 2 \\ 2, & \text{if } i \equiv 0 \pmod 4 \text{ where } i > 5 \text{ and } i = 3 \\ 3, & \text{if } i \equiv 1,2 \pmod 8 \text{ where } i > 5 \text{ and } i = 4,5 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r+6, r \geq 0$, then $v_f(0) = v_f(2) = 2r+1, v_f(1) = v_f(3) = 2r+2$ and $e_f(0) = e_f(1) = e_f(2) = 2r + 1, e_f(3) = 2r + 2$.

Case(vi). For $n \equiv 7 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 6,7 \pmod 8 \text{ where } i > 5 \text{ and } i = 1 \\ 1, & \text{if } i \equiv 0 \pmod 4 \text{ and } i = 2 \\ 2, & \text{if } i \equiv 1,5 \pmod 4 \text{ where } i > 5 \text{ and } i = 3 \\ 3, & \text{if } i \equiv 2,3 \pmod 8 \text{ where } i > 5 \text{ and } i = 4,5 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r+7, r \geq 0$, then $v_f(0) = v_f(1) = v_f(3) = 2r+2, v_f(2) = 2r+1$ and $e_f(0) = e_f(3) = 2r+2, e_f(1) = e_f(2) = 2r+2$.

In the above all cases, clearly $|e_f(i) - e_f(j)| \leq 1$ and $|v_f(i) - v_f(j)| \leq 1$ for every i, j. Hence P_n for $n \geq 3$ is E_4-cordial. □

Illustrations: E_4-cordial labeling of P_{11} is given by

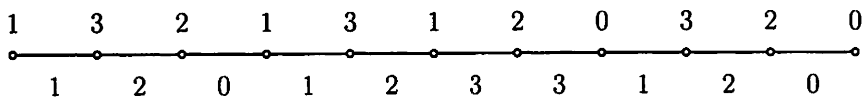


Figure 1:

Theorem 3. C_n for $n \geq 3$ is E_4-cordial if $n \not\equiv 4 \pmod 8$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and let $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_n v_1$ be the edges of the cycle C_n. Define $f : E(C_n) \to \{0,1,2,3\}$ as follows:

Case(i). For $n \equiv 0,1,3,5,6 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 1,6 \pmod 8 \\ 1, & \text{if } i \equiv 3,7 \pmod 8 \\ 2, & \text{if } i \equiv 2,5 \pmod 8 \\ 3, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

From the above labeling we obtain the following:

(a) If $n = 8r, r \geq 1$, then $v_f(0) = v_f(1) = v_f(2) = v_f(3) = 2r$ and $e_f(0) = e_f(1) = e_f(2) = e_f(3) = 2r$.

(b) If $n = 8r+1, r \geq 1$, then $v_f(0) = 2r+1, v_f(1) = v_f(2) = v_f(3) = 2r$ and $e_f(0) = 2r+1, e_f(1) = e_f(2) = e_f(3) = 2r$.

(c) If $n = 8r+3, r \geq 0$, then $v_f(0) = 2r, v_f(1) = v_f(2) = v_f(3) = 2r+1$ and $e_f(0) = e_f(1) = e_f(2) = 2r+1, e_f(3) = 2r$.

(d) If $n = 8r+5, r \geq 0$, then $v_f(0) = v_f(1) = v_f(3) = 2r+1, v_f(2) = 2r+2$ and $e_f(0) = e_f(1) = e_f(3) = 2r+1, e_f(2) = 2r+2$.

(e) If $n = 8r+6, r \geq 0$, then $v_f(0) = v_f(2) = 2r+2, v_f(1) = v_f(3) = 2r+1$ and $e_f(0) = e_f(2) = 2r+2, e_f(1) = e_f(3) = 2r+1$.

Case(ii). For $n \equiv 2 \pmod 8$

$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 1,6 \pmod 8 \ i = n \text{ and } i \neq n-1 \\ 1, & \text{if } i \equiv 3 \pmod 4 \\ 2, & \text{if } i \equiv 2,5 \pmod 8, i \neq n \\ 3, & \text{if } i \equiv 0 \pmod 4, i = n-1 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r+2, r \geq 1$, then $v_f(0) = v_f(2) = 2r+1, v_f(1) = v_f(3) = 2r$ and $e_f(0) = e_f(3) = 2r+1, e_f(1) = e_f(2) = 2r$.

Case(iii). For $n \equiv 7 \pmod 8$

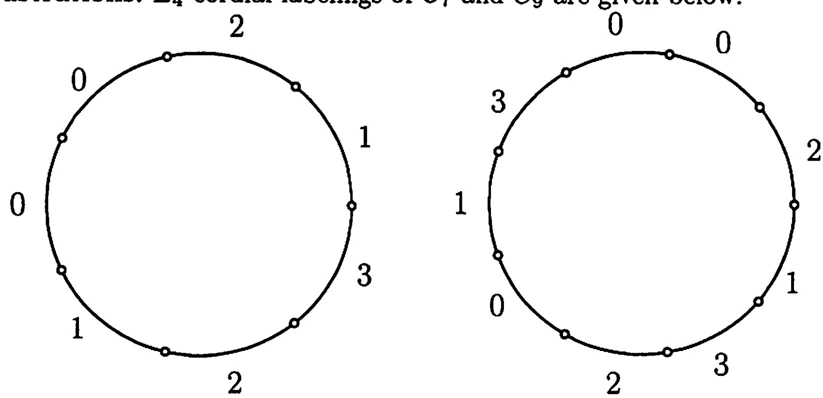
$$f(e_i) = \begin{cases} 0, & \text{if } i \equiv 1,6 \pmod 8 \ i = n \text{ and } i \neq n-1 \\ 1, & \text{if } i \equiv 3,7 \pmod 8 \ i = n-1 \text{ and } i \neq n \\ 2, & \text{if } i \equiv 2,5 \pmod 8 \\ 3, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

From the above labeling we obtain the following:

If $n = 8r+7, r \geq 0$, then $v_f(0) = v_f(1) = v_f(3) = 2r+2, v_f(2) = 2r+1$ and $e_f(0) = e_f(1) = e_f(2) = 2r+2, e_f(3) = 2r+1$.

Clearly $|e_f(i) - e_f(j)| \leq 1$ and $|v_f(i) - v_f(j)| \leq 1$ for every i, j in the above all cases. Hence C_n for $n \geq 3$ is E_4-cordial if $n \not\equiv 4 \pmod 8$. □

Illustrations: E_4-cordial labelings of C_7 and C_9 are given below.



Theorem 4. K_n is E_4-cordial for all $n \geq 3$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_{ji} = v_j v_i$ where $j = 1, 2, \ldots, n-1$; $i = n, n-1, \ldots, j+1$ are the edges of the complete graph K_n. Define $f : E(K_n) \to \{0, 1, 2, 3\}$ as follows:
Case(i). $n \equiv 0 \pmod 4$.

$$f(e_{ji}) = \begin{cases} 0, & \text{if } j \equiv 1, 0 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 2, 7 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 3, 6 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 4, 5 \pmod 8, i \equiv 2 \pmod 4 \\ 1. & \text{if } j \equiv 1, 0 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 2, 7 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 3, 6 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 4, 5 \pmod 8, i \equiv 1 \pmod 4 \\ 2 & \text{if } j \equiv 1, 0 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 2, 7 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 3, 6 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 4, 5 \pmod 8, i \equiv 0 \pmod 4 \\ 3 & \text{if } j \equiv 1, 0 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 2, 7 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 3, 6 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 4, 5 \pmod 8, i \equiv 3 \pmod 4 \end{cases}$$

Then

$$f(v_j) = \left(\sum_{i=n, i \neq j}^{1} f(e_{ji}) \right) \pmod 4 \text{ where } j = 1, 2, \ldots, n,$$

and therefore

$$v_f(i) = \frac{n}{4} \; \forall i = 0, 1, 2, 3.$$

Clearly

$$|e_f(i) - e_f(j)| \leq 1, \; \forall i, j = 0, 1, 2, 3.$$

Case(ii). $n \equiv 1 \pmod 4$.

$$f(e_{ji}) = \begin{cases} 0, & \text{if } j \equiv 1,2 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 3,0 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 4,7 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 5,6 \pmod 8, i \equiv 3 \pmod 4 \\ 1. & \text{if } j \equiv 1,2 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 3,0 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 4,7 \pmod 8, i \equiv 1 \pmod 4 \\ & j = n-1 \text{ and } i \neq n \text{ for } n \equiv 5 \pmod 8 \\ & j \equiv 5,6 \pmod 8, i \equiv 2 \pmod 4 \\ 2 & \text{if } j \equiv 1,2 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 3,0 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 4,7 \pmod 8, i \equiv 0 \pmod 4 \\ & j = n-1 \text{ and } i = n \text{ for } n \equiv 5 \pmod 8 \\ & j \equiv 5,6 \pmod 8, i \equiv 1 \pmod 4 \\ 3 & \text{if } j \equiv 1,2 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 3,0 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 4,7 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 5,6 \pmod 8, i \equiv 0 \pmod 4 \end{cases}$$

Then

$$v_f(i) = \begin{cases} \frac{n-1}{4} & \text{for 3 } i\text{'s} \\ \frac{n+3}{4} & \text{for the remaining one } i \end{cases}$$

and

$$|e_f(i) - e_f(j)| \leq 1, \ \forall i, j = 0, 1, 2, 3.$$

Case(iii). $n \equiv 2 \pmod 4$.

$$f(e_{ji}) = \begin{cases} 0, & \text{if } j \equiv 1,4 \pmod 8, i \equiv 2 \pmod 4 \\ & j = 4; i \neq n \text{ for } n \equiv 6 \pmod 8 \\ & j \equiv 2,3 \pmod 8, i \equiv 3 \pmod 4 \\ & j = 2; i \neq 3 \text{ for } n \equiv 2 \pmod 8 \\ & j \equiv 5,0 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 6,7 \pmod 8, i \equiv 1 \pmod 4 \\ 1. & \text{if } j \equiv 1,4 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 2,3 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 5,0 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 6,7 \pmod 8, i \equiv 0 \pmod 4 \\ 2 & \text{if } j \equiv 1,4 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 2,3 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 5,0 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 6,7 \pmod 8, i \equiv 3 \pmod 4 \\ 3 & \text{if } j \equiv 1,4 \pmod 8, i \equiv 3 \pmod 4 \\ & j = 4; i = n \text{ for } n \equiv 6 \pmod 8 \\ & j \equiv 2,3 \pmod 8, i \equiv 0 \pmod 4 \\ & j = 2; i = 3 \text{ for } n \equiv 2 \pmod 8 \\ & j \equiv 5,0 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 6,7 \pmod 8, i \equiv 2 \pmod 4 \end{cases}$$

Then

$$v_f(i) = \begin{cases} \frac{n+2}{4} & \text{for 2 } i\text{'s} \\ \frac{n-2}{4} & \text{for the remaining 2 } i\text{'s} \end{cases}$$

and

$$|e_f(i) - e_f(j)| \leq 1, \ \forall i, j = 0, 1, 2, 3.$$

Case(iv). $n \equiv 3 \pmod 4$.

$$f(e_{ji}) = \begin{cases} 0, & \text{if } j \equiv 1,6 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 2,5 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 3,4 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 7,0 \pmod 8, i \equiv 0 \pmod 4 \\ 1. & \text{if } j \equiv 1,6 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 2,5 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 3,4 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 7,0 \pmod 8, i \equiv 3 \pmod 4 \\ 2 & \text{if } j \equiv 1,6 \pmod 8, i \equiv 1 \pmod 4 \\ & j \equiv 2,5 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 3,4 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 7,0 \pmod 8, i \equiv 2 \pmod 4 \\ 3 & \text{if } j \equiv 1,6 \pmod 8, i \equiv 0 \pmod 4 \\ & j \equiv 2,5 \pmod 8, i \equiv 2 \pmod 4 \\ & j \equiv 3,4 \pmod 8, i \equiv 3 \pmod 4 \\ & j \equiv 7,0 \pmod 8, i \equiv 1 \pmod 4 \end{cases}$$

Then

$$v_f(i) = \begin{cases} \frac{n+1}{4} & \text{for 3 } i\text{'s} \\ \frac{n-3}{4} & \text{for the remaining one } i \end{cases}$$

and

$$|e_f(i) - e_f(j)| \leq 1, \ \forall i, j = 0, 1, 2, 3.$$

Thus in all cases $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $\forall i, j = 0, 1, 2, 3$.

Hence K_n for $n \geq 3$ is E_4-cordial. □

Illustrations: E_4-cordial labeling of K_8 is as follows:

Labeling of e_{ji}'s and hence v_j's are shown in the following table.

j \ i	8	7	6	5	4	3	2	v_j
1	0	1	2	3	0	1	2	1
2	3	0	1	2	3	0		3
3	1	2	3	0	1			0
4	2	3	0	1				2
5	2	3	0					3
6	1	2						1
7	3							2
8								0

Theorem 5. Every (p, q) graph where $p \geq 3$ is a subgraph of an E_4-cordial graph.

Proof. The result follows from the fact that every graph is a subgraph of K_n and K_n is E_4-cordial for all $n \geq 3$. □

References

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