A STANDARD PROOF OF ANDREWS' CONJECTURE FOR $_4\phi_3$-SERIES

CHUANAN WEI[A,B], XIAOXIA WANG[C]

[A] *Department of Mathematics*
Shanghai Normal University, Shanghai 200234, China
[B] *Department of Information Technology*
Hainan Medical College, Haikou 571199, China
[C] *Department of Mathematics*
Shanghai University, Shanghai 200444, China

ABSTRACT. In terms of Sear's transformation formula for $_4\phi_3$-series, we give standard proofs of a summation formula for $_4\phi_3$-series due to Andrews [Andrews, Adv. Appl. Math. 46 (2011), 15-24] and another summation formula for $_4\phi_3$-series conjectured in the same paper. Meanwhile, other several related results are also derived.

1. INTRODUCTION

For two complex numbers x and q, define the q-shifted factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{i=0}^{n-1}(1-xq^i) \quad \text{when} \quad n \in \mathbb{N}.$$

The fraction form of it reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix} q \Bigg]_n = \frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman [3], define the basic hypergeometric series by

$$_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \Bigg| q; z\right] = \sum_{k=0}^{\infty}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix} \Bigg| q\right]_k z^k,$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then sear's transformation formula for $_4\phi_3$-series (cf. [3, Equation (2.10.4)]) can be expressed as

$$_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, q^{1-n}abc/de \end{matrix} \Bigg| q; q\right] = \left[\begin{matrix} d/a, de/bc \\ d, de/abc \end{matrix} \Bigg| q\right]_n {_4\phi_3}\left[\begin{matrix} q^{-n}, a, e/b, e/c \\ e, de/bc, q^{1-n}a/d \end{matrix} \Bigg| q; q\right]. \quad (1)$$

2010 Mathematics Subject Classification: Primary 05A19 and Secondary 33D15.
Key words and phrases. Basic hypergeometric series; Catalan numbers; Andrews' conjecture for $_4\phi_3$-series.
Email addresses: weichuanan78@163.com (C. Wei), xwang 913@126.com (X. Wang).

Recall Shapilo's identity (cf. [7, Equation (5.12)]):

$$\sum_{k=0}^{n} C_{2k}C_{2n-2k} = 4^n C_n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are Catalan numbers. A deep extension of the last formula due to Andrews [2] can be stated as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{1/2-n}/ab \\ q^{1-n}/a, q^{1-n}/b, q^{1/2}ab \end{matrix} \Big| q;q\right] = q^{-n/2}\left[\begin{matrix} ab \\ a, b \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b, -q^{1/2} \\ ab \end{matrix} \Big| q^{1/2}\right]_n. \quad (2)$$

In the same paper, Andrews made the following conjecture:

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{3/2-n}/ab \\ q^{1-n}/a, q^{2-n}/b, q^{1/2}ab \end{matrix} \Big| q;q\right] \\ &= \left[\begin{matrix} ab/q \\ a, b/q \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b/q, -q^{1/2} \\ ab/q \end{matrix} \Big| q^{1/2}\right]_n \frac{(q^{1/2}-ab)(q-bq^{n/2})}{q^{3/2}-ab^2q^n} \\ &\times \left\{\frac{q^{1-n/2}}{q-b} + \frac{ab(q^{1/2}-bq^{n/2})(q-abq^n)}{(q^{1/2}-b)(q^{1/2}-abq^n)(q-abq^{n/2})}\right\}. \end{aligned} \quad (3)$$

Recently, Guo [5] confirmed (2) and (3) in accordance with the difference method. Different proofs of (3) are given by Ismail and Rahman [6] and Mu [8]. Inspired by these work, we shall give not only standard proofs of the two identities just mentioned but also several related results by using (1).

2. Standard proofs of Andrews' two summation formulas for ${}_4\phi_3$-series

Theorem 1. *For two complex numbers $\{a, b\}$ and a nonnegative integer n, there holds*

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{1/2-n}/ab \\ q^{1-n}/a, q^{-n}/b, q^{1/2}ab \end{matrix} \Big| q;q\right] = \left[\begin{matrix} ab \\ a, qb \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, q^{1/2}b, -q^{1/2} \\ ab \end{matrix} \Big| q^{1/2}\right]_n.$$

Proof. By means of (4), we obtain the following relation:

$$\begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{1/2-n}/ab \\ q^{1-n}/a, q^{-n}/b, q^{1/2}ab \end{matrix} \Big| q;q\right] &= \left[\begin{matrix} q^{1-n}/a^2, qab \\ q^{1-n}/a, qb \end{matrix} \Big| q\right]_n \\ &\times {}_4\phi_3\left[\begin{matrix} q^{-n}, a, q^{1/2}a, q^n a^2 b^2 \\ q^{1/2}ab, qab, a^2 \end{matrix} \Big| q;q\right]. \end{aligned}$$

Evaluating the ${}_4\phi_3$-series on the right hand side by the known identity (cf. [1, Equation (4.3)] and [4, Equation (4.22)]):

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, q^{1/2}a, q^n c^2 \\ q^{1/2}c, qc, a^2 \end{matrix} \Big| q;q\right] = \left[\begin{matrix} -q^{1/2}, q^{n/2}c, q^{-n/2}a/c \\ -a, q^{1/2+n/2}c, q^{-n/2}/c \end{matrix} \Big| q^{1/2}\right]_n,$$

we complete the proof. □

Theorem 2. *For three complex numbers $\{a, b, x\}$ and a nonnegative integer n, there holds*

$$ {}_5\phi_4\left[\begin{matrix} q^{-n}, qx, a, b, q^{1/2-n}/ab \\ x, q^{1-n}/a, q^{1-n}/b, q^{1/2}ab \end{matrix} \Big| q;q\right] = \frac{q^{-n/2}-x}{1-x}\left[\begin{matrix} ab \\ a, b \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b, -q^{1/2} \\ ab \end{matrix} \Big| q^{1/2}\right]_n.$$

Proof. Performing the replacements $a \to b, b \to a$ in Theorem 1, the result reads as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{1/2-n}/ab \\ q^{-n}/a, q^{1-n}/b, q^{1/2}ab \end{matrix} \Big| q; q\right] = \left[\begin{matrix} ab \\ qa, b \end{matrix} \Big| q\right]_n \left[\begin{matrix} q^{1/2}a, b, -q^{1/2} \\ ab \end{matrix} \Big| q^{1/2}\right]_n. $$

Combining the last formula with Theorem 1 by the following relation:

$$ \frac{(qx;q)_k}{(x;q)_k} = \frac{1-xq^k}{1-x} = \frac{(1-q^{-n}/a)(q^{-n}/b-x)}{(1-x)(q^{-n}/b-q^{-n}/a)}\frac{1-q^{k-n}/a}{1-q^{-n}/a} + \frac{(1-q^{-n}/b)(q^{-n}/a-x)}{(1-x)(q^{-n}/a-q^{-n}/b)}\frac{1-q^{k-n}/b}{1-q^{-n}/b}, $$

we finish the proof. □

When $x=0$, Theorem 2 reduces to Andrews' identity offered by (2) exactly. Taking $x = q^{1/2-n}/ab$ in Theorem 2, we recover the known result due to Guo [5, p. 1040]:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{3/2-n}/ab \\ q^{1-n}/a, q^{1-n}/b, q^{1/2}ab \end{matrix} \Big| q; q\right] = \frac{1-abq^{n/2-1/2}}{1-abq^{n-1/2}} \left[\begin{matrix} ab \\ a, b \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b, -q^{1/2} \\ ab \end{matrix} \Big| q^{1/2}\right]_n. \tag{4} $$

Other two related results are displayed as follows.

Corollary 3 ($x = q^{-1/2}ab$ in Theorem 2).

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{1/2-n}/ab \\ q^{1-n}/a, q^{1-n}/b, q^{-1/2}ab \end{matrix} \Big| q; q\right] = q^{-n/2} \left[\begin{matrix} ab \\ a, b \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b, -q^{1/2} \\ q^{-1/2}ab \end{matrix} \Big| q^{1/2}\right]_n. $$

Corollary 4 ($x \to b/q$ and $b \to b/q$ in Theorem 2).

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{3/2-n}/ab \\ q^{1-n}/a, q^{2-n}/b, q^{-1/2}ab \end{matrix} \Big| q; q\right] = \frac{b-q^{1-\frac{n}{2}}}{b-q} \left[\begin{matrix} ab/q \\ a, b/q \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b/q, -q^{1/2} \\ ab/q \end{matrix} \Big| q^{1/2}\right]_n. $$

Theorem 5. *For three complex numbers $\{a, b, x\}$ and a nonnegative integer n, there holds*

$$ {}_5\phi_4\left[\begin{matrix} q^{-n}, qx, a, b, q^{3/2-n}/ab \\ x, q^{1-n}/a, q^{2-n}/b, q^{1/2}ab \end{matrix} \Big| q; q\right] = \left[\begin{matrix} ab/q \\ a, b/q \end{matrix} \Big| q\right]_n \left[\begin{matrix} a, b/q, -q^{1/2} \\ ab/q \end{matrix} \Big| q^{1/2}\right]_n \frac{(q^{1/2}-ab)(q-bq^{n/2})}{(1-x)(q^{3/2}-ab^2q^n)} $$
$$ \times \left\{ \frac{q^{1-n/2}-xbq^{n/2}}{q-b} - \frac{(q^{1/2}-bq^{n/2})(q^{1/2}x-ab)(q-abq^n)}{(q^{1/2}-b)(q^{1/2}-abq^n)(q-abq^{n/2})} \right\}. $$

Proof. Combining (4) with Corollary 4 by the following relation:

$$ \frac{(qx;q)_k}{(x;q)_k} = \frac{1-xq^k}{1-x} = \frac{(1-q^{1-n}/b)(x-q^{-1/2}ab)}{(1-x)(q^{1-n}/b-q^{-1/2}ab)}\frac{1-q^{k+1-n}/b}{1-q^{1-n}/b} + \frac{(1-q^{-1/2}ab)(q^{1-n}/b-x)}{(1-x)(q^{1-n}/b-q^{-1/2}ab)}\frac{1-q^{k-1/2}ab}{1-q^{-1/2}ab}, $$

we complete the proof. $\square$

When $x = 0$, Theorem 5 reduces to Andrews' conjecture given by (3) exactly. Letting $x \to a/q$ and $a \to a/q$ in Theorem 5, we recover the known result due to Guo [5, Equation (4.4)]:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{5/2-n}/ab \\ q^{2-n}/a, q^{2-n}/b, q^{-1/2}ab \end{matrix} \,\middle|\, q; q\right] = \frac{q^{-n/2}(q^{3/2}-ab)}{q^{3/2}-q^n ab} $$
$$ \times \left[\begin{matrix} ab/q \\ a/q, b/q \end{matrix} \,\middle|\, q\right]_n \left[\begin{matrix} q^{-1/2}a, q^{-1/2}b, -q^{1/2} \\ q^{-3/2}ab \end{matrix} \,\middle|\, q^{1/2}\right]_n . $$

Other two related results are laid out as follows.

Corollary 6 ($x = q^{-1/2}ab$ in Theorem 5).

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{3/2-n}/ab \\ q^{1-n}/a, q^{2-n}/b, q^{-1/2}ab \end{matrix} \,\middle|\, q; q\right] $$
$$ = q^{-\frac{n}{2}} \left[\begin{matrix} ab/q \\ a, b/q \end{matrix} \,\middle|\, q\right]_n \left[\begin{matrix} a, q^{-1/2}b, -q^{1/2} \\ ab/q \end{matrix} \,\middle|\, q^{1/2}\right]_n . $$

Corollary 7 ($x = q^{3/2-n}/ab$ in Theorem 5).

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, q^{5/2-n}/ab \\ q^{1-n}/a, q^{2-n}/b, q^{1/2}ab \end{matrix} \,\middle|\, q; q\right] $$
$$ = \left[\begin{matrix} ab/q \\ a, b/q \end{matrix} \,\middle|\, q\right]_n \left[\begin{matrix} a, b/q, -q^{1/2} \\ ab/q \end{matrix} \,\middle|\, q^{1/2}\right]_n \frac{(q^{1/2}-ab)(q-bq^{n/2})}{(q^{3/2}-abq^n)(q^{3/2}-ab^2q^n)} $$
$$ \times \left\{ \frac{(q^{1/2}-bq^{n/2})(q+abq^{n/2})(q-abq^n)}{(q^{1/2}-b)(q^{1/2}-abq^n)} - \frac{q^{1+n/2}(q^{1/2}-a)}{1-q/b} \right\}. $$

According tho the linear methods of establishing Theorems 2 and 5, more related summation formulae can be founded. We shall not offer the corresponding details.

Acknowledgments

The authors are grateful to the reviewer for helpful comments. The work is supported by the Natural Science Foundations of China (Nos. 11301120, 11201241 and 11201291), the Natural Science Foundation of Shanghai (No. 12ZR1443800) and a grant of " The First-class Discipline of Universities in Shanghai ".

References

[1] G.E. Andrews, Connection coefficient problems and partitions, Proc. Symposia in Pure Math. 34 (1979), 1-24.

[2] G.E. Andrews, On Shaprio's Catalan convolution, Adv. Appl. Math. 46 (2011), 15-24.

[3] G. Gasper, M. Rahman, Basic Hypergeometric Series (2nd edition), Cambridge University Press, Cambridge, 2004.

[4] I. Gessel, D. Stanton, Applications of q-Lagrange inversion to basic hypergeometric series, Trans. Amer. Math. Soc. 277 (1983), 173-201.

[5] V.J.W. Guo, Proof of Andrews' conjecture on a ${}_4\phi_3$ summation, J. Differ. Equ. Appl. 19 (2013), 1035-1041.

[6] M.E.H. Ismail, M. Rahman, Connection relations and expansions, Pacific J. Math. 252 (2011), 427-446.

[7] T. Koshy, Catalan Numbers with Applications, Oxford University Press, New York, 2009.

[8] Y.-P. Mu, Andrews' conjecture on a ${}_4\phi_3$ summation and its extensions, Electorn. J. Combin. 18 (2) (2012), #P36.