A NOTE ON THE Q-LUCAS THEOREM

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ABSTRACT. In this note we present an application of q-Lucas theorem, from which the q-binomial rational root theorem obtained by K. R. Slavin can be deduced as a special case.

1. INTRODUCTION

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \prod_{j=1}^{k} \frac{1-q^{n-j+1}}{1-q^j}, & 0 \le k \le n \\ 0, & otherwise \end{cases} \tag{1}$$

In [1], the following q-Lucas theorem was proved and used to derive other new product theorems.

Theorem 1 (q-Lucas). *Let n, k, d be positive integers, and write $n = ad + b$ and $k = rd + s$, where $0 \le b, s \le d-1$, Let ω be a primitive d-th root of unity. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_\omega \tag{2}$$

In this note we will use the q-Lucas theorem to prove an interesting theorem and from which the q-binomial rational root theorem can be deduced as a special case.

2. THE MAIN RESULT

In this section, we will prove the following interesting theorem by using the q-Lucas theorem.

Theorem 2. *Let n, k, d be positive integers, $n > 0$ and $0 \le k \le n$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{e^{\pm 2i\pi m/n}} = \begin{cases} \binom{(m,n)}{\frac{(m,n)k}{n}}, & n \mid km \\ 0, & otherwise \end{cases} \tag{3}$$

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$$\left[\begin{array}{c} n \\ k \end{array}\right]_{e^{\pm 2i\pi m/k}} = \begin{cases} \left(\begin{array}{c} \frac{(m,k)n}{k} \\ (m,k) \end{array}\right), & k \mid mn \\ 0, & otherwise \end{cases} \tag{4}$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{e^{\pm 2i\pi m/(n,k)}} = \begin{cases} \left(\begin{array}{c} \frac{n}{(n,k)} \\ \frac{k}{(n,k)} \end{array}\right), & (m,n,k) = 1 \\ \left(\begin{array}{c} \frac{n(m,n,k)}{(n,k)} \\ \frac{k(m,n,k)}{(n,k)} \end{array}\right), & otherwise \end{cases} \tag{5}$$

where $i^2 = -1$ and (a,b) denotes the greatest common divisor of a and b.

Remark 1. Note that (3) is the q-binomial rational root theorem already proved by Slavin in [2] and Ying-Jie Lin in [3].

Proof. Our proof of result (3) is only repeated from [3] and the proofs of results (4) and (5) follow a similar approach.

Let $\omega_1 = e^{\pm 2i\pi m/n}$. Suppose that ω_1 is a primitive d-th root of unity. Then $d = \frac{n}{(m,n)}$. By the q-Lucas theorem, we have

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{\omega} = \left(\begin{array}{c} \frac{n}{d} \\ r \end{array}\right) \left[\begin{array}{c} 0 \\ s \end{array}\right]_{\omega} \tag{6}$$

where $k = rd + s$ and $0 \le s \le d-1$.

If $n \mid km$, then $n \mid k\,(m,n)$ and $d = \frac{n}{(m,n)} \mid k$, so $r = \frac{k}{d} = \frac{(m,n)k}{n}, s = 0$. Otherwise, $d \nmid k$ and $s > 0$. Since

$$\left[\begin{array}{c} 0 \\ s \end{array}\right]_{\omega} = \begin{cases} 1, s = 0 \\ 0, s > 0 \end{cases}$$

this completes the proof of (3).

To prove (4), we consider $\omega_2 = e^{\pm 2i\pi m/k}$, which is a primitive d-th root of unity. Then one can get $d = \frac{k}{(m,k)}$. By the q-Lucas theorem, we have

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{\omega} = \left(\begin{array}{c} a \\ \frac{k}{d} \end{array}\right) \left[\begin{array}{c} b \\ 0 \end{array}\right]_{\omega} \tag{7}$$

where $n = ad + b$ and $0 \le b \le d-1$.

If $k \mid mn$, then $k \mid (m,k)\,n$ and $d = \frac{k}{(m,k)} \mid n$, so $a = \frac{n}{d} = \frac{(m,k)n}{k}, b = 0$. Otherwise, $d \nmid n$ and $b > 0$. Since

$$\left[\begin{array}{c} b \\ 0 \end{array}\right]_{\omega} = \begin{cases} 1, b = 0 \\ 0, b > 0 \end{cases}$$

this completes the proof of (4).

To prove (5), we consider $\omega_3 = e^{\pm 2i\pi m/(n,k)}$, which is a primitive d-th root of unity. Then one gets

$$d = \frac{(n,k)}{(m,n,k)} \tag{8}$$

By the q-Lucas theorem, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \begin{pmatrix} \frac{n}{d} \\ \frac{k}{d} \end{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_\omega \tag{9}$$

Since

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_\omega = 1$$

together with (8), we completes the proof of (5). □

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