E_3-cordiality of some helm-related graphs

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Abstract

A k-edge labeling of a graph G is a function $f : E(G) \to \{0, \ldots, k-1\}$. Such a labeling induces a labeling on the vertex set $V(G)$ by defining $f(v) := \sum f(e)(\mod k)$, where the summation is taken over all the edges incident on the vertex v. For an edge labeling f, let $v_f(i)$ (respectively $e_f(i)$) be the number of vertices (respectively edges) receiving the label i. A graph G is said to be $\mathbf{E}_k$-cordial if there is an k-edge labeling f of G such that, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$.

A **wheel** W_n is the join of the cycle C_n on n vertices and K_1. A Helm H_n is obtained by attaching a pendent edge to each vertex of the cycle of the wheel W_n.. We prove that (i) Helms (ii) One point union of helms (iii) path union of helms are E_3-cordial.

Introduction

All graphs considered here are finite and simple. By a (p, q)-graph we mean a graph on p vertices and q edges. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A k-edge labeling of a graph G is a function $f : E(G) \to \{0, \ldots, k-1\}$. Such a labeling induces a labeling on the vertex set $V(G)$ by defining $f(v) := \sum f(e)(\mod k)$, where the summation is taken over all the edges incident on the vertex v. For an edge labeling f, let $v_f(i)$ (respectively $e_f(i)$) be the number of vertices (respectively edges) receiving the label i. A graph G is said to be **$\mathbf{E}_k$-cordial** if there is an k-edge labeling f of G such that, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. The map f is then called an E_k-cordial labeling of G. Let f be a 3-edge labeling of a graph G. By $v_f(0,1,2)$ and $e_f(0,1,2)$ we mean the triples $(v_f(0), v_f(1), v_f(2))$ and

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$(e_f(0), e_f(1), e_f(2))$ respectively. These are called the vertex numbers and the edge numbers of the labeling f.

The concept of **E**$_3$-cordial graphs was introduced by Cahit and Yilmaz. They proved that the following graphs are E_3-cordial: P_n $(n \geq 3)$; stars S_n if and only if $n \not\equiv 1 \pmod 3$; K_n $(n \geq 3)$; C_n $(n \geq 3)$; friendship graphs; and fans F_n $(n \geq 3)$. They also prove that S_n $(n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1 \bmod k$ when k is odd or $n \not\equiv 1 \bmod 2k$ when k is even and $k \neq 2$.

In this paper we discuss the helms, one point union of helms, path-union of helms and gear graphs and give **E**$_3$-cordial labelings for them.

Let $\{G_1, \ldots, G_t\}$ be a family of graphs with $n_1, \ldots, n_t$ vertices and $q_1, \ldots, q_t$ edges respectively. The **one point union** of these graphs is obtained by choosing one vertex from each $G_i, 1 \leq i \leq t$ and identifying all of these chosen vertices. If G is the one point union of $G_1, \ldots, G_t$ then one can see that $|V(G)| = \sum_{i=1}^{t} n_i - t + 1$ and $|E(G)| = \sum_{i=1}^{t} q_i$.

For either one point union or path union we first try to obtain such unions with $|V(G)| \equiv 1 (\bmod 3)$ and construct an E_3-labeling such that identified vertex gets the label 0 and all other vertices and edges are equitably labeled. Later for the one point unions of larger families, it is easy to take groups of three and finish the labeling.

E$_3$-Cordiality of Helms

Definition: Helm H_n is a graph defined as follows:

$$\begin{aligned} V(H_n) &= \{v_0, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n\} \\ E(H_n) &= \{v_0 v_i / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n\} \cup \{v_i w_i / 1 \leq i \leq n\} \end{aligned}$$

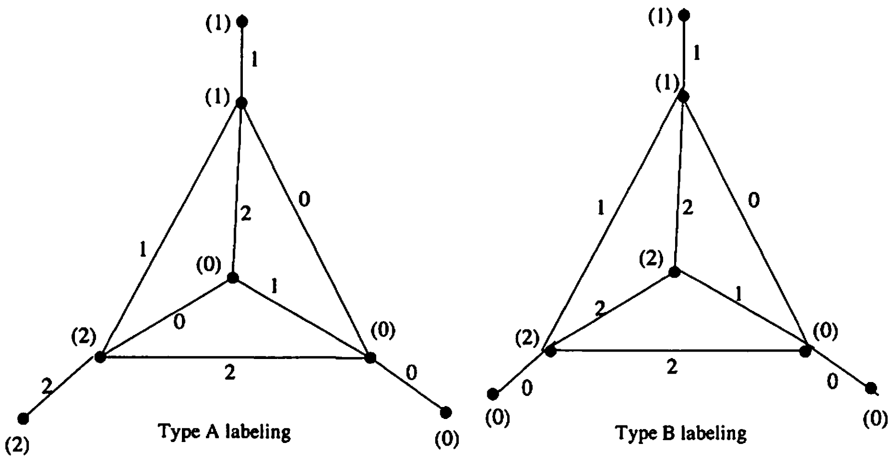
where $(i+1)$ is taken modulo n.

Thus H_n has $(2n+1)$ vertices and $3n$ edges. The edges $p_i = v_i w_i, 1 \leq i \leq n$, are called the pendent edges and the edges $c_i = v_i v_{i+1}, 1 \leq i \leq n$, where $i+1$ is taken modulo n are called the cyclic edges. Finally, the edges $e_i = v_0 v_i, 1 \leq i \leq n$ are called the spokes. The vertex v_0 is called the hub.

A helm H_n is said to be of type 1, 2, 3 if $n \equiv 1,2,3 \pmod 3$ respectively. In what what follows we give more than one labelings, not necessarily E_3-cordial, of the helms. These will be used in the next section. Only for the first case we will give explicit counting for $v_f(0,1,2)$ as well as $e_f(0,1,2)$.

Labelings for Type 3 helms

Let H_n be of Type **3** and let $n = 3y$. The following figure gives an E_3-cordial labeling of H_3. The vertex numbers and the edge numbers for both are $(3,2,2)$ and $(3,3,3)$ respectively.



Let $n = 3x + 6, x \geq 0$. Label the edges as follows: $f(p_1) = 2, f(p_2) = 1, f(p_3) = 0$. If $x > 0$, let $f(p_{3t+4}) = 1 = f(p_{3t+5}), f(p_{3t+6}) = 0, 0 \leq t \leq x - 1$. This means the sequence $1, 1, 0$ is repeated x times. This finishes labeling of the pendent edges upto p_{3x+3}. The last three pendent edges are labeled as $f(p_{3x+4}) = 2, f(p_{3x+5}) = 1, f(p_{3x+6}) = 1$. If $x = 0$, this gives labels of p_4, p_5, p_6.

The edges in the cycle are labeled as follows: $f(c_1) = 1, f(c_2) = 0, f(c_3) = 2, f(c_{3x+4}) = 2, f(c_{3x+5}) = 0 = f(c_{3x+6})$. If $x > 0$, we define $f(c_i) = 2$ for $i = 3t + 4, 3t + 6$ and $f(c_i) = 0$ for $i = 3t + 5, 0 \leq t \leq x - 1$.

Finally the spokes are labeled as $f(e_{3t+1}) = 0, f(e_{3t+2}) = 2, f(e_{3t+3}) = 1, 0 \leq t \leq x$. Thus the sequence $0, 2, 1$ is repeated $x + 1$ times. The last three edges are labeled as $f(e_{3x+4}) = 2, f(e_{3x+5}) = 0, f(e_{3x+6}) = 1$. Clearly,

$e_f(0,1,2) = (3x+6, 3x+6, 3x+6)$.

The vertex numbers are as follows: For a pendent vertex $f(w_i) = f(p_i), 1 \leq i \leq n$. For the cyclic vertices we have

$f(v_1) = f(p_1) + f(e_1) + f(c_1) + f(c_{3x+6}) = 2+0+1+0 \bmod 3 = 0$,
$f(v_2) = f(p_2) + f(e_2) + f(c_2) + f(c_1) = 1+2+0+1 \bmod 3 = 1$,
$f(v_3) = f(p_3) + f(e_3) + f(c_2) + f(c_3) = 0+1+0+2 \bmod 3 = 0$,
$f(v_{3t+4}) = f(p_{3t+4}) + f(e_{3t+4}) + f(c_{3t+3}) + f(c_{3t+4}) = 1+0+2+2 \bmod 3 = 2$,
$f(v_{3t+5}) = f(p_{3t+5}) + f(e_{3t+5}) + f(c_{3t+4}) + f(c_{3t+5}) = 1+2+2+0 \bmod 3 = 2$,
$f(v_{3t+6}) = f(p_{3t+6}) + f(e_{3t+6}) + f(c_{3t+5}) + f(c_{3t+6}) = 0+1+0+2 \bmod 3 = 0, 0 \leq t \leq x-1$. Finally,
$f(v_{3x+4}) = f(p_{3x+4}) + f(e_{3x+4}) + f(c_{3x+3}) + f(c_{3x+4}) = 2+2+2+2 \bmod 3 = 2$,
$f(v_{3x+5}) = f(p_{3x+5}) + f(e_{3x+5}) + f(c_{3x+4}) + f(c_{3x+5}) = 1+0+2+0 \bmod 3 = 0$,
$f(v_{3x+6}) = f(p_{3x+6}) + f(e_{3x+6}) + f(c_{3x+5}) + f(c_{3x+6}) = 1+1+0+0 \bmod 3 = 2$,

The central vertex v_0 has $x+2$ edges of each label incident on it. Thus $f(v_0) = 0$. This shows that $v_f(0,1,2) = (2x+5, 2x+4, 2x+4)$, that is H_{3y} is E_3-cordial. This is called the labeling of type A_3 Thus for $n = 3y, y \geq 1$, we have an E_3-cordial labeling A_3 such that $v_{A_3}(0,1,2) = (2y+1, 2y, 2y), e_{A_3}(0,1,2) = (3y, 3y, 3y)$.

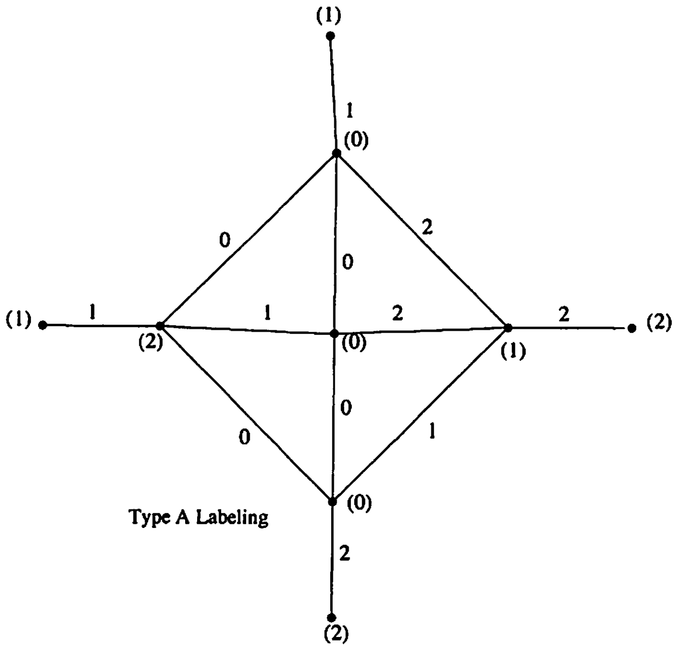
We create another labeling, called the labeling of type B_3, from the labeling of type A_3 by changing the label of the pendent edge p_1 to 0 and that of the spoke e_1 to 2. This is again an E$_3$-cordial labeling in which the hub gets the label 2.

The following table gives the vertex numbers and the edge numbers of these labelings of a helm of H_n type 3 with $n = 3y, y \geq 1$.

f	The label of the Hub	$v_f(0,1,2)$	$e_f(0,1,2)$
A_3	0	$(2y+1, 2y, 2y)$	$(3y, 3y, 3y)$
B_3	2	$(2y+1, 2y, 2y)$	$(3y, 3y, 3y)$

Labelings for helms of Type 1:

The following figures give three E_3-cordial labeling for H_4.

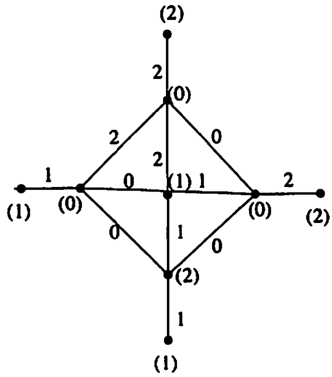


Type A Labeling

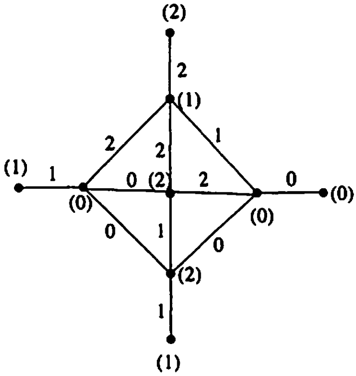
Now consider H_n with $n = 3y + 1, y \geq 2$. Let $y = 1 + x, x \geq 1$, that is, $n = 3x + 4$. Label the edges as follows: $f(p_1) = 2$. If $x > 0$, let $f(p_{3t+2}) = 1 = f(p_{3t+3}), f(p_{3t+4}) = 0, 0 \leq t \leq x - 1$. This means the sequence $1, 1, 0$ is repeated x times. This finishes labeling of the pendent edges up to p_{3x+1}. The last three pendent edges are labeled as $f(p_{3x+2}) = 2, f(p_{3x+3}) = 1, f(p_{3x+4}) = 1$. If $x = 0$, this gives the labels of p_2, p_3, p_4.

The edges in the cycle are labeled as follows: $f(c_1) = 1, f(c_{3x+2}) = 2, f(c_{3x+3}) = 0 = f(c_{3x+4})$. If $x > 0$, we define $f(c_i) = 2$ for $i = 3t+2, 3t+4$ and $f(c_i) = 0$ for $i = 3t + 3, 0 \leq t \leq x - 1$. This means the sequence $2, 0, 2$ is repeated x times.

Finally the spokes are labeled as $f(e_1) = 0, f(e_{3t+2}) = 0, f(e_{3t+3}) = 2, f(e_{3t+4}) = 1, 0 \leq t \leq x - 1$. Thus the sequence $0, 2, 1$ is repeated x times. The last three edges are labeled as $f(e_{3x+2}) = 2, f(e_{3x+3}) = 0, f(e_{3x+4}) = 1$. Clearly, $e_f(0, 1, 2) = (3x + 4, 3x + 4, 3x + 4)$. This is called the labeling



Type B Labeling



Type C Labeling

of type A_1. One can check that the hub gets the label 0 and $v_f(0,1,2) = (2x+3, 2x+3, 2x+3)$.

We give here two more labelings for H_{3y+1}.

The labeling of type B_1 is obtained from the labeling of type A_1 by changing the label of the pendent edge p_2 to 0 and changing the label of the spoke e_2 to 1. The labeling remains E_3-cordial with the hub getting the label 1. The labeling of type C_1 is obtained from the labeling of type A_1 by changing the label of the pendent edge p_1 to 0 and changing the label of the spoke e_2 to 2. The labeling remains E_3-cordial with the hub getting the label 2.

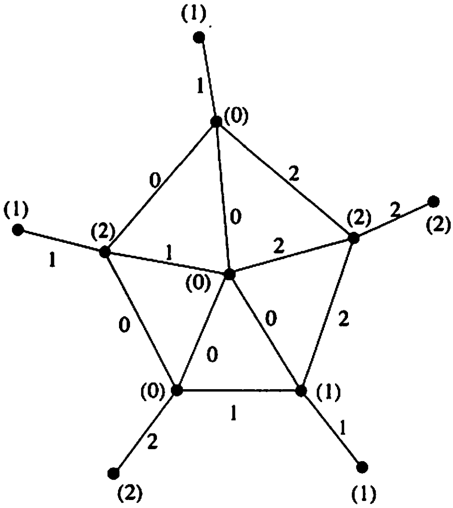
The vertex numbers and edge numbers of these labelings are given by the following table.

f	Label of Hub	$v_f(0,1,2)$	$e_f(0,1,2)$
A_1	0	$(2y+1, 2y+1, 2y+1)$	$(3y+1, 3y+1, 3y+1)$
B_1	1	$(2y+1, 2y+1, 2y+1)$	$(3y+1, 3y+1, 3y+1)$
C_1	2	$(2y+1, 2y+1, 2y+1)$	$(3y+1, 3y+1, 3y+1)$

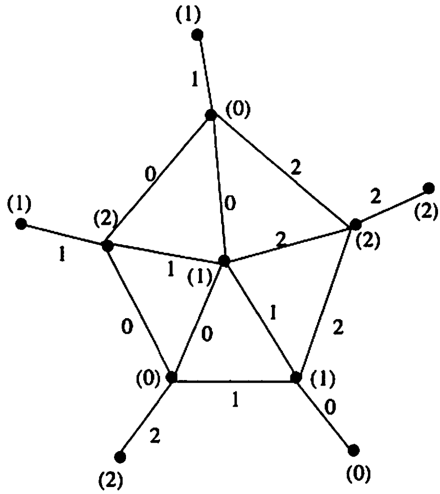
Labelings for helms of Type 2:

The following figures give four labeling of H_5 only three of which are E_3-cordial and the fourth labeling is equitable on only the edges.

Let $n = 3y+2$ and $y = 1+x$, that is, $n = 3x+5, x \geq 1$. Label



Type A Labeling



Type B Labeling

the edges as follows: $f(p_1) = 2, f(p_2) = 1$. If $x > 0$, let $f(p_{3t+3}) = 1 = f(p_{3t+4}), f(p_{3t+5}) = 0, 0 \leq t \leq x-1$. This means the sequence $1, 1, 0$ is repeated x times. This finishes labeling of the pendent edges upto p_{3x+2}. The last three pendent edges are labeled as $f(p_{3x+3}) = 2, f(p_{3x+4}) = 1, f(p_{3x+5}) = 1$.

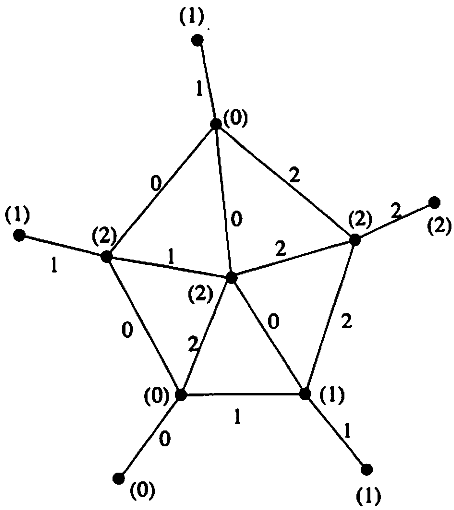
The edges in the cycle are labeled as follows: $f(c_1) = 1, f(c_2) = 2, f(c_{3x+3}) = 2, f(c_{3x+4}) = 0 = f(c_{3x+5})$. If $x > 0$, we define $f(c_i) = 2$ for $i = 3t+3, 3t+5$ and $f(c_i) = 0$ for $i = 3t+4, 0 \leq t \leq x-1$.

Finally the spokes are labeled as $f(e_1) = 0, f(e_2) = 0, f(e_{3t+3}) = 0, f(e_{3t+4}) = 2, f(e_{3t+5}) = 1, 0 \leq t \leq x-1$. Thus the sequence $0, 2, 1$ is repeated x times. The last three edges are labeled as $f(e_{3x+3}) = 2, f(e_{3x+4}) = 0, f(e_{3x+5}) = 1$. Clearly, $e_f(0,1,2) = (3x+5, 3x+5, 3x+5)$.

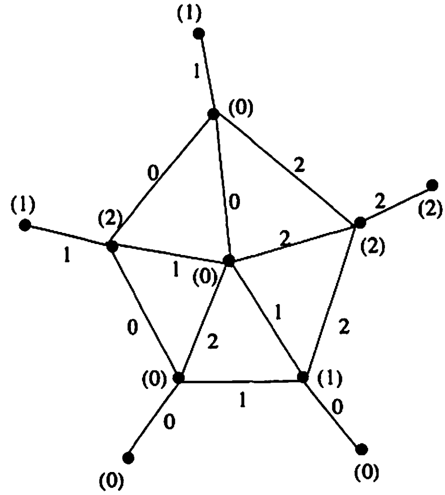
One can check that $v_f(0,1,2) = (2x+3, 2x+4, 2x+4)$, that is H_{3y+2} is E_3-cordial. This is called the type A_2 labeling. The label of the hub is 0.

We obtain three more labelings by making slight changes in the labeling of type A_2.

The labeling of type B_2 is obtained from the labeling of type A_2 by changing the label of the pendent edge p_2 to 0 and that of e_2 to 1. This



Type C Labeling



Type D Labeling

changes the label of w_2 to 0 and that of the hub to 1. The label numbers remain same.

The labeling of type C_2 is obtained from the labeling of type A_2 by changing the label of the pendent edge p_1 to 0 and that of e_1 to 2. This changes the label of w_1 to 0 and that of the hub to 2. The label numbers remain same.

The labeling of type D_2 is obtained from the labeling of type A_2 by changing the label of the pendent edge p_1 to 0 and changing the label of the spoke e_1 to 2 and changing the label of the pendent edge p_2 to 0 and that of the spoke e_2 to 1. The labeling is not E_3-cordial though it assigns labels to the edges equitably. The hub gets the label 0. The label numbers of this labeling are $v_{D_2}(0,1,2) = (2x+5, 2x+3, 2x+3), e_{D_2}(0,1,2) = (3x+5, 3x+5, 3x+5)$.

Hence if $n = 3y+2$ the following table gives the vertex numbers and edge numbers of all the four labelings.

f	Label of the Hub	$v_f(0,1,2)$	$e_f(0,1,2)$
A_2	0	$(2y+1, 2y+2, 2y+2)$	$(3y+2, 3y+2, 3y+2)$
B_2	1	$(2y+1, 2y+2, 2y+2)$	$(3y+2, 3y+2, 3y+2)$
C_2	2	$(2y+1, 2y+2, 2y+2)$	$(3y+2, 3y+2, 3y+2)$
D_2	0	$(2y+3, 2y+1, 2y+1)$	$(3y+2, 3y+2, 3y+2)$

Remark: A damaged helm is a graph $\tilde{H}_n = H_n - s$ where S is a subset of $E(H_n)$. For a labeling $\tilde{f}$ we mean the restriction of f to $\tilde{H}_n$. From the following table one can see that some damaged helms are E_3-cordial.

$\tilde{H}_n$	Labeling f	$v_f(0,1,2)$	$e_f(0,1,2)$
$H_{3y} - \{p_3\}$	$\tilde{A}_3$	$(2y, 2y, 2y)$	$(3y-1, 3y, 3y)$
$H_{3y} - \{p_3, p_1\}$	$\tilde{A}_3$	$(2y, 2y-1, 2y)$	$(3y-1, 3y-1, 3y)$
$H_{3y+1} - \{p_1\}$	$\tilde{A}_1$	$(2y+1, 2y+1, 2y)$	$(3y+1, 3y+1, 3y)$
$H_{3y+1} - \{p_1, p_2\}$	$\tilde{A}_1$	$(2y+1, 2y, 2y)$	$(3y+1, 3y, 3y)$
$H_{3y+2} - \{p_1\}$	$\tilde{A}_2$	$(2y+1, 2y+2, 2y+1)$	$(3y+2, 3y+2, 3y+1)$
$H_{3y+2} - \{p_1, p_2\}$	$\tilde{A}_2$	$(2y+1, 2y+1, 2y+1)$	$(3y+21, 3y+1, 3y+1)$

This prompts us to ask the following questions:

1: How many consecutive pendent edges can one delete and still get an E$_3$-cordial damaged helm?
2: 1: How many consecutive spokes can one delete and still get an E$_3$-cordial damaged helm?
3: What is the maximum number of edges one can delete and still get an E$_3$-cordial damaged helm?

One Point Unions of Helms

In this section we consider one point unions of helms and investigate them for E_3-cordiality.

By the one point union of helms $H_{n_1}, \ldots, H_{n_K}$ we mean vertex disjoint union of $H_{n_1}, \ldots, H_{n_K}$ having a common hub. This is often called a multiple helm of K number of helms. If only 2 helms are involved the then their one point union is called a double helm. If 3 helms are involved then it is called a triple helm.

A multiple helm of K helms is called **homogeneous of type i, i = 1, 2 and 3** if all the K helms are of same type i. Otherwise it is called a **mixed multiple helm.**

Remark: If there are K helms $H_1, \ldots, H_K$ of Type 3 and G is the multiple helm obtained from them, then for all the K helms one can use the E_3-cordial labeling of of type A_3 given in the previous section and get a labeling f_3 of G. If $|V(H_i)| = 6y_i + 1, 1 \leq i \leq K$, then one can check that $|V(G)| = 6\sum y_i + 1, |E(G) = 9\sum y_i + 3K$. If we denote $\sum y_i$ by Y, then $v_{f_3}(0,1,2) = (2Y+1, 2Y, 2Y)$ and $e_{f_3}(0,1,2) = (3Y+K, 3Y+K, 3Y+K)$. Thus f is an E_3-cordial labeling no matter what the value of K is. When we consider a mixed multiple helm G, we can simply assign the labeling f_3 to the homogeneous helm of type 3 within G. •

The following theorem just indicates how the labeling will proceed in the final result.

Theorem: If G is a homogeneous triple helm or a mixed double helm obtained by using a helm of type 1 and a helm of type 2 then G is $\mathbf{E_3}$-cordial.

Proof: If all the three helms are of type **3**, then by the previous remark G is $\mathbf{E_3}$-cordial.

Case 1: Consider 3 helms H_{n_i}, $1 \leq i \leq 3$ of type 1.

Assign the labeling A_1 to H_1, assign the labeling B_1, to H_2 and assign the labeling C_1 to H_3. Call the resulting labeling f_1. If $n_i = 3y_i + 1, 1 \leq i \leq 3$, let $Y = \sum_{i=1}^{3} y_i$. One can check that $|V(G)| = 6Y + 6 + 1, |E(G)| = 9Y + 9$. Moreover, $v_{f_1}(0,1,2) = (2Y+3, 2Y+2, 2Y+2), e_{f_1}(0,1,2) = (3Y+3, 3Y+3, 3Y+3)$ and the hub receives the label 0.

Case 2: The three helms are of type 2. Let the 3 helms be $H_{n_1}, H_{n_2}, H_{n_K}$. Assign the labeling of type A_2 to the helms H_1 and assign the labeling of type D_2 to H_{n_2} and H_{n_3}. Call the resulting labeling f_2. If $n_i = 3y_i + 2, 1 \leq i \leq 3$, let $Y = y_1 + y_2 + y_3$. Clearly $|V(G)| = 6Y + 13, |E(G)| = 9Y + 18$. One can check that the hub gets the label 0 and $v_{f_2}(0,1,2) = (2Y+5, 2Y+4, 2Y+4)$ and $e_{f_2}(0,1,2) = (3Y+6, 3Y+6, 3Y+6)$. Hence G is $\mathbf{E_3}$-cordial.

Case 3: Now suppose G is a double shell obtained by helms H, H' of type

$1,2$ respectively. Let $H = H_{n_1}$ and $H' = H_{n_2}$, where $n_1 = 3y_1 + 1, n_2 = 3y_2+2$. Assign the labellings A_1, D_2 to H, H' respectively. Call the resulting labeling $f_{1,2}$. If $Y = y_1+y_2$, then $|V(G)| = 6Y+7, |E(G)| = 9Y+9$. One can check that $v_{f_{1,2}} = (2Y+3, 2Y+2, 2Y+2)$ and $e_{f_{1,2}} = (3Y+3, 3Y+3, 3Y+3)$. The hub gets the label 0. •

Now we are ready to prove the general result.

Theorem: A multiple helm G obtained by taking K_i helms of type $i, 1 \leq i \leq 3$ is E_3-cordial.

Proof: Let the helms be $H_1, \ldots, H_{K_1}, \ldots, H_{K_1+K_2+K_3}$, where $H_1, \ldots, H_{K_1}$ are of type 1, $H_{K_1+1}, \ldots,$
$H_{K_1+K_2}$ are of type 2 and $H_{K_1+K_2+1}, \ldots, H_{K_1+K_2+K_3}$ are of type 3.

Assign the labeling f_3 to the one point union of $H_{K_1+1}, \ldots, H_{K_1+K_2}$. Suppose $K_1 = 3L_1 + r_1$. form L_1 number of homogeneous triple helms of type 1 using $H_1, \ldots, H_{3L_1}$. Assign the labeling of type f_1 to each triple helm. Suppose $K_2 = 3L_2 + r_1$. form L_2 number of homogeneous triple helms of type 2 using $H_{K_1+1}, \ldots, H_{K_1+3L_2}$. Assign the labeling of type f_2 to each triple helm. So far the hub has been assigned the label 0. Moreover other vertices as well as all the edges receive the labels $0, 1, 2$ equitably. If $r_1 = r_2 = 0$, our labeling process is complete.

If $r_1 = 1, r_2 = 0$, assign the labeling of type A_1 to H_{K_1}. If $r_1 = 0, r_2 = 1$, assign the labeling of type A_2 to $H_{K_1+K_2}$ and if $r_1 = 1 = r_2$, assign the labeling of type $f_{1,2}$ to the one point union of H_{K_1} and $H_{K_1+K_2}$. Finally if $r_1 = 2 = r_2$, assign the labelings of type A_1, B_1 to the two helms of type 1 and the labelings of type A_2, D_2 to the two helms of type 2. This will produce an E_3-cordial labeling of the residual four-tuple helm. •

Path Unions of Helms:

In this section, since the order in which the helms are taken is important, we take Helms of same Type while considering their path union.

Definition: For a natural number m, $PH_m(i)$ is the family of graphs obtained by taking a path of length $m - 1$, that is, with m vertices and attaching a copy of an helm of type i at each vertex of the path, taking care that all helms attached this way are attached at their hubs.

The following theorem shows that each graph in $PH_m(i)$ is E_3-cordial for $1 \leq i \leq 3$. For $G \in PH_m(i)$, let the helms attached to the m vertices of a path be $H_{3y_1+i}, \ldots, H_{3y_m+i}$. Let $Y = \sum_{j=1}^{m} y_j$.

Theorem: Each graph in $PH_m(i)$ is E_3-cordial for all $m \in \mathbb{N}$ and $1 \leq i \leq 3$.

Proof: Case 1 Let $i = 3$. Let $G \in PH_m(3)$.

If $m = 1$, then we have an E_3-cordial labeling A_3 of a helm of type 3.

If $m = 2$, take an edge v_1v_2 and assign the label 2 to this edge. Now attach the first helm to v_1 with the labeling A_3 and the second helm to v_2 with the labeling B_3. Let the resulting labeling be f. Both the labelings A_3 and B_3 are E_3-cordial. The labeling A_3 assigns the label 0 to v_1. Thus, $f(v_1) = 2$. Similarly, $f(v_2) = 1$, since B_3 assigns the label 0 to v_2. Clearly, $|V(G)| = 6Y + 2, |E(G)| = 9Y$ and $v_f(0,1,2) = (2Y+1, 2Y+1, 2Y)$ and $e_f(0,1,2) = (3Y, 3Y, 3Y+1)$, that is, f is an E_3-cordial labeling.

$m = 3p, p \geq 1$. Take a path $\{v_1, v_2, \ldots, v_{3p}\}$ and assign the labels $1,2,0,1,2,0,\ldots,1,2,0,1,2$ to the edges on this path in this order. Attach a helm H_{3y_j} with the labeling A_3 to the vertex $v_j, 1 \leq j \leq 3p$. The point of identification being the hub of that helm in each case. Call the resulting labeling f. The vertices $v_1, v_1, \ldots, v_{3p}$ are assigned the labels $1,0,2,1,0,2,\ldots,1,0,2$ in this order. One can check that $|V(G)| = 6Y + 3p, |E(G)| = 9Y + 3p - 1$ and $v_f(0,1,2) = (2Y+p, 2Y+p, 2Y+p), e_f(0,1,2) = (3Y+p-1, 3Y+p, 3Y+p)$, that is f is an **E**$_3$-cordial labeling.

$m = 3p+1, p \geq 1$. Take the earlier labeling constructed for a path of length $3p-1$. Assign the label 0 to the last edge $v_{3p}v_{3p+1}$. Now attach a helm to v_{3p+1} with the labeling A_3. Call the resulting labeling f. The label of v_{3p} remains 2. Since A_3 assigns the label 0 to the hub, the vertex v_{3p+1} gets the label 0. One can check that $|V(G) = 6Y+3p+1, |E(G)| = 9Y+3p$ and $v_f(0,1,2) = (2Y+p+1, 2Y+p, 2Y+p), e_f(3Y+p, 3Y+p, 3Y+p)$.(the value of Y of course has increased by y_{3p+1})

$m = 3p+2, p \geq 1$. Take the earlier labeling constructed for a path of length $3p-3$. Now four extra edges on the path remain to be labeled. Those are $v_{3p-2}v_{3p-1}, v_{3p-1}v_{3p}, v_{3p}v_{3p+1}, v_{3p+1}v_{3p+2}$. Assign the labels $1, 2, 2, 0$ to them in this order. Attach a helm of type 3 with the labeling of type A_3 to the last four points. Call the resulting labeling f. The label of v_{3p-2} changes from 0 to 1. The labels of the vertices $v_{3p-1}, v_{3p}, v_{3p+1}, v_{3p+2}$ are now $0, 1, 2, 0$ in this order. Thus the vertex and edge numbers are $v_f(0,1,2) = (3Y + p + 1, 3Y + p + 1, 3Y + p)$ and $e_f(0,1,2) = (3Y + p, 3Y + p, 3Y + p + 1)$. The value of Y is again increased by y_{3p+2}.

this completes the proof of the assertion that every $G \in PH_m(3)$ is E_3-cordial for all $m \geq 1$.

Case 2 Let $i = 1$. Let G be in $PH_m(1)$.

If $m = 1$, then we have an $\mathbf{E}_3$-cordial labeling A_1 of a helm of type 3.

If $m = 2$, take and edge v_1v_2 and assign the label 0 to this edge. Now attach the a helm with the labeling of type A_1 to v_1 as well as v_2. Let the resulting labeling be f. Clearly, $v_f(0,1,2) = (2Y + 2, 2Y + 2, 2Y + 2)$ and $e_f(0,1,2) = (3Y + 3, 3Y + 2, 3Y + 2)$, that is, f is an $\mathbf{E}_3$-cordial labeling.

$m = 3p, p \geq 1$. Take a path $\{v_1, v_2, \ldots, v_{3p}\}$ and assign the labels $1, 2, 0, 1, 2, 0, \ldots, 1, 2, 0, 1, 2$ to the edges on this path in this order. Attach a helms of type 1 with the labelings of type $B_1, A_1, C_1, \ldots, B_1, A_1, C_1$ to the vertices $v_1, \ldots, v_{3p}$ respectively. The point of identification being the hub of that helm in each case. As a result these points get the labels $2, 0, 1, \ldots, 2, 0, 1$ respectively. Call the resulting labeling f. One can check that $|V(G)| = 6Y + 9p, |E(G)| = 9Y + 12p - 1$ and $v_f(0,1,2) = (2Y + 3p, 2Y + 3p, 2Y + 3p), e_f(0,1,2) = (3Y + 4p - 1, 3Y + 4p, 3Y + 4p)$, that is f is an $\mathbf{E}_3$-cordial labeling.

$m = 3p + 1, p \geq 1$. Take the earlier labeling constructed for a path of length $3p-1$. Assign the label 0 to the last edge $v_{3p}v_{3p+1}$. Now attach a helm to v_{3p+1} with the labeling A_1. Call the resulting labeling f. The label of v_{3p} remains 2. Since A_3 assigns the label 0 to the hub, the vertex v_{3p+1} gets the label 0. One can check that $|V(G) = 6Y + 9p + 3, |E(G)| = 9Y + 12p + 3$ and $v_f(0,1,2) = (2Y + 3p + 1, 2Y + 3p + 1, 2Y + 3p + 1), e_f(3Y + 4p + 1, 3Y + 4p + 1, 3Y + 4p + 1)$.(the value of Y of course has increased by y_{3p+1})

$m = 3p + 2, p \geq 1$. Take the earlier labeling constructed for a path of length $3p-3$. Now four extra edges on the path remain to be labeled. Those are $v_{3p-2}v_{3p-1}, v_{3p-1}v_{3p}, v_{3p}v_{3p+1}, v_{3p+1}v_{3p+2}$. Assign the labels $1, 2, 2, 0$ to them in this order. Attach a helm of type 3 with the labeling of type A_3 to the last four points. Call the resulting labeling f. The label of v_{3p-2} changes from 0 to 1. The labels of the vertices $v_{3p-1}, v_{3p}, v_{3p+1}, v_{3p+2}$ are now $0, 1, 2, 0$ in this order. Thus, $|V(G)| = 6Y + 9p + 6, |E(G)| = 9Y = 12p + 7$ and the vertex and edge numbers are $v_f(0,1,2) = (2Y + 3p + 2, 2Y + 3p + 2, 2Y + 3p + 2)$ and $e_f(0,1,2) = (3Y + 7p + 3, 3Y + 7p + 2, 3Y + 7p + 2)$.

this completes the proof of the assertion that $PH_m(1)$ is $\mathbf{E}_3$-cordial for all $m \geq 1$.

Case 3 Let $i = 2$. Let G be in $PH_m(2)$.

If $m = 1$, all the three labelings A_2, B_2, C_2 are E_3-cordial. If $m = 2$, take one edge with the label 0. Attach a helm of type 2 with the labeling A_2 to the end vertices at the hubs. Call the resulting labeling f. The label numbers are $v_f(0,1,2) = (2Y+4, 2Y+3, 2Y+3), e_f(3Y+5, 3Y+4, 3Y+4)$.

$m = 3p, p \geq 1$. Take a path $\{v_1, v_2, \ldots, v_{3p}\}$ and assign the labels $1, 2, 0, 1, 2, 0, \ldots, 1, 2, 0, 1, 2$ to the edges on this path in this order. Attach a helms of type 2 with the labelings of type $C_2, A_2, B_2, \ldots, C_2, A_2, B_2$ to the vertices $v_1, \ldots, v_{3p}$ respectively. The point of identification being the hub of that helm in each case. The labels of all the hubs change to 0. as a result we have $3p$ hubs with the label 0. Call the resulting labeling f. One can check that $|V(G)| = 6Y + 15p, |E(G)| = 9Y + 21p - 1$ and $v_f(0,1,2) = (2Y + 5p, 2Y + 5p, 2Y + 5p), e_f(0,1,2) = (3Y + 7p - 1, 3Y + 7p, 3Y + 7p)$, that is f is an $\mathbf{E}_3$-cordial labeling.

$m = 3p + 1, p \geq 1$. Take the earlier labeling constructed for a path of length $3p - 1$. Assign the label 0 to the last edge $v_{3p}v_{3p+1}$. Now attach a helm of type 2 with the labeling A_2 to v_{3p+1}. Call the resulting labeling f. The label of v_{3p} remains 0. Since A_3 assigns the label 0 to the hub, the vertex v_{3p+1} gets the label 0. One can check that $|V(G)| = 6Y + 15p + 5, |E(G)| = 9Y + 21p + 6$ and $v_f(0,1,2) = (2Y + 5p + 1, 2Y + 5p + 2, 2Y + 5p + 2), e_f(3Y + 7p + 2, 3Y + 7p + 2, 3Y + 7p + 2)$.(the value of Y of course has increased by y_{3p+1})

$m = 3p + 2, p \geq 1$. Take the earlier labeling constructed for a path of length $3p - 3$. Now four extra edges on the path remain to be labeled. Those are $v_{3p-2}v_{3p-1}, v_{3p-1}v_{3p}, v_{3p}v_{3p+1}, v_{3p+1}v_{3p+2}$. Assign the labels $1, 2, 2, 0$ to them in this order. Attach a helm of type 3 with the labeling of type A_2, C_2, B_2, A_2 to the last four points. Call the resulting labeling f. The label of v_{3p-2} changes from 0 to 1. The labels of the vertices $v_{3p-1}, v_{3p}, v_{3p+1}, v_{3p+2}$ are all 0. One can check that the vertex and edge numbers are $v_f(0,1,2) = (2Y+5p+3, 3Y+5p+4, 3Y+5p+3)$ and $e_f(0,1,2) = (3Y+7p+4, 3Y+7p+4, 3Y+7p+5)$.

This completes the proof of the assertion that every $G \in PH_m(2)$ is E_3-cordial for all $m \geq 1$.

Thus every $G \in PH_m(i)$is E_3-cordial for $1 \leq i \leq 3$ and for every $m \in \mathbb{N}$.

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