$L(j,k)$-labelings of Cartesian products of three complete graphs*

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Abstract

Given any two positive integers j and k with $j \geq k$, an $L(j,k)$-labeling of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least j, and the difference between labels of vertices that are distance two apart is at least k. The span of an $L(j,k)$-labeling of a graph G is the difference between the maximum and minimum assigned integers . The $\lambda_{j,k}$-number of G is the minimum span taken over all $L(j,k)$-labelings of G. This paper investigates the $\lambda_{j,k}$-numbers of the Cartesian products of three complete graphs.

Keywords: $L(j,k)$-labeling, $\lambda_{j,k}$-number, Cartesian product.

1 Introduction

For any two positive integers j and k with $j \geq k$, an $L(j,k)$*-labeling* f of G is an assignment of integers to the vertices of G such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if $d_G(u,v)=2$, where $d_G(u,v)$ is the length (number of edges) of a shortest path between u and v in G. Given a graph G, for an $L(j,k)$-labeling f of G, elements of the image of f are called labels, and we define the span of f, $span(f)$, to be the absolute difference

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between the maximum and minimum vertex labels of f of f. Without loss of generality we shall assume that the minimum label of $L(j,k)$-labelings of G is 0. Then the span of f is the maximum vertex label. The $\lambda_{j,k}$-*number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G.

Motivated by a special kind of channel assignment problem, Griggs and Yeh [8] first proposed and studied the $L(2,1)$-labeling of a graph. Since then the $\lambda_{2,1}$-numbers of graphs have been studied extensively, see [1,4,6–8,10, 12,14]. And $L(j,k)$-labelings were also investigated in many papers, see [3–6].

Given two graphs G and H, the *Cartesian product* of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x,y) and (x',y') are adjacent if $x = x'$ and $yy' \in E(H)$ or $y = y'$ and $xx' \in E(G)$. Let G^k denote the Cartesian product of k copies of G. Let K_n denote the complete graph on n vertices. Then $K_n^2 = K_n \times K_n$ and $K_n^3 = K_n \times K_n \times K_n$.

Products of graphs have been considered in the attempt of gaining global information from the factors.

The $L(2,1)$-labeling of the Cartesian product of n paths, especially of the Cartesian product of n copies of P_2 (the n-cube Q_n), was investigated by Whittlesey, Georges, and Mauro [14]. In the same paper, they completely determined the $\lambda_{2,1}$-numbers of Cartesian products of two paths. Jha et al. [10] studied the $L(2,1)$-labeling of the Cartesian product of a cycle and a path. The $\lambda_{2,1}$-numbers of the Cartesian product of a cycle and a path were completely computed by Klavžar and Vesel in [11]. Partial results for the $\lambda_{2,1}$-numbers of the Cartesian products of two cycles were obtained in [11]. These partial results are completed in [13]. Georges, Mauro, and Whittlesey [7] determined $L(2,1)$-labeling numbers of Cartesian products of two complete graphs. This result was then extended by Georges, Mauro, and Stein [6] who determined the $L(j,k)$-labeling numbers of Cartesian products of two complete graphs.

Theorem 1.1 *[6] Let j, k, n, and m be integers where $n > m \geq 2$ and $j \geq k$. Then*

(i) $\lambda_{j,k}(K_n \times K_m) = (n-1)j + (m-1)k$, *if* $j/k > m$;

(ii) $\lambda_{j,k}(K_n \times K_m) = (nm-1)k$, *if* $j/k \leq m$.

Theorem 1.2 *[6] Let j, k, and n be integers where $n \geq 2$ and $j \geq k$. Then*

(i) $\lambda_{j,k}(K_n^2) = (n-1)j + (2n-2)k$, *if* $j/k > n-1$;

(ii) $\lambda_{j,k}(K_n^2) = (n^2-1)k$, *if* $j/k \leq n-1$.

Georges, and Mauro [4] also obtained other results on $L(j,k)$-labelling numbers of Cartesian products of complete graphs. In particular, they investigated the $\lambda_{j,k}$-number of K_n^3.

Theorem 1.3 *[4] The $\lambda_{j,k}$-number of $Q_3 \cong K_2^3$ is equal to $3j$ if $j/k \leq 5/2$; and $j + 5k$ if $j/k \geq 5/2$.*

Theorem 1.4 *[4] Suppose n is an odd integer, $n \geq 3$. Then*

(i) $\lambda_{j,k}(K_n^3) = (n-1)(j+3k)$, *if* $j/k \geq 3n-4$;

(ii) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \leq n-2$;

(iii) $\lambda_{j,k}(K_n^3) \leq (n-1)(j+3k)$, *if* $n-2 < j/k < 3n-4$.

Theorem 1.5 *[4] Suppose n is an even integer. Then*

(i) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \leq n/2$;

(ii) $\lambda_{j,k}(K_n^3) \leq \begin{cases} (n^2+2n)k, & \text{if } n/2 < j/k \leq n-2, \\ n(j+3k), & \text{if } n-2 < j/k \leq 2n(n-2), \\ (n-1)j + n(2n-1)k, & \text{if } j/k > 2n(n-2)). \end{cases}$

In [2], the $\lambda_{j,k}$-number of the Cartesian product $\prod_{i=1}^{n} K_{t_i}$ is exactly determined for $n \geq 3$ and relatively prime $t_1, t_2, \cdots, t_n$, where $2 \leq t_1 < t_2 < \cdots < t_n$.

In this paper, we extend the previous work on the $\lambda_{j,k}$-numbers of the Cartesian products of three complete graphs. In Section 3, for $n > m \geq l$ and $n > 2m$, we show that we don't need more labels to label $K_n \times K_m \times K_l$ than to label $K_n \times K_m$ in this case . And we give $\lambda_{j,k}(K_n \times K_m \times K_l) = (nm-1)k$ if $j/k \leq m$, and that $\lambda_{j,k}(K_n \times K_m \times K_l) = (n-1)j + (m-1)k$ if $j/k \geq m$. In Section 4 of this paper, for $n > m \geq l$ and $n = 2m$, we show $\lambda_{j,k}(K_n \times K_m \times K_l) = (nm-1)k$ if $j/k \leq m-1$, and that $\lambda_{j,k}(K_n \times K_m \times K_l) \leq (n-1)(j+k) + (m-1)k$ if $j/k \geq m-1$. We study $\lambda_{j,k}(K_n \times K_m \times K_l)$ for $l \leq m < n < 2m$ in Section 5 and Section 6 .

2 Preliminaries

For two positive integers a and b with $a < b$, denote by $[a, b]$ the set of integers $a, a+1, \ldots, b$. A set of integers is called *k-separated* if and only if any two distinct elements of the set differ by at least k. Given a graph $G(V, E)$, a subset S of V is called 2-*independent* if any two vertices of it are at distance at least 3. The 2-*independence number* of G is the maximum size taken over 2-independent subsets of $V(G)$.

Throughout this paper, j, k, n ,m and l will be positive integers with $n \geq m \geq l \geq 2$ and $j \geq k$.

We shall view the vertices of the graph $K_n \times K_m \times K_l$ as points in the three-dimensional Euclidean space. Each vertex of $K_n \times K_m \times K_l$ will be represented by its coordinate (a, b, c), where a, b, c are nonnegative integers with $0 \leq a \leq n-1$, $0 \leq b \leq m-1$, and $0 \leq c \leq l-1$. For $v = (a, b, c) \in V(K_n \times K_m \times K_l)$, we say that v is a vertex in the a^{th} row, b^{th} column and the c^{th} level of $K_n \times K_m \times K_l$. It is not difficult to see that two vertices are at distance k if their coordinates are different in exactly k components. In other words, any two vertices on a line parallel to some coordinate axis are adjacent; any two vertices on a plane parallel to some coordinate plane but not on any line parallel to some coordinate axis are at distance 2; and any two vertices not on any plane parallel to some coordinate plane are at distance 3. The diameter of $K_n \times K_m \times K_l$ is 3. The 2-independence number of $K_n \times K_m \times K_l$ is l. Thus each label can be used at most l times by any $L(j, k)$-labeling of $K_n \times K_m \times K_l$.

Suppose $n > m$. Let

$$t_0 = \min\{1 \leq t \leq nm \mid 2t \bmod n = 0, t \bmod m = 0\}. \qquad (2.1)$$

Then there exist two positive integers p and q such that $2t_0 = pn$ and $t_0 = qm$. It is easy to see that $m \leq t_0 \leq \frac{nm}{(n,m)}$ and $t_0 | \frac{nm}{(n,m)}$ where (n, m) is a greatest common denominator of the two integers n and m. Let r_0 be the integer such that $nm = r_0 t_0$. From the definition of t_0, we can show that the following two properties holds.

(P1) : if $\frac{n}{(n,m)}$ is even, then $p = \frac{m}{(n,m)}$, $q = \frac{n}{2(n,m)}$, $r_0 = 2(n, m)$ and $t_0 = \frac{nm}{2(n,m)}$;

(P2) : if $\frac{n}{(n,m)}$ is odd, then $p = \frac{2m}{(n,m)}$, $q = \frac{n}{(n,m)}$, $r_0 = (n, m)$ and $t_0 = \frac{nm}{(n,m)}$.

Thus t_0 is well defined. By the properties above, it is easy to see that: (1) if $n > 2m$ then $q > p$ and $t_0 > m$; (2) if $n = 2m$ then $q = p$, $r_0 = 2(n, m) = 2m$ and $t_0 = \frac{nm}{2(n,m)} = m = \frac{n}{2}$; (3) if $m < n < 2m$ and $(n, m) \leq \frac{m}{2}$ then $q < p$ and $t_0 > n > m$.

Lemma 2.1 *For* $0 \leq t_1, t_2 \leq t_0 - 1$ *and* $0 \leq r_1, r_2 \leq r_0 - 1$, *if* $(t_1, r_1) \neq (t_2, r_2)$, *then* $((2t_1 + r_1) \bmod n,\ t_1 \bmod m) \neq ((2t_2 + r_2) \bmod n,\ t_2 \bmod m)$.

Proof. Suppose to the contrary that $((2t_1 + r_1) \bmod n,\ t_1 \bmod m) = ((2t_2 + r_2) \bmod n,\ t_2 \bmod m)$ for some $(t_1, r_1) \neq (t_2, r_2)$. Without loss of generality, we may assume that $t_1 \geq t_2$. Then we obtain that $(2(t_1 - t_2) + (r_1 - r_2)) \bmod n = 0$ and $(t_1 - t_2) \bmod m = 0$. Thus there are two integers x and y such that $t_1 - t_2 = xm$, $x \in \{1, 2, \cdots q - 1\}$ and $2(t_1 - t_2) + (r_1 - r_2) = 2xm + (r_1 - r_2) = yn$.

Since n and m are multiples of (n, m), $(r_1 - r_2)$ must also be a multiple of (n,m). From the properties (P1) and (P2), we know that $r_0 \leq 2(n,m)$. Since $0 \leq |r_1 - r_2| < r_0 \leq 2(n,m)$, $|r_1 - r_2|$ must be 0 or (n,m). If $|r_1 - r_2| = 0$, then $r_1 = r_2$ and $2(t_1 - t_2) = yn$, $(t_1 - t_2) = xm$. Since $(t_1, r_1) \neq (t_2, r_2)$, $t_1 \neq t_2$. This is a contradiction of the minimality of t_0 since $t_1 - t_2 < t_0$. Therefore, we conclude that $|r_1 - r_2| = (n,m)$.

If $\frac{n}{(n,m)}$ is odd, then by (P2), $r_0 = (n,m)$. This is a contradiction since $|r_1 - r_2| < r_0 = (n,m)$. If $\frac{n}{(n,m)}$ is even, then $2xm + (r_1 - r_2) = yn$ cannot hold since $|r_1 - r_2| = (n,m)$ and both $2xm$ and yn are even multiples of (n,m), another contradiction. ∎

Suppose $n > m \geq l$. We define a function g from $V(K_n \times K_m \times K_l)$ to $[0, nm-1]$ as follows.

$$\begin{cases} g\left(((2t+r) \bmod n,\ t \bmod m, 0)\right) = rt_0 + t,\ 0 \leq t \leq t_0 - 1,\ 0 \leq r \leq r_0 - 1; \\ g\left((a,b,c)\right) = g\left(((a+c) \bmod n, (b+c) \bmod m, 0)\right) \text{ for } 0 \leq c \leq l-1. \end{cases} \tag{2.2}$$

Remark 1: By Lemma 2.1, we know that if $(t_1, r_1) \neq (t_2, r_2)$ then $((2t_1 + r_1) \bmod n,\ t_1 \bmod m) \neq ((2t_2 + r_2) \bmod n,\ t_2 \bmod m)$. Therefore, since the number of vertices at level 0 is $t_0 r_0 = nm$, each vertex of $V(K_n \times K_m \times K_l)$ is assigned an integer in $[0, nm-1]$. Furthermore, it is easy to see that the restriction of g to any fixed arbitrary level c is a bijection from the vertices at level c to the integers in $[0, nm-1]$.

The following two lemmas are useful in our proofs. The second one is straight-forward.

Lemma 2.2 [3] *Let j and k be two positive integers with $j \geq k$. For any* *graph G and any positive integer c, we have $\lambda_{cj,ck}(G) = c\lambda_{j,k}(G)$.*

Lemma 2.3 *Let j', j and k be positive integers with $j' \geq j \geq k$. Then for any graph G, we have $\lambda_{j',k}(G) \geq \lambda_{j,k}(G)$.*

3 $\lambda_{j,k}$-numbers of $K_n \times K_m \times K_l$ for $n > 2m$

In this section, we study the $\lambda_{j,k}$-number of $K_n \times K_m \times K_l$ for the case $n > 2m$, and we shall demonstrate that we don't need more labels to label $K_n \times K_m \times K_l$ than to label $K_n \times K_m$ in this case .

We first show that if $n > 2m$ then the mapping g defined in the previous section is an $L(m,1)$-labeling of $K_n \times K_m \times K_l$.

Lemma 3.1 *Suppose $n \geq 2m$. Let h and s be two integers in $[0, nm-1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h-s| < m$ then x and y are different in the first component.*

Proof. We first consider $c = 0$. Let $h = r_1t_0 + t_1$ and $s = r_2t_0 + t_2$. Suppose to the contrary that x and y are equal in the first component. Then $2t_1 + r_1 = 2t_2 + r_2 + in$ for some integer i. That is $2(t_1 - t_2) = in - (r_1 - r_2)$. Since $h \neq s$, x and y are different in the second component. Thus $t_1 \neq t_2$. If $|r_1 - r_2| > 1$ then $|h - s| = |(r_1 - r_2)t_0 + (t_1 - t_2)| \geq ||r_1 - r_2|t_0 - |t_1 - t_2|| = |r_1 - r_2|t_0 - |t_1 - t_2| > t_0 \geq m$. This is a contradiction of our assumption that $0 < |h-s| < m$. Therefore $|r_1 - r_2| \leq 1$. Since

$pn-1 > 2(t_0-1) \geq 2|t_1-t_2| = |in-(r_1-r_2)| \geq |i|n-|r_1-r_2| \geq |i|n-1$,

we clearly have $|i| < p$.

If $|r_1-r_2| = 0$ then since $t_1 \neq t_2$ we have $i \neq 0$. So we have $|2(h-s)| = |2(t_1-t_2)| = |in| \geq n \geq 2m$. This is a contradiction of $0 < |h-s| < m$. If $|r_1-r_2| = 1$ then $|2(h-s)| = |2t_0 + 2(t_1-t_2)| = |2t_0 + in - (r_1-r_2)| \geq |2t_0 - |i|n - 1| = (p-|i|)n - 1 \geq n-1 \geq 2m-1$. This is again a contradiction of $0 < |h-s| < m$. It follows that x and y are different in the first component for $c = 0$.

For $c > 0$, let $x = (a_x, b_x, c)$ and $y = (a_y, b_y, c)$. Then $h = g(x) = g(((a_x+c) \bmod n, (b_x+c) \bmod m, 0))$ and $s = g(y) = g(((a_y+c) \bmod n, (b_y+c) \bmod m, 0))$ by the definition of g. Thus we have $(a_x+c) \bmod n \neq (a_y+c) \bmod n$ by the result above for $c = 0$. Furthermore, $a_x \neq a_y$, i.e, x and y are different in the first component for $c > 0$. ∎

Lemma 3.2 *Suppose $n > m$. Let h and s be two integers in $[0, nm-1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h-s| < m$ then x and y are different in the second component.*

Proof. We first consider $c = 0$. Let $h = r_1 t_0 + t_1$ and $s = r_2 t_0 + t_2$. Suppose to the contrary that x and y are equal in the second component. Then $t_1 - t_2 = im$ for some integer i. Since $qm - 1 = t_0 - 1 \geq |t_1 - t_2| = |i|m$, we clearly have $|i| < q$. If $|r_1 - r_2| > 1$ then we can get the same contradiction as in the proof of Lemma 3.1. Therefore $|r_1 - r_2| \leq 1$.

If $|r_1 - r_2| = 0$ then $i \neq 0$. So we have $|h - s| = |t_1 - t_2| = |i|m \geq m$. This is a contradiction of $0 < |h - s| < m$.

If $|r_1 - r_2| = 1$ then $|h-s| = |t_0 + (t_1 - t_2)| = |t_0 + im| \geq |\, t_0 - |i|m| = |(q-|i|)m| \geq m$. This is again a contradiction of $0 < |h-s| < m$. It follows that x and y are different in the second component for $c = 0$.

With proof similar to that of Lemma 3.1, we can obtain that x and y are different in the first component for $c > 0$. ∎

Lemma 3.3 *Suppose $n > 2m$. Let h and s be two integers in $[0, nm-1]$. And let x and y be two vertices of $K_n \times K_m \times K_l$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h-s| < m$ then $d(x,y) \geq 2$.*

Proof. If x and y are equal in the third component then, by Lemmas 3.1 and 3.2, we have $d(x,y) = 2$. Thus we assume that x and y are different in the third component. If the lemma is not true then $d(x,y) = 1$. This implies that x and y are equal in the first and second components. Let $x = (a, b, c_1)$ and $y = (a, b, c_2)$. And let $h = r_1 t_0 + t_1$ and $s = r_2 t_0 + t_2$. Then

$$\begin{cases} a + c_1 = 2t_1 + r_1 \bmod n, \\ b + c_1 = t_1 \bmod m; \\ a + c_2 = 2t_2 + r_2 \bmod n, \\ b + c_2 = t_2 \bmod m. \end{cases}$$

Therefore, there exist two integers i_1 and i_2 such that $c_1 - c_2 = 2(t_1 - t_2) +$

$(r_1 - r_2) + i_1 n$ and $c_1 - c_2 = (t_1 - t_2) + i_2 m$. It follows that

$$c_1 - c_2 = 2i_2 m - i_1 n - (r_1 - r_2), \quad (3.1)$$

$$t_1 - t_2 = i_2 m - i_1 n - (r_1 - r_2). \quad (3.2)$$

If $|r_1 - r_2| > 1$ then we can get the same contradiction as in the proof of Lemma 3.1. Thus we suppose $|r_1 - r_2| \leq 1$. Without loss of generality, we assume $0 < c_1 - c_2 < m$. By (3.1) and (3.2), we have

$$-2i_2 m + (r_1 - r_2) < -i_1 n < (1 - 2i_2)m + (r_1 - r_2), \quad (3.3)$$

$$-i_2 m < t_1 - t_2 < (1 - i_2)m. \quad (3.4)$$

If $|r_1 - r_2| = 0$ then $|h - s| = |t_1 - t_2| = |i_2 m - i_1 n|$. If $i_2 = 0$ then $i_1 \neq 0$ since otherwise $c_1 - c_2 = 0$. Since $n > 2m$, if $i_2 = 0$ or 1 then $|h - s| = |i_2 m - i_1 n| \geq m$, a contradiction. If $i_2 \neq 0, 1$ then, by (3.4), $|h - s| = |t_1 - t_2| > m$, a contradiction. Thus we assume $|r_1 - r_2| = 1$.

Suppose $r_1 - r_2 = 1$. If $i_2 = q$ then $-pn + 1 < -i_1 n < -pn + m + 1$. This is impossible since $n > 2m$ and i_1 is an integer. If $i_2 = q + 1$ then $-pn - 2m + 1 < -i_1 n < -pn - m + 1$. This is again impossible. Thus $i_2 \neq q, q + 1$. Since $(q - i_2)m < h - s = t_0 + t_1 - t_2 < (q + 1 - i_2)m$, it follows that $|h - s| > m$, a contradiction.

Suppose $r_1 - r_2 = -1$. If $i_2 = -q + 1$ then $pn - 2m - 1 < -i_1 n < pn - m - 1$. This is impossible. If $i_2 = -q$ then $pn - 1 < -i_1 n < pn + m - 1$. Therefore i_1 must be $-p$. By (3.2), $t_1 - t_2 = -qm + pn + 1 = t_0 + 1$. This is a contradiction since $0 \leq t_1, t_2 \leq t_0 - 1$. Thus $i_2 \neq -q, -q + 1$. Since

$(-q-i_2)m < h-s = -t_0+t_1-t_2 < (-q+1-i_2)m$, it follows that $|h-s| > m$, a contradiction. And the lemma follows. ∎

Remark 1 and Lemma 3.3 imply that g is actually an $L(m,1)$-labeling of $K_n \times K_m \times K_l$. By Lemmas 2.2 and 2.3 together with Theorem 1.1, the following theorem holds.

Theorem 3.4 *If $n > 2m$ and $j/k \leq m$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (nm-1)k.$$

Next we deal with the case $n > 2m \geq 4$ and $j/k \geq m$.

Theorem 3.5 *If $n > 2m$ and $j/k \geq m$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (n-1)j + (m-1)k.$$

Proof. By Theorem 1.1, $\lambda_{j,k}(K_n \times K_m \times K_l) \geq \lambda_{j,k}(K_n \times K_m) = (n-1)j + (m-1)k$.

Let (a,b,c) be any vertex of $K_n \times K_m \times K_l$. There are two integers r and t such that $g((a,b,c)) = rm+t$ with $0 \leq t \leq m-1$ and $0 \leq r \leq n-1$. Then we define a mapping L from $V(K_n \times K_m \times K_l)$ to nonnegative integers as: $L((a,b,c)) = rj+tk$. Clearly, the span of L is $(n-1)j+(m-1)k$. Next we show that L is an $L(j,k)$-labeling of $K_n \times K_m \times K_l$.

Let v_1 and v_2 be any two vertices and suppose $g(v_1) = r_1m + t_1$ and $g(v_2) = r_2m+t_2$, where $0 \leq r_1, r_2 \leq n-1$ and $1 \leq t_1, t_2 \leq m-1$. Without loss of generality, assume $g(v_1) \leq g(v_2)$. If v_1 and v_2 are adjacent then, by Lemma 3.3 and Remark 1, $(r_2m+t_2)-(r_1m+t_1)\geq m$ and so $t_2-t_1\geq m-(r_2-r_1)m$. Note that $j\geq mk$, we have

$$\begin{aligned} L(v_2)-L(v_1) &= (r_2j+t_2k)-(r_1j+t_1k) \\ &= (r_2-r_1)j+(t_2-t_1)k \\ &\geq (r_2-r_1)j+[m-(r_2-r_1)m]k \\ &= (r_2-r_1)j-(r_2-r_1-1)mk\geq j. \end{aligned}$$

If v_1 and v_2 are distance two apart then, by Remark 1, $(r_2m+t_2)-(r_1m+t_1)\geq 1$. Note that $j\geq mk$, we have

$$\begin{aligned} L(v_2)-L(v_1) &= (r_2j+t_2k)-(r_1j+t_1k) \\ &= (r_2-r_1)j+(t_2-t_1)k \\ &\geq (r_2-r_1)j+[1-(r_2-r_1)m]k \\ &= (r_2-r_1)j-(r_2-r_1)mk+k\geq k. \end{aligned}$$

∎

4 $\lambda_{j,k}$-numbers of $K_n\times K_m\times K_l$ for $n=2m$

In this section, we study the $\lambda_{j,k}$-numbers of $K_n\times K_m\times K_l$ for $n=2m$.

For $n=2m$, we show that g is an $L(m-1,1)$-labeling of $K_n\times K_m\times K_l$.

Lemma 4.1 *Suppose $n=2m$. Let h and s be two integers in $[0,nm-1]$. And let x and y be two vertices of $K_n\times K_m\times K_l$ such that $g(x)=h$ and $g(y)=s$. If $0<|h-s|<m-1$ then $d(x,y)\geq 2$.*

Proof. The proof of this lemma is the same as that of Lemma 3.3 except for $r_1-r_2=-1$ and $i_2=-q+1$. In this case, we still have $pn-2m-1<-i_1n<pn-m-1$. This implies that $i_1=-p+1$. It follows from (3.2) that $t_1-t_2=t_0-m+1$. Thus $|h-s|=|(r_1-r_2)t_0+t_1-t_2|=|-m+1|=m-1$.

∎

Remark 1 and Lemma 4.1 imply that g is actually an $L(m-1,1)$-labeling of $K_n \times K_m \times K_l$. By Lemmas 2.2 and 2.3 together with Theorem 1.1, the following theorem holds.

Theorem 4.2 *If $n = 2m$ and $j/k \leq m-1$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (nm-1)k.$$

For $n = 2m$ and $j/k \geq m-1$, we use the mapping g to construct an $L(j,k)$-labeling of $K_n \times K_m \times K_l$ with span $(n-1)j + (n+m-2)k$.

Let (a,b,c) be any vertex of $K_n \times K_m \times K_l$. Suppose $g((a,b,c)) = rm+t$ with $0 \leq t \leq m-1$ and $0 \leq r \leq n-1$. Then we define a mapping L from $V(K_n \times K_m \times K_l)$ to nonnegative integers as: $L((a,b,c)) = rj + (r+t)k$. Clearly, the span of L is $(n-1)j+(n+m-2)k$. With a proof similar to that of Theorem 3.5, we can show that L is an $L(j,k)$-labeling of $K_n \times K_m \times K_l$. Thus we have the following theorem.

Theorem 4.3 *If $n = 2m$ and $j/k \geq m-1$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) \leq (n-1)j + (n+m-2)k.$$

5 $\lambda_{j,k}$-numbers of $K_n \times K_m \times K_l$ for $m < n < 2m$

In this section, we study the $\lambda_{j,k}$-numbers of $K_n \times K_m \times K_l$ for $m < n < 2m$.

Let $d = n-m-1$. We show that the mapping g is an $L(d,1)$-labeling of $K_n \times K_m \times K_l$ when $m < n < 2m$.

Lemma 5.1 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm - 1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h-s| < d$ then x and y are different in the first component.*

Proof. Let $h = r_1 t_0 + t_1$ and $s = r_2 t_0 + t_2$. Suppose to the contrary that x and y are equal in the first component. From the proof of Lemma 3.1, we know that $|h - s| \geq \frac{n-1}{2}$. This contradicts $0 < |h - s| < d = n - m - 1$. ∎

The following lemma is an immediate corollary of Lemma 3.2.

Lemma 5.2 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm - 1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h - s| < d$ then x and y are different in the second component.*

With a proof similar to that of Lemma 3.3, we can show the following lemma. We omit the proof here.

Lemma 5.3 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm - 1]$. And let x and y be two vertices of $K_n \times K_m \times K_l$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h - s| < d$ then $d(x, y) \geq 2$.*

Remark 1 and Lemma 5.3 imply that g is actually an $L(d, 1)$-labeling of $K_n \times K_m \times K_l$. By Lemmas 2.2 and 2.3 together with Theorem 1.1, the following theorem holds.

Theorem 5.4 *If $m < n < 2m$ and $j/k \leq d = n - m - 1$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (nm - 1)k.$$

By defining an $L(j,k)$-labeling L as: $L((a,b,c)) = rj + [(m-d)r + t)]k$ if $g((a,b,c)) = rm + t$ with $0 \leq r \leq n-1$ and $0 \leq t \leq m-1$, for $j/k \geq d$ where the integers $j \geq k$, with a proof similar to that of Theorem 3.5, one can show the following theorem.

Theorem 5.5 *If $m < n < 2m$ and $j/k \geq d$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) \leq (n-1)j + [(m-d)(n-1) + (m-1)]k.$$

6 Another method for $m < n < 2m$

In this section, we study the $\lambda_{j,k}$-number of $K_n \times K_m \times K_l$ by another method for $m < n < 2m$.

Suppose $n > m$. Let

$$t_0 = \min\{1 \leq t \leq nm \mid t \bmod n = 0, 2t \bmod m = 0\}. \tag{6.1}$$

Then there exist two positive integers p and q such that $t_0 = pn$ and $2t_0 = qm$. It is easy to see that $t_0 \leq \frac{nm}{(n,m)}$ and $t_0 | \frac{nm}{(n,m)}$. Let r_0 be the integer such that $nm = r_0 t_0$. From the definition of t_0, we can show that the following two properties holds.

(Q1) : if $\frac{m}{(n,m)}$ is even, then $p = \frac{m}{2(n,m)}$, $q = \frac{n}{(n,m)}$, $r_0 = 2(n,m)$ and $t_0 = \frac{nm}{2(n,m)}$;

(Q2) : if $\frac{m}{(n,m)}$ is odd, then $p = \frac{m}{(n,m)}$, $q = \frac{2n}{(n,m)}$, $r_0 = (n,m)$ and $t_0 = \frac{nm}{(n,m)}$.

By the properties above, it is easy to see that (1) if $n > 2m$ then $q > 4p$ and $t_0 > n$; (2) if $n = 2m$ then $q = 4p$ and $r_0 = (n,m) = m$ and $t_0 = n$; (3) if $m < n < 2m$ and $(n,m) \leq m/2$ then $q < 4p$ and $t_0 > n > m$. We next suppose that $m < n < 2m$.

Let ϕ be a mapping from $V(K_n \times K_m \times K_l)$ to $\{0, 1, 2, \ldots nm - 1\}$ defined as:

$$\begin{cases} \phi((t \bmod n, (2t + r) \bmod m, 0)) = rt_0 + t, \ 0 \leq t \leq t_0 - 1, \ 0 \leq r \leq r_0 - 1; \\ \phi((a, b, c)) = h(((a + c) \bmod n, (b + c) \bmod m, 0)) \text{ for } \ 0 \leq c \leq l - 1. \end{cases} \tag{6.2}$$

Remark 2: With a proof similar to that of Remark 1, it is easy to see that each vertex of $V(K_n \times K_m \times K_l)$ is assigned an integer in $[0, nm - 1]$. Furthermore, it is easy to see that the restriction of ϕ to any fixed arbitrary level c is a bijection from the vertices at level c to the integers in $[0, nm - 1]$.

Let $d = \lfloor (m + 1)/2 \rfloor$. With a proof similar to that of Lemmas 3.1, 3.2 and 3.3, we can show the following three lemmas.

Lemma 6.1 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm - 1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h - s| < d$ then x and y are different in the first component.*

Lemma 6.2 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm - 1]$. And let x and y be two vertices in level $c(\geq 0)$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h - s| < d$ then x and y are different in the second* *component.*

Lemma 6.3 *Suppose $m < n < 2m$. Let h and s be two integers in $[0, nm-1]$. And let x and y be two vertices of $K_n \times K_m \times K_l$ such that $g(x) = h$ and $g(y) = s$. If $0 < |h - s| < d$ then $d(x, y) \geq 2$.*

By Remark 2 and Lemma 6.3, ϕ is an $L(d, 1)$-labeling of $K_n \times K_m \times K_l$ when $m < n < 2m$.

Theorem 6.4 *If $m < n < 2m$ and $j/k \leq d = \lfloor (m+1)/2 \rfloor$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (nm - 1)k.$$

suppose $m < n < 2m$, m is odd and $j/k \geq d = \lfloor (m+1)/2 \rfloor$. With a proof similar to that of Theorem 3.5, we can show the following theorem.

Theorem 6.5 *If $m < n < 2m$, m is odd and $j/k \geq d$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) \leq (n-1)j + [(m-d)(n-1) + (m-1)]k.$$

By Theorem 5.4 and 6.4, we have the following theorem.

Theorem 6.6 *If $m < n < 2m$ and $j/k \leq d = \max\{n-m-1, \ \lfloor (m+1)/2 \rfloor\}$, then*

$$\lambda_{j,k}(K_n \times K_m \times K_l) = (nm - 1)k.$$

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