Error-tolerance pooling design in a finite vector space

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Abstract

Most of pooling designs are always constructed by the "containment matrix". But we are interested in considering non-containment relationship. In [J. Guo, K. Wang, Pooling designs with surprisingly high degree of error correction in a finite vector space, Discrete Appl Math], Guo and Wang gave a construction by the use of non-containment relationship. In this paper, we generalize Guo-Wang's designs and obtain a new family of pooling designs. Our designs and Guo-Wang's designs have the same numbers of items and pools, but the error-tolerance property of our designs is better than that of Guo-Wang's designs.

Key words: Pooling design; Finite vector space; Error-tolerance property

1 Introduction

A pooling design is usually represented by a binary matrix whose columns are indexed with items and rows are indexed with pools. An entry at cell (i, j) is 1 if and only if the jth pool is contained by the ith item, and 0,

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otherwise. A mathematical model with error-correcting presented in [1] is an s^e-disjunct matrix. A binary matrix M is said to be s^e-disjunct if given any $s+1$ columns of M with one designated, there are $s+1$ rows with a 1 in the designate column and 0 in each of the other s columns (see [2]). An s^e-disjunct matrix can be employed to discern s defectives, detect e errors and correct $\lfloor e/2 \rfloor$ errors (see [3]). The s^e-disjunct matrix has become an important tool for determining pooling design.

In 1996 and 1997 Macula gave constructions by containment relation of subset in a finite set (see [4], [2]). And then the constructions of disjunct matrices by means of the containment relation of subspaces in finite classical spaces were given by many authors. Until 2011, Guo and Wang used the method of cardinal number of the intersection of subsets being a constant in finite set to obtain a construction with high degree of error correction (see [5]).

Next, we'll give an introduction about some fundamental knowledge used in this paper.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. Let n be a positive integer, denote by $\mathbb{F}_q^{(n)}$ the n-dimensional vector space over $\mathbb{F}_q$. For positive integers $m \leq n$, let $\begin{bmatrix} [n] \\ m \end{bmatrix}_q$ be all of the m-dimensional subspaces of $\mathbb{F}_q^{(n)}$.

Given integers m, n and prime power q. Then Gaussian coefficient denoted by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{n(n-1)\cdots(n-m+1)}{m!}, & \text{if } q = 1; \\ \frac{\prod_{i=n-m+1}^{n} (q^i - 1)}{\prod_{i=1}^{m} (q^i - 1)}, & \text{if } q \neq 1. \end{cases}$$

For convenience, we write $\binom{n}{m}$ to substitute $\begin{bmatrix} n \\ m \end{bmatrix}_1$. And we let $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$ whenever $m \leq 0$ or $n < m$.

In 2012, inspired by the concluding remark iv in [5], we have a new family of pooling designs with a higher degree of error correction under some conditions ([6]). Recently, they generalized the method of [5] in a finite vector space ([7]).

Lemma 1.1. *([7]) For* $\max\{0, r+m-n\} \leq j \leq r$ *and* $m \leq n$. *Let* P *be a given* m*-dimensional subspace of* $\mathbb{F}_q^{(n)}$ *and* Q_0 *be a given* j*-dimensional with* $Q_0 \subseteq P$. *Then the number of* r*-dimensional subspaces of* $\mathbb{F}_q^{(n)}$ *intersecting* P *at* Q_0 *is* $f(j, r, n; m) = q^{(r-j)(m-j)} \begin{bmatrix} n-m \\ r-j \end{bmatrix}_q$. *Moreover, for the integer* $0 \leq \alpha \leq n+j-m-r$, *the function* $f(j, r, n; m+\alpha)$ *about* α *is decreasing.*

Definition 1.2. *([7]) For positive integers* $1 \leq d < k < n$ *and* $\max\{0, d+k-n\} \leq i \leq d$. *Let* $M_q(i; d, k, n)$ *be the binary matrix with rows indexed with* $\begin{bmatrix} [n] \\ d \end{bmatrix}_q$ *and columns indexed with* $\begin{bmatrix} [n] \\ k \end{bmatrix}_q$ *such that* $M_q(A, B) = 1$ *if and only if* $\dim(A \cap B) = i$.

Theorem 1.3. *([7]) Let* i, d, k, n *be positive integers with* $\lfloor (d+1)/2 \rfloor \leq i \leq d < k$ *and* $n-k-s(k+d-2i) \geq d-i$. *If* $k-i \geq 2$ *and* $1 \leq s \leq q(q^{k-1}-1)/(q^{k-i}-1)$, *then* $M_q(i; d, k, n)$ *is an* s^e*-disjunct matrix, where*

$$\begin{aligned} e &= q^{(d-i)(k+s(k+d-2i)-i)} \begin{bmatrix} n-k-s(k+d-2i) \\ d-i \end{bmatrix}_q \\ &\quad (q^{k-i} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q - (s-1)q^{k-i-1} \begin{bmatrix} k-2 \\ i-1 \end{bmatrix}_q) - 1. \end{aligned}$$

In this paper, we take the method of [7] to study the corresponding problems of [6] in a finite vector space and give a construction as follows.

Definition 1.4. *For positive integers* $1 \leq i \leq d < k < n$, *let* I *be a nonempty subset of* $\{i, i+1, \ldots, d\}$, *and let* $M_q(I; d, k, n)$ *be the binary matrix with rows indexed with* $\begin{bmatrix} [n] \\ d \end{bmatrix}_q$ *and columns indexed with* $\begin{bmatrix} [n] \\ k \end{bmatrix}_q$ *such that* $M_q(A, B) = 1$ *if and only if* $\dim(A \cap B) \in I$.

Remark **1.1.** *If* $I = \{i\}$, *then* $M_q(I; d, k, n) = M_q(i; d, k, n)$.

Theorem 1.5. *Let* i, d, k, n *be positive integers with* $\lfloor (d+1)/2 \rfloor \leq i \leq d < k$, *and let* $\varepsilon = k+(\bar{s}-1)(k-2i+d)$, $n_{(i)} = q^{k-i} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q - (\bar{s}-1)q^{k-i-1} \begin{bmatrix} k-2 \\ i-1 \end{bmatrix}_q$. *Suppose* $I = \{i, i+1, \ldots, d\}$. *If* $1 \leq \bar{s} \leq \min\{\lceil \frac{q^k-q}{q^{k-i}-1} \rceil, \lfloor \frac{n-d-k}{k+d-2i} \rfloor + 1\}$, *then* $M_q(I; d, k, n)$ *is an* $\bar{s}^{e_1}$*-disjunct matrix, where*

$$e_1 = n_{(i)} q^{(d-i)\varepsilon} \begin{bmatrix} n-i-\varepsilon \\ d-i \end{bmatrix}_q - 1.$$

2 Proof of Theorem 1.5

Proof. Let $C_0, C_1, \cdots, C_{\bar{s}}$ be any $\bar{s}+1$ distinct columns of $M_q(I; d, k, n)$. To obtain the maximum numbers of subspaces $\begin{bmatrix} [n] \\ i \end{bmatrix}_q$ in

$$C_0 \cap \bigcup_{j=1}^{\bar{s}} C_j = \bigcup_{j=1}^{\bar{s}} (C_0 \cap C_j),$$

we may assume that $\dim(C_0 \cap C_j) = k-1$ and $\dim(C_0 \cap C_j \cap C_l) = k-2$, for any $j, l \in [\bar{s}] := \{1, 2, \ldots, \bar{s}\}$ and $j \neq l$. So the number of i-dimensional subspaces of C_0 not contained in $C_1, C_2, \cdots, C_{\bar{s}}$ is at least

$$\begin{aligned} n_{(i)} &= \begin{bmatrix} k \\ i \end{bmatrix}_q - \bar{s}\begin{bmatrix} k-1 \\ i \end{bmatrix}_q + (\bar{s}-1)\begin{bmatrix} k-2 \\ i \end{bmatrix}_q \\ &= q^{k-i}\begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q - (\bar{s}-1)q^{k-i-1}\begin{bmatrix} k-2 \\ i-1 \end{bmatrix}_q . \end{aligned}$$

Let D be a d-dimensional subspaces of $\mathbb{F}_q^{(n)}$, such that $\dim(D \cap C_0) \geq i$, Suppose $\dim(D \cap C_j) \geq i$ for $j \in [\bar{s}]$. By $D \cap C_1 + D \cap C_{j'} \subseteq D$, $j' \in \{2, 3, \cdots, \bar{s}\}$, we have

$$\begin{aligned} \dim(C_1 \cap C_{j'}) &\geq \dim(D \cap C_1 \cap C_{j'}) \\ &= \dim(D \cap C_1) + \dim(D \cap C_{j'}) - \dim(D \cap C_1 + D \cap C_{j'}) \\ &\geq 2i - d, \end{aligned}$$

then

$$\begin{aligned} & \dim(C_1 + C_2 + \cdots + C_{\bar{s}}) \\ = \; & \dim(C_1 + C_2 + \cdots + C_{\bar{s}-1}) + \dim C_{\bar{s}} - \dim((C_1 + C_2 + \cdots + C_{\bar{s}-1}) \cap C_{\bar{s}}) \\ \leq \; & \dim(C_1 + C_2 + \cdots + C_{\bar{s}-1}) + k - (2i - d) \\ \leq \; & \dim C_1 + (\bar{s}-1)(k - 2i + d) \\ = \; & k + (\bar{s}-1)(k - 2i + d). \end{aligned}$$

So we know that $\dim(C_1 + C_2 + \cdots + C_{\bar{s}}) \leq k + (\bar{s}-1)(k+d-2i)$. Briefly, we denote $\varepsilon = k + (\bar{s}-1)(k+d-2i)$. Let P_0 be a given i-dimensional subspaces of C_0 not contained in $C_1, C_2, \cdots, C_{\bar{s}}$. Clearly, $\dim(P_0 + C_1 + C_2 + \cdots + C_{\bar{s}}) \leq$

$i+\varepsilon$. Then the number of d-dimensional subspaces D in $\mathbb{F}_q^{(n)}$ satisfying $D \cap (P_0 + C_1 + C_2 + \cdots + C_{\bar{s}}) = P_0$, by Lemma 1.1, is at least

$$f(i,d,n;i+\varepsilon) = q^{(d-i)\varepsilon} \begin{bmatrix} n-i-\varepsilon \\ d-i \end{bmatrix}_q .$$

It is easily to see that $D \cap C_0 \supseteq P_0$ and $D \cap C_j \subset P_0$ for each $j \in [\bar{s}]$. Therefore, the number of d-dimensional subspaces D in $\mathbb{F}_q^{(n)}$ satisfying $\dim(D \cap C_0) \in I$ and $\dim(D \cap C_j) < i$ for each $j \in [\bar{s}]$ is at least

$$n_{(i)} q^{(d-i)\varepsilon} \begin{bmatrix} n-i-\varepsilon \\ d-i \end{bmatrix}_q .$$

Since $e_1 = n_{(i)} q^{(d-i)\varepsilon} \begin{bmatrix} n-i-\varepsilon \\ d-i \end{bmatrix}_q - 1 \geq 0$, $n_{(i)} > 0$, which implies that

$$1 \leq \bar{s} \leq \min\{\lceil \frac{q^k-q}{q^{k-i}-1} \rceil, \lfloor \frac{n-d-k}{k+d-2i} \rfloor + 1\}.$$

This proofs Theorem 1.5. □

3 Comparison of test efficiency

Theorem 3.1. *For positive integers* $\lfloor (d+1)/2 \rfloor \leq i \leq d < k < n$, *and* $1 \leq s^* \leq \min\{\lceil \frac{q^k-q}{q^{k-i}-1} \rceil, \lfloor \frac{n-d-k+i}{k+d-2i} \rfloor\}$. *If* $k+d > 3i$, *then* $e_1 > e$.

Proof. Suppose $k+d > 3i$, then $\frac{n-d-k}{k+d-2i} + 1 > \frac{n-d-k+i}{k+d-2i}$, and $n-i-k-(s^*-1)(k+d-2i) > n-k-s^*(k+d-2i)$. So

$$q^{(d-i)(k+(s^*-1)(k-2i+d))} \begin{bmatrix} n-i-k-(s^*-1)(k-2i+d) \\ d-i \end{bmatrix}_q$$

$$> q^{(d-i)(k+s^*(k-2i+d)-i)} \begin{bmatrix} n-k-s^*(k-2i+d) \\ d-i \end{bmatrix}_q .$$

Therefore $e_1 > e$ as claimed. □

Example 1. Let us consider $M_2(\{3,4,5\};5,7,30)$ and $M_2(3;5,7,30)$. We have

s^*	1	2	3
e_1+1	$3.127290634 \times 10^{19}$	$2.754497577 \times 10^{19}$	$2.354854839 \times 10^{19}$
$e+1$	$3.127228003 \times 10^{19}$	$2.750967678 \times 10^{19}$	$2.16397962 \times 10^{19}$

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