MULTI-RESTRAINED STIRLING NUMBERS

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ABSTRACT. Given positive integers n, k, and m, the (n,k)-th m-restrained Stirling number of the first kind is the number of permutations of an n-set with k disjoint cycles of length $\leq m$. Inverting the matrix consisting of the (n,k)-th m-restrained Stirling number of the first kind as the $(n+1,k+1)$-th entry, the (n,k)-th m-restrained Stirling number of the second kind is defined. In this paper, the multi-restrained Stirling numbers of the first and the second kinds are studied to find their explicit formulae, recurrence relations, and generating functions. Also, a unique expansion of multi-restrained Stirling numbers for all integers n and k, and a new generating function for the Stirling numbers of the first kind are introduced.

1. INTRODUCTION

For any positive integers n and k, the (n,k)-th Stirling number of the first kind, denoted by $S_1(n,k)$, is defined as $(-1)^{n-k}$ times the number of permutations of an n-set with k disjoint cycles, and the (n,k)-th Stirling number of the second kind, denoted by $S_2(n,k)$, is defined as the number of partitions of an n-set with k nonempty subsets. By convention, the Stirling numbers are defined for zero n and k: $S_i(n,0) = S_i(0,k) = 0$ for any positive integers n and k, and $S_i(0,0) = 1$, for $i = 1$ or 2.

The n-th falling factorial of an indeterminate x, $[x]_n = x(x-1)(x-2)\cdots(x-n+1)$, can be expended as a sum of the powers of x, with $S_1(n,k)$ as the coefficient of x^k. Then, the n-th power of x, x^n, can be solved as a sum of the falling factorials of x, with $S_2(n,k)$ as the coefficient of $[x]_k$. That is,

$$[x]_n = \sum_{k=0}^{n} S_1(n,k)x^k : x^n = \sum_{k=0}^{n} S_2(n,k)[x]_k. \tag{1.1}$$

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Then, the matrices consisting of $S_i(n,k)$ as the $(n+1,k+1)$-th entry for each $i=1$ and 2 are inverse to each other, i.e.

$$[S_1(n,k)]_{n,k\geq 0} = [S_2(n,k)]^{-1}_{n,k\geq 0}\,.$$

The multi-restricted numbers of the second kind are defined by restricting the size of each subset in partitions of a set. For any positive integers m, n, and k, the (n,k)-th m-restricted number of the second kind, denoted by $M_2^m(n,k)$, is defined as the number of partitions of an n-set with k nonempty subsets, each of size $\leq m$ [4]. For zero n and k, the multi-restricted of the second kind are also defined similarly to the the Stirling numbers: $M_2^m(0,0) = 0$ and $M_2^m(n,0) = 0 = M_2^m(0,k)$ for any positive integers n and k.

Inverting the matrix consisting of the multi-restricted numbers of the second kind, the multi-restricted numbers of the first kind are defined. The (n,k)-th m-restricted number $M_1^m(n,k)$ of the first kind is the $(n+1,k+1)$-th entry in the inverse of the matrix consisting of $M_2^m(n,k)$ as the $(n+1,k+1)$-entry.

By the definition, the (n,k)-th m-restricted number of the second kind is the same as the (n,k)-th Stirling numbers of the second kind, if $m \geq n$ or $m > n-k$. Hence, the matrix $[M_2^m(n,k)]_{n,k\geq 0}$ is a lower triangular matrix whose diagonal entries are all 1's. So is the matrix $[M_1^m(n,k)]_{n,k\geq 0}$, and the (n,k)-th m-restricted number of the first kind is also the same as the (n,k)-th Stirling number of the first kind, if $m \geq n$ or $m > n-k$. Furthermore, the m-restricted numbers of the first kind and second kind are the same as Stirling numbers of the first kind and second kind, if m is large enough.

However, the m-restricted numbers of the first kind do not take alternating signs if $m \geq 3$, while the Stirling numbers of the first kind do. Moreover, the signless multi-restricted numbers of the first kind do not count the number of permutations with a restriction on the length of each cycle, while the multi-restricted numbers of the second kind do count the number of partitions with a restriction on the size of each subset. That is, the signless multi-restricted numbers of the first kind are not the signless Stirling numbers of the first kind with the restriction on the length of each cycle in permutations, while the multi-restricted numbers of the second kind are the Stirling numbers of the second kind with the restriction on the size of each subset in partitions.

In this paper, we study the so-called *(signless) multi-restrained Stirling numbers of the first kind* as the number of permutations of an n-set with k disjoint cycles, each of the length $\leq m$. Section 2 includes the formal definitions of the (signed and signless) multi-restrained Stirling numbers of the first and second kinds, their basic properties, and their general schemes. Section 3 includes the explicit formulae for the multi-restrained Stirling numbers of the first kind and the signless multi-restrained Stirling numbers of the first kind. Section 4 includes two different recurrence relations for the multi-restrained Stirling numbers of the first kind and second kind, and a unique expansion of the multi-restrained Stirling numbers for the negative integers n and k. Section 5 includes a generating function for the multi-restrained Stirling numbers of the first kind, and a new generating function for the Stirling numbers of the first kind.

2. Multi-restrained Stirling numbers

The (n, k)-th signless Stirling number of the first kind, denoted by $C(n, k)$, is the number of permutations of an n-set with k disjoint cycles, i.e.

$$C(n,k) = |S_1(n,k)|. \tag{2.1}$$

Since the Stirling numbers of the first kind takes alternating signs, $C(n, k)$ can be also expressed as $C(n,k) = (-1)^{n-k} S_1(n,k)$, that is

$$S_1(n,k) = (-1)^{n-k} C(n,k). \tag{2.2}$$

Now, we define the number of permutations of an n-set with k disjoint cycles restraining the length of each cycle, and obtain a new series of numbers approaching to the signless and signed Stirling numbers of the first kind.

Definition 2.1. For all positive integers n, k, and m, the (n, k)-th signless m-restrained Stirling number of the first kind, denoted by ${}^mC(n,k)$, is defined as the number of permutations of an n-set with k disjoint cycles of length $\leq m$. For the zero n and k, ${}^mC(n,k)$ is defined as

(1) ${}^mC(0,0) = 1$;
(2) ${}^mC(n,k) = 0$, for either $n = 0$ or $k = 0$.

Definition 2.2. For any nonnegative integers n and k and any positive integer m, the (n, k)-th m-restrained Stirling number of the first kind, denoted by ${}^mS_1(n,k)$, is defined as

$${}^mS_1(n,k) = (-1)^{n-k} \cdot {}^mC(n,k). \tag{2.3}$$

The following table shows the 3-restrained Stirling numbers of the first kind for $n, k = 0, 1, 2, \ldots, 7$.

(2.4)

${}^3S_1(n,k)$	$k=0$	1	2	3	4	5	6	7
$n=0$	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	−1	1	0	0	0	0	0
3	0	2	−3	1	0	0	0	0
4	0	0	11	−6	1	0	0	0
5	0	0	−20	35	−10	1	0	0
6	0	0	40	−135	85	−15	1	0
7	0	0	0	490	−525	175	−21	1

Since the signless restrained Stirling numbers of the first kind count the number of permutations with the restriction on the length of each cycle, whereas the signless Stirling numbers of the first kind count such numbers without any restriction, the signless restrained Stirling numbers of the first kind cannot exceed the signless Stirling numbers of the first kind. For some n, k, and m, the restrained Stirling numbers of the first kind are even the same as the Stirling numbers of the first kind. Especially when m is large enough, no cycle length may exceed m.

Lemma 2.3. *For any nonnegative integers n and k and any positive integer m,*

(1) $|{}^mS_1(n,k)| \le |S_1(n,k)|$;
(2) ${}^mS_1(n,k) = S_1(n,k)$ *if* $m \ge n$ *or* $m > n-k$;
(3) $\lim_{m\to\infty} {}^mS_1(n,k) = S_1(n,k)$.

Proof. By Definition 2.1 and 2.2, the results are straightforward except the case of $m > n-k$ in (2). Suppose there exist m, n, and k such that $m > n-k$ and ${}^mS_1(n,k) \ne S_1(n,k)$. Then, $|{}^mS_1(n,k)| \ne |S_1(n,k)|$, so $|{}^mS_1(n,k)| < |S_1(n,k)|$ by (1). This implies there is a permutation of an n-set with k disjoint cycles, at least one of whose lengths is greater than m. If one cycle has length $m+1$, the remaining $n-(m+1)$ elements should compose $k-1$ disjoint cycles, but this is impossible, because $n-(m+1) = (n-m)-1 < k-1$ from the condition $m > n-k$. Hence, ${}^mS_1(n,k) = S_1(n,k)$ if $m > n-k$. □

There is no permutation of an n-set with k disjoint cycles of length at most m, if $n < k$ and $n > km$. The identity permutation is the only permutation of an n-set having n disjoint cycles of length 1. Hence, ${}^mC(n,k) = 0$ if $n < k$ or $n > km$, and ${}^mC(n,n) = 1$ for any positive integer m. By Definition 2.2, we can obtain the following.

Lemma 2.4. *For any nonnegative integers n and k, and any positive integer m,*

(1) ${}^mS_1(n,k) = 0$ *if* $n < k$ *or* $n > km$;
(2) ${}^mS_1(n,n) = 1$.

Lemma 2.4 shows that for any fixed integer m, the matrix consisting of ${}^mS_1(n,k)$ as the $(n+1, k+1)$-th entry is a lower triangular matrix whose main diagonal entries are all 1's. This matrix is invertible. Using its inverse matrix, we define the multi-restrained Stirling numbers of the second kind.

Definition 2.5. Given any positive integer m, let ${}^mS_1$ be the matrix consisting of ${}^mS_1(n,k)$ as the $(n+1, k+1)$-th entry for any nonnegative integers n and k, and let ${}^mS_2$ be the inverse of ${}^mS_1$, i.e. ${}^mS_2 = ({}^mS_1)^{-1}$. Then, the (n,k)-th m-restrained Stirling number of the second kind, denoted by ${}^mS_2(n,k)$, is defined to be the $(n+1, k+1)$-th entry of ${}^mS_2$. That is,

$$[{}^mS_2(n,k)]_{n,k\geq 0} = [{}^mS_1(n,k)]^{-1}_{n,k\geq 0} \tag{2.5}$$

The following table shows the 3-restrained Stirling numbers of the second kind for $n, k = 0, 1, \dots 7$.

(2.6)

${}^3S_2(n,k)$	$k=0$	1	2	3	4	5	6	7
$n=0$	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	3	1	0	0	0	0
4	0	−5	7	6	1	0	0	0
5	0	−65	−15	25	10	1	0	0
6	0	−455	−455	0	65	15	1	0
7	0	−1295	−4725	−1715	140	140	21	1

Since ${}^mS_1$ is a lower triangular matrix whose main diagonal entries are all 1's, so is its inverse matrix ${}^mS_2$.

Lemma 2.6. *For any nonnegative integers n and k, and any positive integer m,*

(1) ${}^mS_2(n,n) = 1$;
(2) ${}^mS_2(n,k) = 0$ *if just one of* n *or* k *is zero;*
(3) ${}^mS_2(n,k) = 0$ *if* $n < k$.

Furthermore, the lower triangularity of the matrix ${}^mS_1$ provides the identity between the multi-restrained Stirling numbers of the second kind and the Stirling numbers of the second kind for some n, k, and m, as the identity between the multi-restrained Stirling numbers of the first and the Stirling numbers of the first kind in Lemma 2.3.

Lemma 2.7. *For any positive integer m, and any nonnegative integers n and k,*

(1) ${}^mS_2(n,k) = S_2(n,k)$ *if* $m \geq n$ *or* $m > n-k$;
(2) $\lim_{m\to\infty} {}^mS_2(n,k) = S_2(n,k)$.

3. Explicit Formulae

The explicit formula for the signless multi-restrained numbers of the first kind can be easily derived from the explicit formula for the signless Stirling numbers of the first kind,

$$C(n,k) = \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1+k_2+\cdots+k_n=k}} \frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}(k_1!)(k_2!)\cdots(k_n!)}. \tag{3.1}$$

Since k_i is just the number of the cycles of length i in each permutation, we let $k_i = 0$ if $i > m$ in order to count the number of permutations of an n-set with k disjoint cycles of length $\leq m$.

Theorem 3.1. *For any positive integer m and nonnegative integers n and k, the (n,k)-th m-restrained Stirling numbers of the first kind is*

$$^{m}C(n,k) = \sum_{\substack{k_1+2k_2+\cdots+mk_m=n \\ k_1+k_2+\cdots+k_m=k}} \frac{n!}{1^{k_1}2^{k_2}\cdots m^{k_m}(k_1!)(k_2!)\cdots(k_m!)}. \tag{3.2}$$

Proof. For any positive integer p, let $1^{k_1}|2^{k_2}|\cdots|p^{k_p}$ be the partition of an n-set with k_i i-subsets where $i = 1,2,\cdots,p$. Since the number of cycles of length i in an i-subset is $(i-1)!$, there are $((i-1)!)^{k_i}$ products of the k_i disjoint cycles of length i, where each cycle is from one of the k_i i-subsets. Hence, the number of products of the cycles, each from one of the partition subsets in the form of $1^{k_1}|2^{k_2}|\cdots|p^{k_p}$ is $(0!)^{k_1}(1!)^{k_2}\cdots((p-1)!)^{k_p}$. Since the number of partitions of an n-set into the subsets formed of $1^{k_1}|2^{k_2}|\cdot|p^{k_p}$ is

$$\frac{n!}{(1!)^{k_1}(2!)^{k_2}\cdots(p!)^{k_p}(k_1!)(k_2!)\cdots(k_p!)} \tag{3.3}$$

[2], the number of permutations of an n-set with k_i disjoint cycles of length i where $i = 1,2,\cdots,p$ is the product of $(0!)^{k_1}(1!)^{k_2}\cdots((p-1)!)^{k_p}$ and (3.3), i.e.

$$\frac{n!}{1^{k_1}2^{k_2}\cdots m^{k_m}(k_1!)(k_2!)\cdots(k_p!)} \tag{3.4}$$

To count the number of all the permutations of an n-set with k disjoint cycles of length $\leq m$, we have to add (3.4) satisfying $k_1+2k_2+\cdots+mk_m = n$ and $k_1+k_2+\cdots+k_m = k$. □

By Definition 2.2, the explicit formula for the multi-restrained Stirling numbers of the first kind is as follows.

Corollary 3.2. *For any positive integer m and nonnegative integers n and k, the (n,k)-th m-restrained Stirling number of the first kind is*

(3.5)
$$ {}^mS_1(n,k) = \sum_{\substack{k_1+2k_2+\cdots+mk_m=n \\ k_1+k_2+\cdots+k_m=k}} \frac{(-1)^{n-k}n!}{1^{k_1}2^{k_2}\cdots m^{k_m}(k_1!)(k_2!)\cdots(k_m!)}.$$

4. Recurrence Relations

Consider 1 as an element of an n-set. Then, 1 is contained in exactly one of the cycles in each permutation of an n-set. The number of cycles of an n-set of length i containing the element 1 is

$$\binom{n-1}{i-1}\cdot(i-1)! = (n-1)(n-2)\cdots(n-i+1), \tag{4.1}$$

because $\binom{n-1}{i-1}$ counts the choices of $i-1$ elements from the $n-1$ elements, and $(i-1)!$ counts the number of cycles of length i using $i-1$ chosen elements together with 1. Now, there are $n-i$ elements remaining to form the remaining cycles that complete a permutation of the n-set. Since we consider permutations of an n-set with k cycles of length $\leq m$ only, we have to make $k-1$ additional cycles of length $\leq m$ with the remaining $n-i$ elements. By adding all the cases where the element 1 belongs to a cycle of length 1 through m, we count the number of all permutations of an n-set with k cycles of length $\leq m$. Hence,

$$ {}^mC(n,k) = \sum_{i=1}^{m}(n-1)(n-2)\cdots(n-i+1)\cdot {}^mC(n-i,k-1) \tag{4.2}$$

We can also prove this recurrence relation for the signless multi-restrained Sitrling numbers of the first kind using the explicit formula (3.2).

Theorem 4.1. *For any positive integer m and nonnegative integers n and k,*

$$ {}^mC(n,k) = \sum_{i=1}^{m}[n-1]_{i-1}{}^mC(n-i,k-1) \tag{4.3}$$

Proof. ${}^mC(n,k)$ is the sum of

$$\frac{n!}{1^{k_1}2^{k_2}\cdots m^{k_m}(k_1!)(k_2!)\cdots(k_m!)} \tag{4.4}$$

where $\sum_{i=1}^m ik_i = n$ and $\sum_{i=1}^m k_i = k$. Considering $n! = n\cdot(n-1)! = \sum_{i=1}^m ik_i\cdot(n-1)!$, (4.4) is the same as

$$\sum_{i=1}^{m}\frac{(n-1)(n-2)\cdots(n-i+1)\cdot(n-i)!}{1^{k_1}\cdots i^{k_{i-1}}i^{k_i-1}i^{k_{i+1}}\cdots m^{k_m}(k_1!)\cdots(k_{i-1})!(k_i-1)!(k_{i+1})\cdots(k_m!)}.$$

Let $p_j = k_j$ if $j \neq i$ and $p_j = k_j - 1$ if $j = i$. Then, $\sum_{j=1}^{m} jp_j = \sum_{i=1}^{m} ik_i - i = n - i$ and $\sum_{j=1}^{m} p_j = \sum_{i=1}^{m} k_i - 1 = k - 1$. Hence, ${}^mC(n,k)$ is the sum of

$$(4.5) \qquad \sum_{i=1}^{m} \frac{(n-1)(n-2)\cdots(n-i+1)\cdot(n-i)!}{1^{p_1}2^{p_2}\cdots m^{p_m}(p_1!)(p_2!)\cdots(p_m!)}$$

where $\sum_{j=1}^{m} jp_j = n - i$ and $\sum_{j=1}^{m} p_j = k - 1$. Since $[n-1]_{i-1} = (n-1)(n-2)\cdots(n-i+1)$ and

$${}^mC(n-i,k-1) = \sum_{\substack{p_1+2p_2+\cdots+mp_m=n-i \\ p_1+p_2+\cdots+p_m=k-1}} \frac{(n-i)!}{1^{p_1}2^{p_2}\cdots m^{p_m}(p_1!)(p_2!)\cdots(p_m!)},$$

(4.3) is obtained. □

Definition 2.2 and Theorem 4.1 provide the recurrence relation for the multi-restrained Stirling numbers of the first kind as follows.

Corollary 4.2. *For any positive integer m and nonnegative integers n and k,*

$$(4.6) \qquad {}^mS_1(n,k) = \sum_{i=1}^{m} (-1)^{i-1}[n-1]_{i-1}\,{}^mS_1(n-i,k-1)$$

Proof. By Definition 2.2, ${}^mC(n,k) = (-1)^{k-n}\cdot{}^mS_1(n,k)$ and ${}^mC(n-i,k-1) = (-1)^{k-n+i-1}\cdot{}^mS_1(n-i,k-1)$. Since $(-1)^{2(k-n)} = 1$, (4.3) yields (4.6). □

Let S_n and S_{n-1} be the set of permutations of an n-set, $\overline{n} \doteq \{1,2,3,\cdots,n-1,n\}$, and its subset, $\overline{n-1} = \{1,2,3,\cdots,n-1\}$, respectively. Then, each permutation in S_n can be expressed as a product of a permutation σ in S_{n-1} and either the 1-cycle (n) or the transposition $(n\ a)$ for some $a \in \overline{n-1}$, i.e. for any π in S_n, π can be expressed as case (1) $(n)\sigma$ or case (2) $(n\ a)\sigma$.

To count the number of permutations of an n-set with k disjoint cycles, we need to add the case (1) and the case (2). In the case (1), the $\sigma \in S_{n-1}$ should have $k-1$ disjoint cycles, because the 1-cycle (n) is disjoint with every cycle in σ. In the case (2), the $\sigma \in S_{n-1}$ should have k disjoint cycles, because the transposition $(n\ a)$ is not disjoint with one of the cycles in σ.

Since the length of each cycle cannot be greater than m, case (2) should be examined more closely. Without loss of generality, let $(a_1\ a_2\ \cdots\ a_i\ a)$ be a cycle in σ. Then, $(n\ a)(a_1\ a_2\ \cdots\ a_i\ a) = (a_1\ a_2\ \cdots\ a_i\ a\ n)$ so the length of the product cycle is one more than the length of the cycle before multiplying $(n\ a)$ in σ. Hence, if the cycle containing the element a in $\sigma \in S_{n-1}$ has length m, $\pi = (n\ a)\sigma$ has length $m+1$, that we have to disregard. Let's call it case (3): $\pi \in S_n$ is expressed as a product of

$(n\ a)$, $(a_1\ a_2\ \cdots\ a_{m-1}\ a)$, and any permutation of the $(n-1-m)$-set, $\underline{n-1}-\{a_1,a_2,\cdots,a_{m-1},a\}$, with $k-1$ cycles of length $\le m$.

The case (1) provides ${}^mC(n-1,k-1)$ and the case (2) provides $(n-1)\cdot{}^mC(n-1,k)$, because there are $n-1$ choices for the element a. The case (3) provides $(n-1)(n-2)\cdots(n-m)\cdot{}^mC(n-1-m,k-1)$, because there are $n-1$ choices for the element a and there are $(n-2)\cdots(n-m)$ choices for the cycle $(a_1\ a_2\ \cdots\ a_{m-1})$. By adding the case (1) and the case (2), and subtracting the case (3), we can have another recurrence relation for the signless multi-restrained Stirling numbers of the first kind as in Theorem 4.3. We can also prove this recurrence relation explicitly using the recurrence relation in Theorem 4.1.

Theorem 4.3. *For any positive integer m and nonnegative integers n and k,*

$$\begin{aligned}{}^mC(n,k) = &\ {}^mC(n-1,k-1)+(n-1)\cdot{}^mC(n-1,k)\\ &-(n-1)(n-2)\cdots(n-m)\cdot{}^mC(n-m-1,k-1)\end{aligned}$$

Proof. By adding and subtracting $(n-1)(n-2)\cdots(n-m)\cdot{}^mC(n-m-1,k-1)$ to (4.3), we have

$$\begin{aligned}{}^mC(n,k) = &\ {}^mC(n-1,k-1)\\ &+\{(n-1)\cdot{}^mC(n-2,k-1)\\ &+\cdots\\ &+(n-1)(n-2)\cdots(n-m+1)\cdot{}^mC(n-m,k-1)\\ &+(n-1)(n-2)\cdots(n-m+1)(n-m)\cdot{}^mC(n-m-1,k-1)\}\\ &-(n-1)(n-2)\cdots(n-m+1)(n-m)\cdot{}^mC(n-m-1,k-1)\end{aligned}$$

Factoring $(n-1)$ out, the middle part $\{\cdots\}$ becomes

$$(n-1)\cdot\sum_{i=1}^{m}[n-1-i]_{i-1}\cdot{}^mC(n-1-i,k-1),$$

which is $(n-1)\cdot{}^mC(n-1,k)$ by (4.3) □

Definition 2.2 and Theorem 4.3 provide another recurrence relation for the multi-restrained Stirling numbers of the first kind as follows.

Corollary 4.4. *For any positive integer m and nonnegative integers n and k,*

$$\begin{aligned}{}^mS_1(n,k) = &\ {}^mS_1(n-1,k-1)-(n-1)\cdot{}^mS_1(n-1,k)\\ &-(-1)^m(n-1)(n-2)\cdots(n-m)\cdot{}^mS_1(n-m-1,k-1).\end{aligned}$$

To obtain a recurrence relation for the multi-restrained Stirling numbers of the second kind, we recall [3, Corollary 4.2] as follows.

Proposition 4.5. [3, Corollary 4.2] *Let $[a(n,k)]_{n,k\geq 0}$ be an infinite invertible matrix with $a(0,0) = \pm 1$, and with only finitely many non-zero elements in each row. Set $a(n,k) = 0$ if just one of n and k is negative. Suppose that $a(n,k)$ satisfies the graded recurrence relation*

$$a(n+1,k+1) = \sum_{i=0}^{r-1} f_i(n)a(n-i,k) \tag{4.7}$$

for all integers n,k with $n \geq 0$. Then there is a unique extension of $a(n,k)$ to all integers n,k such that the relation (4.7) is always satisfied and

$$a(-n,-k) = (-1)^{n+k} b(k,n) \tag{4.8}$$

for all non-negative integers n,k, where $[b(n,k)]_{n,k\in N}$ is the matrix inverse to $[a(n,k)]_{n,k\in \mathbf{N}}$. Finally, we have

$$b(n-1,k-1) = \sum_{i=0}^{r-1} f_i(k+i-1)b(n,k+i) \tag{4.9}$$

for all natural numbers n and k.

Since ${}^mS_1(n,k)$ satisfies every condition for $a(n,k)$ in Proposition 4.5 and the matrices $[{}^mS_1(n,k)]$ and $[{}^mS_2(n,k)]$ are inverse to each other, we can provide a recurrence relation for the multi-restrained Stirling numbers of the second kind.

Corollary 4.6. *For any positive integer m and for any integers n and k,*
(4.10)

$${}^mS_2(n,k) = {}^mS_2(n-1,k-1) - \sum_{i=1}^{m-1} (-1)^i (k)\cdots(k+i-1)\cdot {}^mS_2(n,k+i)$$

Proof. The recurrence relation (4.6) can be rewritten as

$$\begin{aligned}{}^mS_1(n+1,k+1) &= \sum_{i=1}^{m} (-1)^{i+1}(n)(n-1)\cdots(n+2-i)\cdot {}^mS_1(n+1-i,k)\\ &= \sum_{i=0}^{m-1} (-1)^i (n)(n-1)\cdots(n+1-i)\cdot {}^mS_1(n-i,k).\end{aligned}$$

Hence, we can apply (4.9) of Proposition 4.5 to have

$${}^mS_2(n-1,k-1) = \sum_{i=0}^{m-1} (-1)^i (k+i-1)(k+i-2)\cdots(k)\cdot {}^mS_2(n,k+i), \tag{4.11}$$

which can be solved for ${}^mS_2(n,k)$ as (4.10). □

Proposition 4.5 also provides an extension of the multi-restrained Stirling numbers of the first kind, ${}^mS_1(n,k)$, and the second kind, ${}^mS_2(n,k)$, for all integers n and k.

Corollary 4.7. *For any positive integer m and all integers n and k,*

(1) ${}^{m}S_1(n,k) = (-1)^{n-k} \cdot {}^{m}S_2(-k,-n)$
(2) ${}^{m}S_2(n,k) = (-1)^{n-k} \cdot {}^{m}S_1(-k,-n)$

Proof. Applying (4.8), we have

$${}^{m}S_1(-n,-k) = (-1)^{n-k} \cdot {}^{m}S_2(k,n)$$

for any nonnegative integers n and k. We adjust n and k to provide (1), and we solve (1) for ${}^{m}S_2(n,k)$ to have (2), for all integers n and k. □

Since both the ${}^{m}S_1(n,k)$ and ${}^{m}S_2(n,k)$ for all integers n and k are built to satisfy the recurrence relation, the properties stated in Lemma 2.3, Lemma 2.4, Lemma 2.6, and Lemma 2.7 can be extended for all integers n and k. The general schemes for the m-restrained Stirling numbers of the first kind and the second kind can be shown in the following box. S_i and ${}^{m}S_i$ represents the Stirling numbers and the m-restrained Stirling numbers of the i-th kind, respectively, for $i = 1$ and 2. The blank cells represents 0. Note that m diagonal entries of the m-restrained Stirling numbers are the same as the Stirling numbers.

$$\begin{array}{cccccccccccccc}
1 \\
S_i & 1 \\
\cdots & S_i & 1 \\
S_i & \cdots & S_i & 1 \\
{}^{m}S_i & \ddots & \ddots & \ddots & \ddots \\
{}^{m}S_i & {}^{m}S_i & S_i & \cdots & S_i & 1 \\
{}^{m}S_i & {}^{m}S_i & {}^{m}S_i & S_i & \cdots & S_i & 1 \\
 & & & & & & & 1 \\
 & & & & & & & & 1 \\
 & & & & & & & & S_i & 1 \\
 & & & & & & & & \vdots & S_i & 1 \\
 & & & & & & & & S_i & \vdots & S_i & \ddots \\
 & & & & & & & & {}^{m}S_i & S_i & \vdots & \ddots & 1 \\
 & & & & & & & & {}^{m}S_i & {}^{m}S_i & S_i & \ddots & S_i & 1
\end{array}$$

m-restrained Stirling numbers

Applying Corollary 4.7 to Corollary 4.4, another recurrence relation for the multi-restrained Stirling numbers of the second kind can be obtained as follows.

Corollary 4.8. *For any positive integer m and all integers n and k,*

$$\begin{aligned} {}^mS_2(n,k) = &\ {}^mS_2(n-1,k-1) + k \cdot {}^mS_2(n-1,k) \\ &+ (-1)^m k(k+1)\cdots(k+m-1) \cdot {}^mS_2(n,k+m) \end{aligned}$$

Proof. Applying ${}^mS_1(n,k) = (-1)^{n-k} \cdot {}^mS_2(-k,-n)$ to the recurrence relation in Corollary 4.4, and multiplying both sides by $(-1)^{n-k}$, we have

$$\begin{aligned} {}^mS_2(-k,-n) = &\ {}^mS_2(1-k,1-n) + (n-1) \cdot {}^mS_2(-k,1-n) \\ &- (n-1)(n-2)\cdots(n-m) \cdot {}^mS_2(1-k,1+m-n). \end{aligned}$$

Replacing k and n with $1-n$ and $1-k$ respectively, we have

$$\begin{aligned} {}^mS_2(n-1,k-1) = &\ {}^mS_2(n,k) - k \cdot {}^mS_2(n-1,k) \\ &- (-k)(-k-1)\cdots(-k-m+1) \cdot {}^mS_2(n,k+m). \end{aligned}$$

Since $(-k)(-k-1)\cdots(-k-m+1) = (-1)^m k(k+1)\cdots(k+m-1)$, we just need to solve for ${}^mS_2(n,k)$. □

5. Generating functions

The chain rule for the higher order differentiation of composite functions by Faá di Bruno [1] can be modified for a special function as follows.

Proposition 5.1. *For any function $g(t)$ and for any indeterminate x, let $f(t) = (g(t))^x$. Then, for any positive integer n, the n-th derivative of $f(t)$ with respect to t is*

(5.1)
$$f^{(n)}(t) = \sum_{k=1}^{n} \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1+k2+\cdots+k_n=k}} \frac{n!}{k_1!k_2!\cdots k_n!} \prod_{i=1}^{n} \left(\frac{g^{(i)}(t)}{i!}\right)^{k_i} (g(t))^{x-k}[x]_k.$$

Using this modified Faá di Bruno formula (5.1), we can find a generating function for the signless multi-restrained Stirling numbers of the first kind.

Theorem 5.2. *For any positive integer m, let $g(t) = 1+t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^m}{m}$, and for any indeterminate x, let $f(t) = (g(t))^x$. Then, for any positive integer n, the n-th derivative of $f(t)$ with respect to t at $t=0$ is*

$$f^{(n)}(0) = \sum_{k=1}^{n} {}^mC(n,k)[x]_k. \tag{5.2}$$

Proof. Since $g^{(j)}(t) = 0$ for $j > m$, the coefficient for $[x]_k$ of $f^{(n)}(t)$ in (5.1) is

$$\sum_{\substack{k_1+2k_2+\cdots+mk_m=n \\ k_1+k2+\cdots+k_m=k}} \frac{n!}{k_1!k_2!\cdots k_m!} \prod_{i=1}^{m} \left(\frac{g^{(i)}(t)}{i!}\right)^{k_i} (g(t))^{x-k}, \tag{5.3}$$

for each $k = 1, 2, \cdots, n$. Since $g(0) = 1$ and $g^{(i)}(0) = (i-1)!$ for each $i = 1, 2, \cdots, m$, we have

$$\prod_{i=1}^{m} \left(\frac{g^{(i)}(0)}{i!} \right)^{k_i} (g(0))^{x-k} = \frac{1}{1^{k_1} 2^{k_2} \cdots m^{k_m}}.$$

Hence, the coefficient of $[x]_k$ in $f^{(n)}(0)$ for each $k = 1, 2, \cdots, m$ is

$$\sum_{\substack{k_1 + 2k_2 + \cdots + mk_m = n \\ k_1 + k2 + \cdots + k_m = k}} \frac{n!}{k_1! k_2! \cdots k_m! \cdot 1^{k_1} 2^{k_2} \cdots, m^{k_m}}$$

which is the same as ${}^mC(n,k)$ in (3.2). □

Since ${}^mS_1(n,k) = (-1)^{n-k} \cdot {}^mC(n,k)$, we can also have a generating function for the multi-restrained Stirling numbers of the first kind.

Corollary 5.3. *For any positive integer m and for any indeterminate x, let $g(t) = 1 + t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + (-1)^{m-1}\frac{t^m}{m}$ and let $f(t) = (g(t))^x$. Then, for any positive integer n, the n-th derivative of $f(t)$ with respect to t at $t = 0$ is*

$$f^{(n)}(0) = \sum_{k=1}^{n} {}^mS_1(n,k)[x]_k. \tag{5.4}$$

Proof. Since $g(0) = 1$ and $g^{(i)}(0) = (-1)^{i-1}(i-1)!$, the coefficient of $[x]_k$ in $f^{(n)}(t)$, (5.3), is

$$\sum_{\substack{k_1 + 2k_2 + \cdots + mk_m = n \\ k_1 + k2 + \cdots + k_m = k}} \frac{n!}{k_1! k_2! \cdots k_m!} \cdot \prod_{i=1}^{m} \left(\frac{(-1)^{i-1}}{i} \right)^{k_i}. \tag{5.5}$$

Since

$$\prod_{i=1}^{m} \left((-1)^{i-1}\right)^{k_i} = (-1)^{(k_1+2k_2+\cdots+mk_m)-(k_1+k_2+\cdots+k_m)} = (-1)^{(n-k)},$$

(5.5) becomes $(-1)^{n-k} \cdot {}^mC(n,k)$ as desired. □

The property, $\lim_{m\to\infty} {}^mS_1(n,k) = S_1(n,k)$ for any integers n and k, suggests a new generating function for the Stirling numbers of the first kind as

$$\lim_{m\to\infty} \left(1 + t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + (-1)^{m-1}\frac{t^m}{m}\right)^x = (1 + \log(1+t))^x.$$

Theorem 5.4. *For any indeterminate x, let $f(t) = (1+\log(1+t))^x$. Then, for any positive integer n, the n-th derivative of $f(t)$ with respect to t at $t=0$ is*

$$f^{(n)}(0) = \sum_{k=1}^{n} S_1(n,k)[x]_k. \tag{5.6}$$

Proof. Using the binomial series identity, we have

$$\begin{aligned} f(t) = [1+\log(1+t)]^x &= \sum_{k=0}^{\infty} \binom{x}{k} [\log(1+t)]^k \\ &= \sum_{k=0}^{\infty} \frac{[\log(1+t)]^k}{k!} [x]_k. \end{aligned}$$

By Theorem 8.3 in p 282 of [2], we can replace $\frac{[\log(1+t)]^k}{k!}$ with $\sum_{p=k}^{\infty} S_1(p,k)\frac{t^p}{p!}$:

$$f(t) = \sum_{k=0}^{\infty} \left(\sum_{p=k}^{\infty} S_1(p,k) \frac{t^p}{p!} \right) [x]_k,$$

which is the same as

$$f(t) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} S_1(n,k)[x]_k \right) \frac{t^n}{n!},$$

because $S_1(n,0) = 0$ for $n \neq 0$. □

This generating function for the Stirling numbers of the first kind provides a new generating function for the signless Stirling number of the first kind as well.

Corollary 5.5. *For any indeterminate x, let $f(t) = (1-\log(1-t))^x$. Then, for any positive n, the n-th derivative of $f(t)$ with respect to t at $t=0$ is*

$$f^{(n)}(0) = \sum_{k=1}^{n} C(n,k)[x]_k. \tag{5.7}$$

Proof. Since $C(n,0) = 0$ for $n \neq 0$ and $C(n,k) = (-1)^{n-k} S_1(n,k)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} C(n,k)[x]_k \right) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} S_1(n,k)[x]_k \right) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} (-1)^{n-k} S_1(n,k) \frac{t^n}{n!} \right) [x]_k. \end{aligned}$$

Since $(-1)^{n-k}S_1(n,k)t^n = (-1)^k S_1(n,k)(-t)^n$,

$$\sum_{n=0}^{\infty}(-1)^{n-k}S_1(n,k)\frac{t^n}{n!} = (-1)^k\left(\sum_{n=0}^{\infty}S_1(n,k)\frac{(-t)^n}{n!}\right)$$
$$= (-1)^k\left(\frac{[-\log(1-t)]^k}{k!}\right)(-1)^k = \frac{[-\log(1-t)]^k}{k!},$$

by Corollary 8.1 in p 283 [2]. Hence,

$$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}C(n,k)[x]_k\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\frac{[-\log(1-t)]^k}{k!}[x]_k$$
$$= \sum_{k=0}^{\infty}\binom{x}{k}[-\log(1-t)]^k = (1-\log(1-t))^x.$$

□

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