Binomial Structures Associated with a q^2-Analogue Operator

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Abstract

Employing q-commutive structures, we develop binomial analysis and combinatorial applications induced by an important operator in analogue Fourier analysis associated with well-known q-series of L. J. Rogers.

1 Introduction

The q-analogue binomial analysis induced by the ∂_q operator,

$$\partial_q f(z) \equiv \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}, \qquad (1)$$

(originating in q^2-Fourier analysis c.f. [3], [4]) which we shall show is naturally associated with two well-known q-identities of L. J. Rogers, formulas (94) and (99) in Slater's famous list, [6], $\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q;q)_{2m+1}} =$

$$\prod_{m=1}^{\infty} \frac{(1-q^{10m-3})(1-q^{10m-7})(1-q^{20m-4})(1-q^{20m-16})(1-q^{10m})}{(1-q^m)}, \qquad (2)$$

and $\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q;q)_{2m}} =$

$$\prod_{m=1}^{\infty} \frac{(1-q^{10m-1})(1-q^{10m-9})(1-q^{20m-8})(1-q^{20m-12})(1-q^{10m})}{(1-q^m)}, \qquad (3)$$

requires some novel q-commutative constructions and yields interesting variants of results of classical q-binomial and partition analysis.

The Binomial combinatorics induced by (1) appears to be difficult to develop because the complexity of the associated Pascal formula does not allow standard commutative and q-commutative methods to be employed. However, we will show that, in the appropriate algebraic context, fundamental results with natural combinatorial interpretations follow easily. After presenting the relevant Binomial developments, applications will be given to integer partition generating functions, commutative q-identities and a Rogers-Ramanujan-type partition result.

We use the standard notation of q-analysis adopted from [2]. In particular, $0 < q < 1$, $(a;q)_k \equiv \prod_{j=0}^{k-1}(1-aq^j)$, $(a;q)_0 \equiv 1$, $(a;q)_\infty \equiv \lim_{k\to\infty}(a;q)_k$, the q-derivative, $\mathcal{D}_q f(z) \equiv \frac{f(z)-f(qz)}{(1-q)z}$. $[j]_q \equiv 1+q+q^2+\cdots+q^{j-1} = \frac{1-q^j}{1-q}$ and $[j]_q! \equiv [1]_q[2]_q\cdots[j]_q$. $\lfloor r \rfloor$ is the greatest integer $\leq r$, $\lceil r \rceil$ is the least integer $\geq r$. The parity operator on integers is $\pi(k) \equiv \begin{cases} 1 \text{ if k is odd,} \\ 0 \text{ if k is even .}\end{cases}$ Note: $\pi(nk) = \pi(n)\pi(k)$.

Classically, the q-derivative is linked to q-binomial analysis by:

$$\mathcal{D}_q^n(z^n) = [n]_q! = \frac{(q;q)_n}{(1-q)^n}. \tag{4}$$

Computing the m-fold composition of ∂_q on z^n gives:

$$\partial_q^m(z^n) = \frac{\prod_{k=n-m+1}^{n}(-1)^{k-1}(1-q^{(-1)^{k-1}k})}{(1-q)^m} z^{n-m}, \quad 1 \leq m \leq n,$$

and $\partial_q^m(z^n) = 0, \quad m > n$.

Thus the analogue of (4) for ∂_q is:

$$\partial_q^n(z^n) = \frac{\prod_{k=1}^{n}(-1)^{k-1}(1-q^{(-1)^{k-1}k})}{(1-q)^n}.$$

This leads us to define

$$\{a;q\}_n \equiv \prod_{k=1}^{n}(-1)^{k-1}(1-(aq^{k-1})^{(-1)^{k-1}}. \tag{5}$$

It is easy to check that :

$$\{q;q\}_n = q^{-\lfloor\frac{n}{2}\rfloor(\lfloor\frac{n}{2}\rfloor+1)}(q;q)_n. \tag{6}$$

Using this construction, we see that the operator (1) is closely linked to the q-identities of Rogers listed at the beginning of the paper. In fact, combining (2) and (3) with (6), gives us:

Lemma 1.

$$\sum_{n=0}^{\infty}\frac{1}{\{q;q\}_n} = \frac{1}{(q,q^9;q^{10})_\infty} + \frac{1}{(q^3,q^7;q^{10})_\infty} = \frac{(q,q^9;q^{10})_\infty + (q^3,q^7;q^{10})_\infty}{(q,q^3,q^7,q^9;q^{10})_\infty}.$$

We discuss combinatorial interpretations of this formula in the last section.

Define

$$\{k\}_q \equiv \frac{(-1)^{k-1}(1-q^{(-1)^{k-1}k})}{1-q} = \begin{cases} \sum_{i=0}^{k-1} q^i \text{ if k is odd,} \\ \sum_{i=1}^{k} q^{(-1)^i} \text{ if k is even ,} \end{cases} \tag{7}$$

$$\{j\}_q! \equiv \prod_{k=1}^{j} \{k\}_q, \text{ and } \{0\}_q! \equiv 1.$$

These definitions give

$$\lim_{q\uparrow 1}\{n\}_n = n, \text{ and } \lim_{q\uparrow 1}\{n\}_q! = n!$$

Recall the standard q-analogue binomial coefficient (corresponding to the q-derivative $\mathcal{D}_q$)

$$\binom{n}{k}_q \equiv \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{8}$$

Its q-analogue Pascal formula, $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{(n-k)}\binom{n-1}{k-1}_q$,

follows easily from (8), and the subtraction formula:

$$[n]_q - [k]_q = q^k[n-k]_q.$$

In our case, the subtraction formula takes the form:

Lemma 2.

$$q^{n\pi(k)(1-\pi(n))-k\pi(n)(1-\pi(k))}\{n\}_q - \{k\}_q = q^{n\pi(nk)-k(1-\pi(n))(1-\pi(k))}\{n-k\}_q.$$

Proof. For each possible combination of parities of n, k, n-k, the subtraction formulas are easily computed using the definition of $\{n\}_q$. (We also use $\pi(nk) + \pi(k)(1-\pi(n)) = \pi(k)$.) The resulting formulas combine to that given above. □

Define:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q \equiv \frac{\{n\}_q!}{\{k\}_q!\{n-k\}_q!}, \tag{9}$$

Using (6) it is easy to see that

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = q^{-\pi(k(n-k))n} q^{-2\lfloor \frac{k}{2} \rfloor \lfloor \frac{n-k}{2} \rfloor} \binom{n}{k}_q. \tag{10}$$

We get a q-analogue as q approaches 1:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q \to \binom{n}{k} \text{ as } q \to 1.$$

Applying the subtraction formula of Lemma (2) to 10, we obtain the analogue of the Pascal formula for this setting:

Corollary 1.

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = q^{-n\pi(k)(1-\pi(n))+k\pi(n)(1-\pi(k))} \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_q$$

$$+q^{-n\pi(k)+k\pi(n)(1-\pi(k))(2\pi(n)-1)} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_q.$$

Despite the complexity of this Pascal formula, in the next sections, we will prove a relevant Binomial theorem and commuative q-identitiy and partition applications of these analogue Binomial coefficients.

2 Binomial Expansions

Recall the famous q-binomial formula presented in Schutzenberger, [5]:

Theorem 1. *If x and y are two elements in an associative algebra with* **1** *over a field F and if $q \neq 0$ in F is such that $xy = qyx$, then for any a,b in F:*

$$(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k}_q (by)^{n-k}(ax)^k. \tag{11}$$

We want to find the appropriate analogue of this theorem for $\{{n \atop k}\}_q$.

If z is an element in an associative algebra, A, with unit **1** over a field, F, with $zz^{-1} = z^{-1}z = 1$, and if u is any element in A, define

$$u^{[n]_z} \equiv \begin{cases} 1 & \text{if n} = 0 \\ (u^{[n-1]_z})u & \text{if n is odd} \\ z^{-1}(u^{[n-1]_z})uz & \text{if n is even} \end{cases}. \tag{12}$$

Observe that whenever z commutes with u, then $u^{[n]_z} = u^n$.

Our next theorem includes an analogue of Theorem 1 for $\{{n \atop k}\}_q$, and lets us compare binomial expansions with varying amounts of q-commutivity. In particular, it shows the effect of varying q-commutativity on binomial coefficients.

Theorem 2. *If z is an element in an associative algebra, A, with unit $\mathbf{1}$, over a field F, with $zz^{-1} = z^{-1}z = \mathbf{1}$, and $q \neq 0 \in F$, then for a,b in F, and x,y in A:*
1)if x, y, z all commute,

$$(ax+by)^{[n]_z} = \sum_{k=0}^{n} \binom{n}{k} (by)^{[n-k]_z} (ax)^{[k]_z},$$

2)if $xy = qyx$, z commutes with x and y,

$$(ax+by)^{[n]_z} = \sum_{k=0}^{n} \binom{n}{k}_q (by)^{[n-k]_z} (ax)^{[k]_z},$$

3)if $xy{=}qyx$, $zx{=}qxz$, $zy{=}qyz$,

$$(ax+by)^{[n]_z} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (by)^{[n-k]_z} (ax)^{[k]_z}.$$

Proof. In cases 1) and 2), z commutes with x and y, so $u^{[n]_z} = u^n$ for $u = x$, or y, or $x+y$. Thus 1) follows from the Binomial Theorem, and 2) from Theorem 1.

We establish case 3). For notational simplicity, assume $a = b = 1$; the extension to the general case is left to the reader. Rewriting 3), using definition (12), it is enough to show that: $(z^{-1})^{\lfloor \frac{n}{2} \rfloor}((x+y)^2 z)^{\lfloor \frac{n}{2} \rfloor}(x+y)^{\pi(n)} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (z^{-1})^{\lfloor \frac{(n-k)}{2} \rfloor}(y^2 z)^{\lfloor \frac{(n-k)}{2} \rfloor} y^{\pi(n-k)} (z^{-1})^{\lfloor \frac{k}{2} \rfloor}(x^2 z)^{\lfloor \frac{k}{2} \rfloor} x^{\pi(k)}$.

This, in turn, follows from Theorem 1 together with the facts, proved by induction, that if z^{-1},z,x,y satisfy the hypotheses, then $(z^{-1})x = q^{-1}x(z^{-1})$, $(z^{-1})y = q^{-1}y(z^{-1})$, and for $0 \leq n \in \mathbb{Z}$: $x^n = (z^{-1}z)^{\lfloor \frac{n}{2} \rfloor} x^n$ $= q^{2+\cdots+2(\lfloor \frac{n}{2} \rfloor)}(z^{-1})^{\lfloor \frac{n}{2} \rfloor}(x^2 z)^{\lfloor \frac{n}{2} \rfloor} x^{\pi(n)}$, while for $0 \leq k \leq n \in \mathbb{Z}$: $y^k x^{n-k} = (z^{-1}z)^{\lfloor \frac{n}{2} \rfloor} x^{n-k} y^k = (z^{-1}z)^{\lfloor \frac{n-k}{2} \rfloor} x^{n-k} (z^{-1}z)^{\lfloor \frac{k}{2} \rfloor} y^k = q^{2+4+\cdots+2(\lfloor \frac{n-k}{2} \rfloor)}$ $(z^{-1})^{\lfloor \frac{n-k}{2} \rfloor}(x^2 z)^{\lfloor \frac{n-k}{2} \rfloor} x^{\pi(n-k)} q^{2+\cdots+2(\lfloor \frac{k}{2} \rfloor)}(z^{-1})^{\lfloor \frac{k}{2} \rfloor}(y^2 z)^{\lfloor \frac{k}{2} \rfloor} y^{\pi(k)}$.

□

Theorem 2 yields numerical q-identities involving $\left\{ {n \atop k} \right\}_q$. Using countably infinite matrix realizations of the associative algebra, the theorem provides an infinite matrix identity which gives numerical identities in corresponding matrix entries. To illustrate the procedure, we establish:

Corollary 2.

$$\text{For } p{=}0 \text{ or } 1\text{:} \quad \prod_{m=0}^{n-1} [(q^{4(n-m)};q^2)_2 q^{-2n-2(n-m+1)-1}](q-q^{-1})^p = \tag{13}$$

$$\sum_{j=0}^{1}\sum_{k=0}^{n-j+pj}\begin{Bmatrix}2n+p\\2k+j\end{Bmatrix}_q\prod_{m=0}^{k-1}[(q^{-2(n-m)-1};q)_2q^{2(m+1)}](1-q^{2(n-k)+1})^j$$

$$\times\prod_{m=0}^{(n-k)-1}[(q^{2((n-k)-m-j+pj)};q)_2q^{-2(m+1)}](q^{2-p}-1)^{p+j-2pj}.$$

Proof. The countably infinite matrices: X, Y, Z^{-1}, Z, with i,j entries

$$X_{ij}=(q^i-1)\delta_{i-1,j},\quad Y_{ij}=(1-q^{-i})\delta_{i-1,j},\quad Z^{-1}_{ij}=q^{-i}\delta_{i,j},\quad Z_{ij}=q^i\delta_{i,j},$$

together with the identity matrix, generate an algebra which satisfies the hypotheses of Theorem 2. Direct computation shows that e.g.

$$((Z^{-1})^k(YZ)^k(Z^{-1})^{n-k}(XZ)^{n-k})_{ij}$$

$$=\prod_{m=0}^{k-1}[(q^{-i+2m};q)_2q^{2m+2}]\prod_{m=0}^{n-k-1}[(q^{i-2k-2m-1};q)_2q^{-2m-2}]\delta_{i-2n,j}.$$

Developing all relevant formulas, Theorem 2 gives an infinite matrix identity.

To obtain the p=0 formula, equate the (2n+1,1) entries in the infinite matrix identity; for the p=1 formula, equate the (2n+1,0) entries. □

3 Generating Functions and a Rogers-Ramanujan type result

Theorem 3. *The generating function for partitions with m parts, the largest $\lfloor\frac{m}{2}\rfloor$ of which are both not less than 2 and have minimal difference 2 with all other parts, is given by $\frac{1}{(q,q)_m}$.*

Proof. We adapt a classical argument, see e.g. [1]. Let $s_1\le\cdots\le s_{\lceil\frac{m}{2}\rceil}\le r_1\le r_2\le\cdots\le r_{\lfloor\frac{m}{2}\rfloor}$ be a partition satisfying the hypothesis. Write $n=s_1+\cdots+s_{\lceil\frac{m}{2}\rceil}+r_1+r_2+\cdots+r_{\lfloor\frac{m}{2}\rfloor}$ with $2\le r_1\le r_2-2, r_2\le r_4-2$, etc.. Define $0\le n_1\le n_2\le\cdots\le n_m$ uniquely by $s_j\equiv n_j$, for $j=1,\ldots,\lceil\frac{m}{2}\rceil$. and $r_j\equiv n_{\lceil\frac{m}{2}\rceil+j}+2j$ for $j=1,\ldots,\lfloor\frac{m}{2}\rfloor$. This defines a correspondence between partitions of n with m parts, the largest $\lfloor\frac{m}{2}\rfloor$ of which are '2-distinct' parts, and partitions of $n-2(\sum_1^{\lfloor\frac{m}{2}\rfloor}j)=n-\lfloor\frac{m}{2}\rfloor(\lfloor\frac{m}{2}\rfloor+1)$ into m non-negative parts. Using this correspondence, we see that generating function for partitions with m parts, the largest $\lfloor\frac{m}{2}\rfloor$ of which are both not less than 2 and have minimal difference 2 with all other parts, can be written:

$$\sum_{0\le n_1\le\cdots\le n_m}q^{n_1+\cdots+n_{\lceil\frac{m}{2}\rceil}+(n_{\lceil\frac{m}{2}\rceil+1}+2)+(n_{\lceil\frac{m}{2}\rceil+2}+4)+\cdots}$$

$$= \sum_{0\le n_1\le\cdots\le n_m} q^{2+\cdots+2\lfloor\frac{m}{2}\rfloor+n_1+\cdots+n_m}$$

$$= q^{\lfloor\frac{m}{2}\rfloor(\lfloor\frac{m}{2}\rfloor+1)} \sum_{0\le n_1\le\cdots\le n_m} q^{n_1+\cdots+n_m}$$

$$= q^{\lfloor\frac{m}{2}\rfloor(\lfloor\frac{m}{2}\rfloor+1)} \sum_{0\le n_1\le\cdots\le n_{m-1}} q^{n_1+\cdots+n_{m-1}} \frac{q^{n_{m-1}}}{1-q}$$

$$= \frac{q^{\lfloor\frac{m}{2}\rfloor(\lfloor\frac{m}{2}\rfloor+1)}}{(1-q)} \sum_{0\le n_1\le\cdots\le n_{m-2}} q^{n_1+\cdots+n_{m-2}} \frac{q^{2(n_{m-2})}}{(1-q^2)} = \cdots = \frac{q^{\lfloor\frac{m}{2}\rfloor(\lfloor\frac{m}{2}\rfloor+1)}}{(q;q)_m}.$$

□

Theorem 3 and standard partition interpretations applied to Lemma 1 provide a Rogers-Ramanujan type partition result:

Corollary 3. *The number of partitions of n into parts all equal to 1 or 9 modulo 10, or all equal to 3 or 7 modulo 10 is equal to the number of partitions of n with m parts, the largest* $\lfloor\frac{m}{2}\rfloor$ *of which are both not less than 2 and have minimal difference 2 with all other parts.*

References

[1] G. Andrews, K. Eriksson *Integer Partitions* Cambridge Univ. Press, Cambridge, 2004.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series,* Encyclopedia Math. Appl., vol. 35, Cambridge Univ. Press, Cambridge, 1990..

[3] R. L. Rubin, *A q^2-Analogue Operator for q^2-Analogue Fourier Analysis,* J. Math. Anal. Appl. **212** (1997), 571-582.

[4] R. L. Rubin, *Duhamel Solutions of Non-Homogeneous q^2-Analogue Wave Equations,* Proc. Am. Math. Soc. **135** (2007),777-786.

[5] M.P. Schutzenberger, *Une interpretation de certaines solutions de l'equation fonctionnelle:* $F(x+y) = F(x)F(y)$. C. R. Acad. Sci. Paris 236(1953), 352-353.

[6] L. J. Slater, *Further Identities of the Rogers-Ramanujan Type* Proc. London Math. Soc. 54(1952), 147-167.