Asymmetric Cycle Avoidance Online Ramsey Games in Random Graphs*

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Abstract

Consider the following one-person game: let $S = \{F_1, F_2, \dots, F_r\}$ be a family of forbidden graphs. The edges of a complete graph are randomly shown to the Painter one by one, and he must color each edge with one of r colors when it is presented, without creating some fixed monochromatic forbidden graph F_i in the i-th color. The case of all graphs F_i being cycles is studied in this paper. We give a lower bound on the threshold function for online S-avoidance game, which generalizes the results of Marciniszyn, Spöhel and Steger for the symmetric case. [Combinatorics, Probability and Computing, Vol. 18, 2009: 271-300.]
Keywords: *Random graph; Cycle; Online Ramsey game; Ramsey property*

1. Introduction

Consider the following online Ramsey game. The edges of a complete graph of order n are shown to the Painter one by one in a random order, and the Painter must color the edge as soon as it is present, using one of r available colors. The Painter's goal is to color as many edges as possible, without creating any monochromatic copy of some fixed graph F. This online Ramsey game was introduced by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali [4] for the case $F = K_3$ and $r = 2$. The question we are interested in

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is that how long the Painter can 'survive' in such an online game, i.e., how many random edges can be colored without creating a monochromatic copy of F. The word 'online' is used to emphasize the fact that in each step the Painter has to decide how to color the new edge before seeing any further edges that appear later in the game. The term 'asymptotically almost surely' (*a.a.s.*) means 'with probability tending to 1 as $n \to \infty$'. Let $N_0 = N_0(F, r, n)$ be a threshold function for this game, meaning that there exists a strategy for the Painter such that he *a.a.s.* survives with this strategy for $N \ll N_0$ edges, and moreover the Painter *a.a.s.* loses the game for $N \gg N_0$ edges, regardless of his strategy. Marciniszyn, Spöhel and Steger [8] proved the existence of such a threshold function.

Let $G(n, N)$ denote the uniform model of random graphs on n vertices and N edges. Let $v(G) = |V(G)|$, $e(G) = |E(G)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S, and $G \setminus S = G[V(G) \setminus S]$. P_k is a path on k vertices, and C_k is a cycle of length k. For online K_l-avoidance game with two colors, Marciniszyn et al. showed that the threshold is $n^{(2-\frac{2}{l-1})(1+\frac{2}{l(l-1)})}$ [6]. They also gave a general upper bound for online F-avoidance game in the case of two colors [7]. Further more, for general $r \geq 2$, they proved that a lower bound for online F-avoidance game is $n^{2-\frac{1}{h(F,r)}}$, where $\lim_{r\to\infty} h(F,r) = \max_{H\subseteq F} \frac{e(H)-1}{v(H)-2}$ [8]. Belfrage, Mütze and Spöhel [2] presented techniques for deriving upper bounds on the threshold using a deterministic two-player game, and showed that the best bound derived in this way is the threshold of the game for F being a forest. Balogh and Butterfield [1] studied the online K_3-avoidance game, by a direct application of the techniques in [2], they proved that the upper bound on the threshold with r colors is $n^{\frac{3}{2}-c_r}$ (c_r is a constant related to r), which is the first result that separates the online threshold function from the offline bound for $r \geq 3$. Using a high performance computing network, Gordinowicz and Prałat [5] studied the triangle avoidance game on a small number of vertices.

In this paper we study the asymmetric version of the online Ramsey game for cycle avoidance. Let $\mathcal{S} = \{F_1, F_2, \ldots, F_r\}$ be a family of forbidden graphs such that if the Painter plays the online game with $r \geq 2$ available colors, his objective is to color as many edges as possible without creating a monochromatic copy of F_i in the i-th color. We refer to it as the online $\mathcal{S}$-avoidance game, denoted its threshold function by $N_0(\mathcal{S}, r, n)$. We call the number of properly colored edges N its *duration*.

Theorem 1 characterizes a threshold function for the offline version of the asymmetric game involving cycles. Since Painter has more information available when choosing colors in offline games, it will yield higher upper bounds than that of online games, as in Theorem 2.

Theorem 1. [9] *Let $r \geq 2$, $k_1 \geq k_2 \geq \ldots \geq k_r \geq 3$, and $\mathcal{P}$ = 'every r-edge-coloring of G contains a monochromatic copy of C_{k_i} in the i-th color for some i', then there exist positive constants c and C such that*

$$\lim_{n\to\infty} \mathbb{P}[G(n,m) \in \mathcal{P}] = \begin{cases} 1 & \text{if } m > Cn^{2-1/m_2(C_{k_{r-1}},C_{k_r})}, \\ 0 & \text{if } m < cn^{2-1/m_2(C_{k_{r-1}},C_{k_r})}. \end{cases}$$

where $m_2(C_{k_{r-1}}, C_{k_r}) := \frac{k_r}{k_r+(k_{r-1}-2)/(k_{r-1}-1)-2}$.

Theorem 2 is the main result of this paper, which gives lower bounds on online games.

Theorem 2. *If $\mathcal{S} = \{C_{k_1}, C_{k_2}, \ldots, C_{k_r}\}$ for $k_1 \geq k_2 \geq \ldots \geq k_r$, then the online $\mathcal{S}$-avoidance game has a threshold $N_0(\mathcal{S}, r, n)$ that satisfies*

$$N_0(\mathcal{S}, r, n) \geq n^{2-1/d(C_{k_1},C_{k_2},\ldots,C_{k_r})},$$

where $d(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) := \frac{\prod_{i=1}^{r} k_i}{\sum_{i=1}^{r}(k_i-2)\prod_{j=1}^{i-1} k_j+2}$.

In order to prove this theorem, we first present some useful notation and preliminaries in Section 2. In Section 3, the lower bound on $N_0(\mathcal{S}, r, n)$ for $r = 2$ is proved, and it is generalized to the case of r colors in Section 4.

2. Notation and Preliminaries

For a graph F, we give the following definitions which are similar to [8]. The standard density measure for graphs is $d(F) = e(F)/v(F)$ which is exactly half of the average degree, the maximum density measures m motivated by Theorem 3 and m_2 are as follows:

$$m(F) := \max_{H\subseteq F} \frac{e(H)}{v(H)}, \quad m_2(F) := \max_{H\subseteq F} \frac{e(H)-1}{v(H)-2}.$$

We call F a 1-*balanced* graph if $m(F) = e(F)/v(F)$, and 2-*balanced* if $m_2(F) = (e(F)-1)/(v(F)-2)$. For example, the cycles are both 1-balanced and 2-balanced. For graphs F_1 and F_2, analogously to [8], we define the density measures of asymmetric version as follows:

$$d(F_1, F_2) := \frac{e(F_2)}{v(F_2) + 1/m(F_1) - 2}, \text{ and}$$

$$m(F_1, F_2) := \max_{H\subseteq F_2} \frac{e(H)}{v(H) + 1/m(F_1) - 2},$$

that is, $m(F_1, F_2) = \max_{H \subseteq F_2} d(F_1, H)$. We set $d(F_1, F_2) = 0$ if F_1 or F_2 is empty.

Then we consider the asymmetric $\{F_1, F_2\}$-avoidance game with two colors, say red and blue, and assume that Painter uses one color, say red, in every move if this does not create a red copy of F_2. Clearly, the game will end with a blue copy of F_1, which is forced by a surrounding red structure. More precisely, when the game is over, the so far colored $G(n, N)$ contains a blue copy of F_1, each edge of which changed to red would complete a red copy of F_2. We will show that this greedy strategy yields the claimed lower bounds of Theorem 2 in the case $r = 2$. Similar to [8], for nonempty graphs F_1 and F_2, we define the family $\mathcal{F}(F_1, F_2)$ whose graphs have an 'inner' (blue) copy of F_1, each edge of which completes an 'outer' (red) copy of F_2. Here the colors should only provide the intuitive connection to the greedy strategy, the members of the family $\mathcal{F}(F_1, F_2)$ are not associate with a coloring. The family of graphs $\mathcal{F}(F_1, F_2)$ are formally defined as follows:

Definition 1. *For graphs $F_1 = (V, E)$ and F_2, let $\mathcal{F}(F_1, F_2) := \{F' : F' = (V \dot\cup U, E \dot\cup D)$, where F' is a minimal graph such that for all $f \in E$ there are sets $U(f) \subseteq V \dot\cup U$ and $D(f) \subseteq D$ with $(f \dot\cup U(f), \{f\} \dot\cup D(f)) \cong F_2\}$.*

In the definition, we take F' as any minimal graph with respect to subgraph inclusion, i.e., F' does not have a subgraph which satisfies the same properties. This ensures in particular that $\mathcal{F}(F_1, F_2)$ is finite. We will call V and E the sets of *inner vertices* and *inner edges* respectively, while $U(f)$ and $D(f)$ the sets of *outer vertices* and *outer edges* related to $f \in E$ respectively, where each edge $f \in E$ with its endvertices together with $U(f)$ and $D(f)$ forms a copy of F_2.

To illustrate the definition, we take a graph $F' \in \mathcal{F}(C_5, C_4)$ as an example, shown in Figure 1. We can see that the inner vertices set $V = \{v_1, v_2, v_3, v_4, v_5\}$, and the inner edges set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$ whose elements are drawn dashed. Let f_i be the inner edges, $f_i = v_iv_{i+1}$ for $1 \leq i \leq 4$, and $f_5 = v_5v_1$. Then, for each inner edge of E, the outer vertices $U(f_i) = \{u_{2i-1}, u_{2i}\}$, $1 \leq i \leq 4$, and $U(f_5) = \{u_9, u_{10}\}$, the outer edges $D(f_i) = \{v_iu_{2i-1}, u_{2i-1}u_{2i}, u_{2i}v_{i+1}\}$, $1 \leq i \leq 4$, and $D(f_5) = \{v_5u_9, u_9u_{10}, u_{10}v_1\}$. Note that inner vertices $v \in V$ may also serve as outer vertices $U(f)$ for a nonincident inner edge f. For example, v_1 is an inner vertex, and it is also an outer vertex in the outer copy associated with edge f_4, as shown in Figure 2. Hence, every outer copies contains exactly one inner edge and two or more inner vertices.

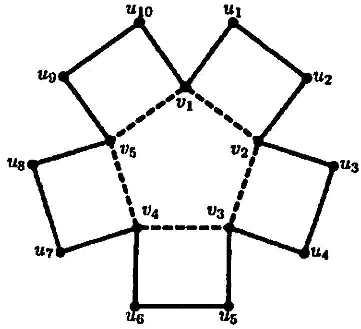

Figure 1: A graph $F' \in \mathcal{F}(C_5, C_4)$. Note that it is also a distinguished graph.

Among the graphs $F' \in \mathcal{F}(F_1, F_2)$, there are some distinguished ones F^* in which no outer copies overlap. The graph shown in Figure 1 is a distinguished one in $\mathcal{F}(C_5, C_4)$. For $F^* \in \mathcal{F}(F_1, F_2)$, we have $|U(f)| = v(F_2) - 2$ and $|D(f)| = e(F_2) - 1$. Hence, F^* has exactly $e(F_1)(v(F_2) - 2) + v(F_1)$ vertices and $e(F_1)(e(F_2) - 1) + e(F_1) = e(F_1)e(F_2)$ edges. If F_1 is 1-balanced graph, then $d(F_1) = m(F_1)$, and this yields

$$d(F^*) = \frac{e(F_1)e(F_2)}{e(F_1)(v(F_2) - 2) + v(F_1)} = \frac{e(F_2)}{v(F_2) + 1/m(F_1) - 2} = d(F_1, F_2). \tag{2.1}$$

The following Theorem obtained by Bollobás will be used in the proof of lower bounds on $N_0(\mathcal{S}, r, n)$.

Theorem 3. [3] *If F is a nonempty graph and $\mathcal{P}$ ='G contains a copy of F', then*

$$\lim_{n\to\infty} \mathbb{P}[G(n,m) \in \mathcal{P}] = \begin{cases} 1 & \text{if } m \gg n^{2-1/m(F)}, \\ 0 & \text{if } m \ll n^{2-1/m(F)}. \end{cases}$$

In our proofs we will frequently use the following easy to check proposition.

Proposition 1. *For $a, c \in R$ and $b > d > 0$, we have*

$$(a)\quad \frac{a}{b} \le \frac{c}{d} \Leftrightarrow \frac{a-c}{b-d} \le \frac{a}{b}, \text{or} \quad \frac{a}{b} \ge \frac{c}{d} \Leftrightarrow \frac{a-c}{b-d} \ge \frac{a}{b},$$

$$(b)\quad \frac{a}{b} \le \frac{c}{d} \Rightarrow \frac{a+c}{b+d} \ge \frac{a}{b}.$$

3. Proof of the lower bounds for $N_0(\mathcal{S}, 2, n)$

In this section, the special case of Theorem 2 will use mostly $F_1 = C_k$ and $F_2 = C_l$ for $k \geq l \geq 3$. First we say that the standard density measure of F^* is also the maximum one.

Lemma 1. *If $F_1 = C_k$ and $F_2 = C_l$, $k \geq l \geq 3$, then*

$$m(F^*) = d(F^*) = d(C_k, C_l).$$

Proof. Note that C_l is a 1-balanced graph and $m(C_l) = 1$. By (2.1),

$$d(F^*) = d(C_k, C_l) = \frac{e(C_l)}{v(C_l) + 1/m(C_k) - 2} = \frac{l}{l-1}.$$

Let $H \subset F^*$, we will prove that $d(H) \leq d(F^*)$ as follows. If $H \subset F^*[V(H)]$, then $d(H) < d(F^*[V(H)])$, so it leaves the case $H = F^*[V(H)]$, that is, H is a subgraph of F^* by removing some vertices from F^*. Suppose that H is obtained by removing the outer vertices set ΔV_{out} in x outer copies of C_l, and inner vertices set ΔV_{in} of the inner copy of C_k, that is $H \cong F^* \setminus (\Delta V_{out} \cup \Delta V_{in})$, where $\lceil |\Delta V_{out}|/(v(C_l) - 2)\rceil \leq x \leq e(C_k)$.

If we remove p_i vertices in the i-th outer copy of C_l, then at least $p_i + 1$ edges are deleted. So at least $\sum_{i=1}^{x}(p_i + 1) = |\Delta V_{out}| + x$ outer edges in total are deleted. Note that each inner vertex is incident to two edges in the inner copy of C_k and two edges in the outer copy of C_l. If we remove $|\Delta V_{in}|$ inner vertices, at least $|\Delta V_{in}|$ inner edges are deleted. If there are just $|\Delta V_{in}|$ edges are deleted (that is, all inner vertices are removed), then H is the set of some paths and $d(H) \leq 1 \leq d(F^*)$. Hence we only consider that at least $|\Delta V_{in}| + 1$ edges are deleted, there are two subcases.

Case 1. $|\Delta V_{in}| < x$, that is $|\Delta V_{in}|/x < 1$. We have

$$d(H) = \frac{e(H)}{v(H)} \leq \frac{e(F^*) - (|\Delta V_{out} \cup \Delta V_{in}| + x + 1)}{v(F^*) - |\Delta V_{out} \cup \Delta V_{in}|}.$$

Note that $|\Delta V_{in}| < x$ and $x \geq |\Delta V_{out}|/(l-2)$, which implies that $l - 2 \geq |\Delta V_{out}|/x$, then

$$\begin{aligned}\frac{|\Delta V_{out} \cup \Delta V_{in}| + x + 1}{|\Delta V_{out} \cup \Delta V_{in}|} &= 1 + \frac{1 + 1/x}{|\Delta V_{out}|/x + |\Delta V_{in}|/x} \\ &\geq 1 + \frac{1 + 1/x}{(l-2) + |\Delta V_{in}|/x} > 1 + \frac{1}{l-1} = \frac{e(F^*)}{v(F^*)},\end{aligned}$$

that is,

$$\frac{|\Delta V_{out} \cup \Delta V_{in}| + x + 1}{|\Delta V_{out} \cup \Delta V_{in}|} \geq \frac{e(F^*)}{v(F^*)}.$$

By Proposition 1(a), we have

$$\frac{e(F^*) - (|\Delta V_{out} \cup \Delta V_{in}| + x + 1)}{v(F^*) - |\Delta V_{out} \cup \Delta V_{in}|} \le \frac{e(F^*)}{v(F^*)}. \tag{3.1}$$

Case 2.. $|\Delta V_{in}| \ge x$. Note that there are at least $|\Delta V_{in}| - x + 1$ inner vertices in ΔV_{in} incident to the outer edges which do not belong to these x outer C_l. So at least $|\Delta V_{in}| - x + 1$ outer edges are deleted after removing $|\Delta V_{out} \cup \Delta V_{in}| + x + 1$ edges as in Case 1. Thus we have

$$d(H) = \frac{e(H)}{v(H)} \le \frac{e(F^*) - (|\Delta V_{out} \cup \Delta V_{in}| + x + 1 + (|\Delta V_{in}| - x + 1))}{v(F^*) - |\Delta V_{out} \cup \Delta V_{in}|}.$$

Note that $|\Delta V_{in}| \ge x$ and $x \ge |\Delta V_{out}|/(l-2)$, which implies that $l - 2 \ge |\Delta V_{out}|/x$, and hence we have

$$\begin{aligned}\frac{|\Delta V_{out} \cup \Delta V_{in}| + x + 1 + (|\Delta V_{in}| - x + 1)}{|\Delta V_{out} \cup \Delta V_{in}|} &= 1 + \frac{|\Delta V_{in}| + 2}{|\Delta V_{out}| + |\Delta V_{in}|} \\ &\ge 1 + \frac{|\Delta V_{in}|/x + 2/x}{(l-2) + |\Delta V_{in}|/x} \\ &= 1 + \frac{1 + (|\Delta V_{in}| - x + 2)/x}{(l-1) + (|\Delta V_{in}| - x)/x}.\end{aligned}$$

Since $1/(l-1) < 1$ and $\frac{(|\Delta V_{in}| - x + 2)/x}{(|\Delta V_{in}| - x)/x} > 1$, by Proposition 1($b$),

$$1 + \frac{1 + (|\Delta V_{in}| - x + 2)/x}{(l-1) + (|\Delta V_{in}| - x)/x} \ge 1 + \frac{1}{l-1} = \frac{e(F^*)}{v(F^*)},$$

that is, $\frac{|\Delta V_{out} \cup \Delta V_{in}| + x + 1 + (|\Delta V_{in}| - x + 1)}{|\Delta V_{out} \cup \Delta V_{in}|} \ge \frac{e(F^*)}{v(F^*)}$. By Proposition 1($a$),

$$\frac{e(F^*) - (|\Delta V_{out} \cup \Delta V_{in}| + x + 1 + (|\Delta V_{in}| - x + 1))}{v(F^*) - |\Delta V_{out} \cup \Delta V_{in}|} \le \frac{e(F^*)}{v(F^*)}. \tag{3.2}$$

By (3.1) and (3.2), $d(H) \le d(F^*)$. Hence $m(F^*) = d(F^*)$, and the lemma follows. □

For a graph $F \in \mathcal{F}(C_k, C_l)$, $k \ge l \ge 3$, we will show that if its outer copies are overlapped, then it is at least as dense as F^*.

Lemma 2. *For $F \in \mathcal{F}(C_k, C_l)$, $k \ge l \ge 3$, we have*

$$d(F) \ge m(F^*).$$

Proof. Since $d(F^*) = m(F^*)$ by Lemma 1, it is sufficient to prove $d(F) \ge d(F^*)$. Some auxiliary notation is introduced as follows. For each inner edge of F, we can find the only outer copy of C_l corresponding to it, and we can assume that the inner edges are $f_i = v_{i-1}v_i$ for $2 \leq i \leq k$ and $f_1 = v_k v_1$. Denote the outer C_l corresponding to f_i by C_l^i $(1 \leq i \leq k)$. Take the graph shown in Figure 2 as an example, $C_4^1 = v_6u_5v_4v_1$, $C_4^2 = v_1u_1u_2v_2$, $C_4^3 = v_2u_2v_5v_3$, $C_4^4 = v_3u_3v_1v_4$, $C_4^5 = v_4u_5u_4v_5$ and $C_4^6 = v_5u_4u_5v_6$.

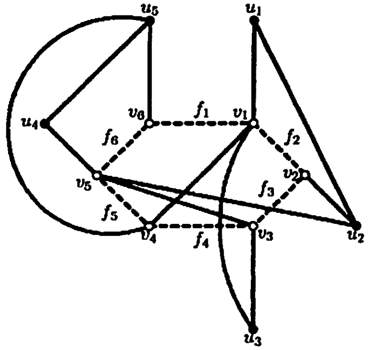

Figure 2: A graph in $\mathcal{F}(C_6, C_4)$. Note that the inner vertex v_1 also serves as an outer vertex in the outer copy associated with edge f_4.

Recall that $D(f_i)$ is the edge set of the outer edges of C_l^i, and $U(f_i)$ is its outer vertex set. Note that each outer copy of C_l contains two inner vertices. For each f_i, we define

$$\Delta_E^i := D(f_i) \cap \left(\dot{\bigcup}_{j<i} D(f_j) \right),$$

$$\Delta_V^i := U(f_i) \cap \left(\dot{\bigcup}_{j<i} U(f_j) \cup \{v_1, \ldots, v_{i-2}, v_k\} \right),$$

$$\Delta_{V'}^i := U(f_i) \cap \left(\{v_{i+1}, \ldots, v_{k-1}\} \setminus \dot{\bigcup}_{j<i} U(f_j) \right).$$

Intuitively, Δ_E^i contains the edges overlapped when outer C_l^i is merged with the preceding outer copies $C_l^j\,(j < i)$, and $\Delta_V^i \cup \Delta_{V'}^i$ contains all the vertices overlapped in this merge operation. Δ_V^i contains the vertices overlapped by the preceding vertices of $C_l^j\,(j < i)$, some of which are possibly inner vertices. The vertex set $\Delta_{V'}^i$ is used to record the inner vertex of C_l^j which are overlapped by an outer vertex of C_l^i for $j > i$ at the first time. In the graph shown in Figure 2, the nonempty edge (vertex) sets are as follows: $\Delta_E^3 = \{v_2u_2\}$, $\Delta_E^6 = \{v_6u_5, u_5u_4, u_4u_5\}$; $\Delta_V^3 = \{u_2\}$, $\Delta_V^4 = \{v_1\}$,$\Delta_V^5 = \{u_5\}$, $\Delta_V^6 = \{u_4, u_5\}$; $\Delta_{V'}^1 = \{v_4\}$, $\Delta_{V'}^3 = \{v_5\}$.

We call an outer copy of C_l^i *full* if $|\Delta_E^i| = l - 1$ and $|\Delta_V^i| = l - 2$, that is, it is merged entirely with the preceding outer copies. And we call an outer copy of C_l^i *trivial* if $|\Delta_E^i| = 0$ and $|\Delta_V^i| = 0$, that is, it is independent from the preceding outer copies. Otherwise, we call them *non-trivial*. In the graph shown in Figure 2, C_4^1 and C_4^2 are trivial, C_4^3, C_4^4 and C_4^5 are non-trivial, and C_4^6 is full. If an outer C_l^i is non-trivial, we can easily prove that $|\Delta_E^i| \le |\Delta_V^i|$, and then $0 \le |\Delta_E^i| \le l-2$, $1 \le |\Delta_V^i| \le l-2$. Let X_i and A_i denote $|\Delta_E^i|$ and $|\Delta_V^i|$ respectively when C_l^i is full. Let Y_i and B_i denote $|\Delta_E^i|$ and $|\Delta_V^i|$ respectively when C_l^i is not full. Hence

$$
\begin{aligned}
d(F) &= \frac{e(F^*) - |\Delta_E^1| - |\Delta_E^2| - \ldots - |\Delta_E^k|}{v(F^*) - (|\Delta_V^1| + |\Delta_{V'}^1|) - (|\Delta_V^2| + |\Delta_{V'}^2|) - \ldots - (|\Delta_V^k| + |\Delta_{V'}^k|)} \\
&= \frac{e(F^*) - |\Delta_E^1| - |\Delta_E^2| - \ldots - |\Delta_E^k|}{v(F^*) - |\Delta_V^1| - |\Delta_V^2| - \ldots - |\Delta_V^k| - \sum_{i=1}^k |\Delta_{V'}^i|} \\
&= \frac{e(F^*) - \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i}{v(F^*) - \sum_{i=1}^k A_i - \sum_{i=1}^k B_i - \sum_{i=1}^k |\Delta_{V'}^i|}.
\end{aligned} \tag{3.3}
$$

Case 1. Suppose that there does not exist any full C_l. Since $|\Delta_E^i|/|\Delta_V^i| \le 1 \le d(F^*)$ (or $|\Delta_E^i| = |\Delta_V^i| = 0$) and (3.3), we have

$$
\begin{aligned}
d(F) &= \frac{e(F^*) - \sum_{i=1}^k Y_i}{v(F^*) - \sum_{i=1}^k B_i - \sum_{i=1}^k |\Delta_{V'}^i|} \\
&= \frac{e(F^*) - \sum_{i=1}^k |\Delta_E^i|}{v(F^*) - \sum_{i=1}^k |\Delta_V^i| - \sum_{i=1}^k |\Delta_{V'}^i|} \\
&\ge \frac{e(F^*) - \sum_{i=1}^k |\Delta_E^i|}{v(F^*) - \sum_{i=1}^k |\Delta_V^i|} \\
&\ge \frac{e(F^*)}{v(F^*)}. \quad \text{(by Proposition 1}(a))
\end{aligned}
$$

Hence we have $d(F) \ge d(F^*)$.

Case 2. Suppose that there exists a full C_l. We will prove

$$
\frac{\sum_{i=1}^k X_i + \sum_{i=1}^k Y_i}{\sum_{i=1}^k A_i + \sum_{i=1}^k B_i + \sum_{i=1}^k |\Delta_{V'}^i|} \le \frac{e(F^*)}{v(F^*)} = \frac{l}{l-1}. \tag{3.4}
$$

Assume that there exists a full C_l^i for $1 \le i < k$. Since all the outer edges of C_l^i are common, then v_i must be in some $\Delta_{V'}^j$, where $j < i$, hence

$$
\frac{|\Delta_E^i|}{|\Delta_V^i| + |\Delta_{V'}^j|} \le \frac{l-1}{l-2+1} = \frac{l-1}{l-1} = 1.
$$

Assume that C_l^k is full. Note that the edges of the first full C_l are merged with at least two preceding outer C_l, and thus at least one of the outer C_l, say $C_l^{k'}$, is non-trivial. If $|\Delta_E^{k'}| = 0$, then $|\Delta_V^{k'}| = 1$, thus

$$\frac{|\Delta_E^k| + |\Delta_E^{k'}|}{|\Delta_V^k| + |\Delta_V^{k'}|} = \frac{l-1+0}{l-2+1} = \frac{l-1}{l-1} = 1.$$

If $|\Delta_E^{k'}| \geq 1$, then $|\Delta_V^{k'}| \geq |\Delta_E^{k'}| \geq 1$, and by Proposition 1$(a)$ we have

$$\frac{|\Delta_E^k| + |\Delta_E^{k'}|}{|\Delta_V^k| + |\Delta_V^{k'}|} = \frac{(|\Delta_E^k|+1) + (|\Delta_E^{k'}|-1)}{(|\Delta_V^k|+1) + (|\Delta_V^{k'}|-1)} \leq \frac{(l-1+1)}{(l-2+1)} = \frac{l}{l-1}.$$

Hence we have

$$\begin{aligned}
&\frac{\sum_{i=1}^{k} X_i + \sum_{i=1}^{k} Y_i}{\sum_{i=1}^{k} A_i + \sum_{i=1}^{k} B_i + \sum_{i=1}^{k} |\Delta_{V'}^i|} \\
&= \frac{\sum_{i=1,i\neq k'}^{k-1} X_i + \sum_{i=1,i\neq k'}^{k-1} Y_i + |\Delta_E^k| + |\Delta_E^{k'}|}{\sum_{i=1,i\neq k'}^{k-1} A_i + \sum_{i=1,i\neq k'}^{k-1} B_i + \sum_{i=1}^{k} |\Delta_{V'}^i| + |\Delta_V^k| + |\Delta_V^{k'}|} \\
&= \frac{\sum_{i=1,i\neq k'}^{k-1} X_i + \sum_{i=1,i\neq k'}^{k-1} Y_i + (|\Delta_E^k| + |\Delta_E^{k'}|)}{(\sum_{i=1,i\neq k'}^{k-1} A_i + \sum_{i=1}^{k} |\Delta_{V'}^i|) + \sum_{i=1,i\neq k'}^{k-1} B_i + (|\Delta_V^k| + |\Delta_V^{k'}|)}. \qquad (3.5)
\end{aligned}$$

Note that

$$\frac{\sum_{i=1,i\neq k'}^{k-1} X_i}{\sum_{i=1,i\neq k'}^{k-1} A_i + \sum_{i=1}^{k} |\Delta_{V'}^i|} \leq 1,$$

$$\frac{\sum_{i=1,i\neq k'}^{k-1} Y_i}{\sum_{i=1,i\neq k'}^{k-1} B_i} \leq 1 \text{ or } \sum_{i=1,i\neq k'}^{k-1} Y_i = \sum_{i=1,i\neq k'}^{k-1} B_i = 0,$$

$$\frac{|\Delta_E^k| + |\Delta_E^{k'}|}{|\Delta_V^k| + |\Delta_V^{k'}|} \leq \frac{l}{l-1}.$$

By Proposition 1(b), we have

$$\frac{\sum_{i=1,i\neq k'}^{k-1} X_i + \sum_{i=1,i\neq k'}^{k-1} Y_i + (|\Delta_E^k| + |\Delta_E^{k'}|)}{(\sum_{i=1,i\neq k'}^{k-1} A_i + \sum_{i=1}^{k} |\Delta_{V'}^i|) + \sum_{i=1,i\neq k'}^{k-1} B_i + (|\Delta_V^k| + |\Delta_V^{k'}|)} \leq \frac{l}{l-1},$$

and together with (3.5), this implies (3.4). Hence by Proposition 1(a), (3.3) and (3.4), we have

$$d(F) = \frac{e(F^*) - \sum_{i=1}^{k-1} X_i - \sum_{i=1}^{k-1} Y_i}{v(F^*) - \sum_{i=1}^{k-1} A_i - \sum_{i=1}^{k-1} B_i - \sum_{i=1}^{k} |\Delta_{V'}^i|} \geq \frac{e(F^*)}{v(F^*)} = d(F^*).$$

Thus the lemma holds. □

We conclude this section by giving the two colors version of Theorem 2.

Theorem 4. *If $\mathcal{S} = \{C_k, C_l\}$ for $k \geq l$, then the online $\mathcal{S}$-avoidance game has a threshold $N_0(\mathcal{S}, 2, n)$ that satisfies*

$$N_0(\mathcal{S}, 2, n) \geq n^{1+\frac{1}{l}}.$$

Proof. By Lemma 2, we have $m(F) \geq d(F) \geq m(F^*)$, thus $n^{2-1/m(F^*)} \leq n^{2-1/m(F)}$. If $N \ll n^{2-1/m(F)}$, then $G(n, N)$ will *a.a.s* contain no copy of F by Theorem 3. Hence $G(n, N)$ contains no graph $F \in \mathcal{F}(C_k, C_l)$ when $N \ll n^{2-1/m(F^*)} \leq n^{2-1/m(F)}$. That is, Painter can color the graph for N steps with a proper strategy in the $\mathcal{S}$-avoidance game. Hence, by Lemma 1, we have $N_0(\mathcal{S}, 2, n) \geq n^{2-1/m(F^*)} = n^{1+\frac{1}{l}}$, where $\mathcal{S} = \{C_k, C_l\}$. □

4. Proof of Theorem 2 for r Colors

First, we focus our attention on the asymmetric online game with three colors, say yellow, red and blue, that is, the Painter avoids first F_3 in yellow, then F_2 in red, and eventually F_1 in blue. We call this strategy the greedy (F_3, F_2, F_1)-avoidance strategy. By the same argument as before, when the game is over, the board contains a red-blue copy of a member of the family $\mathcal{F}(F_1, F_2)$, each edge of which completes an entirely yellow copy of F_3.

Let $\mathcal{F}(F_1, F_2, F_3)$ denote the class of all graphs that have an inner (blue-red) copy of a graph of $\mathcal{F}(F_1, F_2)$, each edge of which also completes an outer (yellow) copy of F_3. Similarly as in the case of two colors, the members of the family $\mathcal{F}(F_1, F_2, F_3)$ are not associated with a coloring. For example, a graph of $\mathcal{F}(C_5, C_4, C_3)$ is shown in Figure 3.

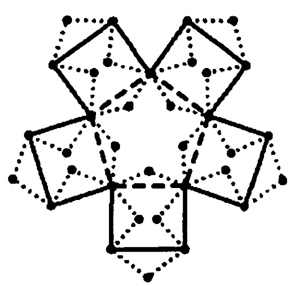

Figure 3: A graph in $\mathcal{F}(C_5, C_4, C_3)$.

This motivates the following inductive definitions $\mathcal{F}(F_1, \ldots, F_r)$ and $d(F_1, F_2, \ldots, F_r)$ for general r.

Definition 2. *For any graphs F_i, $1 \le i \le r$, define* $\mathcal{F}(F_1, F_2, \ldots, F_r)$

$$:= \begin{cases} F_1 & r = 1, \\ \mathcal{F}(F_1, F_2) & r = 2, \\ \{F^r \in \mathcal{F}(F^{r-1}, F_r) : F^{r-1} \in \mathcal{F}(F_1, F_2, \ldots, F_{r-1})\} & r \ge 3. \end{cases}$$

By the same argument as the case of three colors, if Painter uses the greedy $\langle F_r, \ldots, F_1 \rangle$-avoidance strategy in the game with r colors, at the end of the game the board will contain a copy of a graph in $\mathcal{F}(F_1, F_2 \ldots, F_r)$.

Definition 3. *For any graphs F_i, $1 \le i \le r$, define*

$$d(F_1, F_2, \ldots, F_r) := \begin{cases} \dfrac{e(F_r)}{v(F_r)} & r = 1, \\ \dfrac{e(F_r)}{v(F_r) + 1/m(F_1, F_2, \ldots, F_{r-1}) - 2} & r \ge 2, \end{cases}$$

where $m(F_1, F_2, \ldots, F_{r-1}) := \max_{H \subseteq F_{r-1}} d(F_1, F_2, \ldots, F_{r-2}, H)$.

We briefly call $m(F_1, F_2, \ldots, F_r)$ the *maximum density* of graphs in $\mathcal{F}(F_1, F_2, \ldots, F_r)$.

Analogously to the distinguished graph F^* for two colors, let $(F^r)^*$ be a graph in $\mathcal{F}(F_1, \ldots, F_r)$ in which no outer copies overlap, and the inner copy is $(F^{r-1})^*$, its structure is shown in Figure 4. By a similar argument as in the case of two colors, these graphs have exactly $e((F^{r-1})^*)(v(F_r) - 2) + v((F^{r-1})^*)$ vertices and $e((F^{r-1})^*)(e(F_r) - 1) + e((F^{r-1})^*) = e((F^{r-1})^*) e(F_r)$ edges. Hence we have

$$d((F^r)^*) = \frac{e((F^r)^*)}{v((F^r)^*)} = \frac{e((F^{r-1})^*)e(F_r)}{e((F^{r-1})^*)(v(F_r) - 2) + v((F^{r-1})^*)} = \frac{e(F_r)}{v(F_r) + 1/d((F^{r-1})^*) - 2}.$$

In the following lemma, we state that the maximum density of graphs in $\mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$ are no more than $m_2(C_{k_r})$, where the outer cycle C_{k_r} is the smallest one.

Lemma 3. $m(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) \le m_2(C_{k_r})$ *for* $k_1 \ge k_2 \ge \ldots \ge k_r$.

Proof. We prove the lemma by induction.
(1) For $r = 2$, since cycles are 1-balanced graphs, we have

$$m(C_{k_1}, C_{k_2}) = d(C_{k_1}, C_{k_2}) = \frac{k_2}{k_2 + m(C_{k_1}) - 2} = \frac{k_2}{k_2 - 1}$$

by equality (2.1). Hence

$$m(C_{k_1}, C_{k_2}) = \frac{k_2}{k_2 - 1} \le \frac{k_2 - 1}{k_2 - 2} = m_2(C_{k_2}).$$

(2) Suppose that $m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}}) \le m_2(C_{k_{r-1}})$ for $r \ge 3$. We will show that $m(C_{k_1}, C_{k_2}, \dots, C_{k_r}) \le m_2(C_{k_r})$. Let $H \subseteq C_{k_r}$, by Definition 3,

$$m(C_{k_1}, C_{k_2}, \dots, C_{k_r}) = \max_{H \subseteq C_{k_r}} \frac{e(H)}{v(H) + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}}) - 2}}.$$

Since $m_2(C_{k_r}) \ge \dfrac{e(H) - 1}{v(H) - 2}$, we have $\max_{H \subseteq C_{k_r}} \frac{e(H)}{v(H) + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}})} - 2} \le \max_{H \subseteq C_{k_r}} \frac{e(H)}{\frac{e(H)-1}{m_2(C_{k_r})} + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}})}}$. By the hypothesis,

$$\max_{H \subseteq C_{k_r}} \frac{e(H)}{\frac{e(H)-1}{m_2(C_{k_r})} + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}})}} \le \max_{H \subseteq C_{k_r}} \frac{e(H)}{\frac{e(H)-1}{m_2(C_{k_r})} + \frac{1}{m_2(C_{k_{r-1}})}}.$$

Since cycles are 1-balanced graphs and $k_{r-1} \ge k_r$,

$$m_2(C_{k_{r-1}}) = \frac{k_{r-1} - 1}{k_{r-1} - 2} \le \frac{k_r - 1}{k_r - 2} = m_2(C_{k_r})$$

and

$$\max_{H \subseteq C_{k_r}} \frac{e(H)}{\frac{e(H)-1}{m_2(C_{k_r})} + \frac{1}{m_2(C_{k_{r-1}})}} \le \max_{H \subseteq C_{k_r}} \frac{e(H)}{\frac{e(H)-1}{m_2(C_{k_r})} + \frac{1}{m_2(C_{k_r})}} = m_2(C_{k_r}),$$

that is, $m(C_{k_1}, C_{k_2}, \dots, C_{k_r}) \le m_2(C_{k_r})$. This complete the induction step, and the proof is complete. □

With Lemma 3 at hand, it is easy to prove the following equality, which will be used in the proof of Lemma 5 and Theorem 2.

Lemma 4. $m(C_{k_1}, C_{k_2}, \dots, C_{k_r}) = d(C_{k_1}, C_{k_2}, \dots, C_{k_r})$.

Proof. Let $H \subseteq C_{k_r}$, then

$$\frac{e(C_{k_r}) - e(H)}{\left(v(C_{k_r}) + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}})} - 2\right) - \left(v(H) + \frac{1}{m(C_{k_1}, C_{k_2}, \dots, C_{k_{r-1}})} - 2\right)}$$
$$= \frac{e(C_{k_r}) - e(H)}{v(C_{k_r}) - v(H)} = \frac{(e(C_{k_r}) - 1) - (e(H) - 1)}{(v(C_{k_r}) - 2) - (v(H) - 2)}. \tag{4.1}$$

Since cycles are 2-balanced graphs,

$$\frac{e(C_{k_r})-1}{v(C_{k_r})-2} = m_2(C_{k_r}) \geq \frac{e(H)-1}{v(H)-2}.$$

By Proposition 1(a),

$$\frac{(e(C_{k_r})-1)-(e(H)-1)}{(v(C_{k_r})-2)-(v(H)-2)} \geq \frac{e(C_{k_r})-1}{v(C_{k_r})-2} = m_2(C_{k_r}).$$

By Lemma 3, $m(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) \leq m_2(C_{k_r})$. Then by (4.1),

$$\frac{e(C_{k_r})-e(H)}{\left(v(C_{k_r})+\frac{1}{m(C_{k_1},C_{k_2},\ldots,C_{k_{r-1}})}-2\right)-\left(v(H)+\frac{1}{m(C_{k_1},C_{k_2},\ldots,C_{k_{r-1}})}-2\right)}$$

$$\geq m_2(C_{k_r}) \geq m(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) = \frac{e(C_{k_r})}{v(C_{k_r})+\frac{1}{m(C_{k_1},C_{k_2},\ldots,C_{k_{r-1}})}-2},$$

hence

$$\frac{e(C_{k_r})}{v(C_{k_r})+\frac{1}{m(C_{k_1},C_{k_2},\ldots,C_{k_{r-1}})}-2} \geq \frac{e(H)}{v(H)+\frac{1}{m(C_{k_1},C_{k_2},\ldots,C_{k_{r-1}})}-2},$$

that is, $d(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}}, C_{k_r}) \geq d(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}}, H)$. By Definition 3, we have $m(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$. □

The following lemma is a generalization of Lemma 1.

Lemma 5. *If $F_i = C_{k_i}$ for $1 \leq i \leq r$ and $k_1 \geq k_2 \geq \ldots \geq k_r$, then $m((F^r)^*) = d((F^r)^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$.*

Proof. We prove it by induction.
(1) For $r = 2$, by Lemma 1, we have $m((F^2)^*) = d((F^2)^*) = d(C_{k_1}, C_{k_2})$.
(2) Suppose that $m((F^{r-1})^*) = d((F^{r-1})^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}})$ for $r \geq 3$. We will show that $m((F^r)^*) = d((F^r)^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$.

First we will prove that $d((F^r)^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$. Note that $e((F^r)^*) = e(C_{k_r})e((F^{r-1})^*)$ and $v((F^r)^*) = e((F^{r-1})^*)(v(C_{k_r})-2) + v((F^{r-1})^*)$. Hence

$$d((F^r)^*) = \frac{e(C_{k_r})e((F^{r-1})^*)}{e((F^{r-1})^*)(v(C_{k_r})-2)+v((F^{r-1})^*)}.$$

By the hypothesis and Lemma 4, we have $d((F^{r-1})^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}}) = m(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}})$, thus

$$\begin{aligned} d((F^r)^*) &= \frac{e(C_{k_r})}{v(C_{k_r})+1/d((F^{r-1})^*)-2} \\ &= \frac{e(C_{k_r})}{v(C_{k_r})+1/m(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}})-2}, \end{aligned}$$

and by Definition 3, we have

$$d((F^r)^*) = d(C_{k_1}, C_{k_2}, \ldots, C_{k_r}). \tag{4.2}$$

Now, we will prove that $m((F^r)^*) = d((F^r)^*)$. Let ΔV_{out} (ΔE_{out}) denote the vertex (edge) set that is removed from the outer C_{k_r} of $(F^r)^*$, and ΔV_{in} (ΔE_{in}) denote the vertex (edge) set that is removed $V((F^{r-1})^*)$ of $(F^r)^*$. Note that $(F^{r-1})^*$ is the graph obtained by removing all the outer vertices and edges from $(F^r)^*$. We define the graphs as follows,

$$H^r_{out} := (V((F^r)^*) \setminus \Delta V_{out}, E((F^r)^*) \setminus \Delta E_{out}),$$
$$H^r := (V((F^r)^*) \setminus (\Delta V_{out} \cup \Delta V_{in}), E((F^r)^*) \setminus (\Delta E_{out} \cup \Delta E_{in})).$$

The structures of $(F^r)^*$, H^r_{out} and H^r are shown in Figure 4. We approach

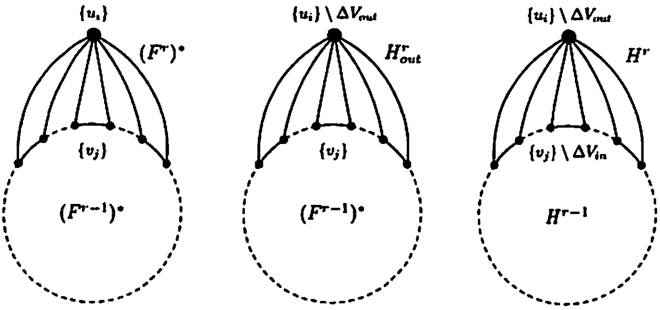

Figure 4: The structures of $(F^r)^*, H^r_{out}$ and H^r.

$(F^r)^*$ and H^r_{out} similarly as F^* and H in the proof of Lemma 1. By arguments similar to the proof of Lemma 1, we can prove that $d((F^r)^*) \geq d(H^r_{out})$ and $d(H^r_{out}) \geq d(H^r)$. Hence $d((F^r)^*) \geq d(H^r)$, which implies that $m((F^r)^*) = d((F^r)^*)$. Together with (4.2), the proof is complete. □

The following lemma is a generalization of Lemma 2, and the techniques used in the proof is similar to that of Lemma 2.

Lemma 6. *For each $F^r \in \mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$, where $k_1 \geq k_2 \geq \ldots \geq k_r$, we have $d(F^r) \geq m((F^r)^*)$.*

Proof. We will prove the lemma by induction.
(1) For $r = 2$, by Lemma 2, we have $d(F^2) \geq m((F^2)^*)$.
(2) Suppose that $d(F^{r-1}) \geq m((F^{r-1})^*)$. We will show that $d(F^r) \geq m((F^r)^*)$.

Let $F^r_{out} \in \mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$ be the graph to which no outer edges of C_{k_r} are merged, and $F^{r-1} \in \mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}})$ be the inner copy of F^r_{out}. We first prove that $d(F^r_{out}) \geq m((F^r)^*)$. Note that

$$d(F^r_{out}) = \frac{e(F^{r-1}) + (k_r - 1)e(F^{r-1})}{v(F^{r-1}) + (k_r - 2)e(F^{r-1})} = \frac{k_r}{k_r + 1/d(F^{r-1}) - 2} \tag{4.3}$$

and

$$d((F^r)^*) = \frac{e((F^{r-1})^*) + (k_r - 1)e((F^{r-1})^*)}{v((F^{r-1})^*) + (k_r - 2)e((F^{r-1})^*)} = \frac{k_r}{k_r + 1/d((F^{r-1})^*) - 2}.$$

By induction, we have $d(F^{r-1}) \geq m((F^{r-1})^*) \geq d((F^{r-1})^*)$, hence $d(F^r_{out}) \geq d((F^r)^*)$. Since $d((F^r)^*) = m((F^r)^*)$ by Lemma 5, it follows that $d(F^r_{out}) \geq m((F^r)^*)$.

Next, we will prove $d(F^r) \geq d(F^r_{out})$. Note that F^r and F^r_{out} have the same inner subgraph F^{r-1}. It is different that F^r maybe has the outer C_{k_r} whose vertices and edges are overlapped. We divide the edges of F^{r-1} into several sets as follows. First, we consider the edge set of the inner C_{k_1} of F^2, then we consider the outer edges of F^i for $2 \leq i \leq r-1$. Denote

$$E^1 = E(F^1) = E(C_{k_1}),$$
$$E^i = E(F^i) \setminus E(F^{i-1}), \quad 2 \leq i \leq r-1,$$

then $\bigcup_{1 \leq i \leq r-1} E^i = E(F^{r-1})$. Analogously to Lemma 2, we will assign a partial order on the edges of E^i (if an edge is common, we only record it at the first time). We define $\Delta^{i,j}_E$, $\Delta^{i,j}_V$ and $\Delta^{i,j}_{V'}$ for each outer $C^{i,j}_{k_r}$ corresponding to E^i for $1 \leq j \leq |E^i|$, similarly to Δ^i_E, Δ^i_V and $\Delta^i_{V'}$ in Lemma 2. Then the inequality

$$\frac{\sum_{j=1}^{|E_i|} |\Delta^{i,j}_E|}{\sum_{j=2}^{|E_i|}(|\Delta^{i,j}_V| + |\Delta^{i,j}_{V'}|)} \leq \frac{k_r}{k_r - 1}, \quad 1 \leq i \leq r-1 \tag{4.4}$$

holds. Let $F^r_{\{E^i\}}$ denote a graph in which the outer C_{k_r} corresponding to E^i do not overlap, then $F^r_{\{E^1, E^2, \ldots, E^{r-1}\}} = F^r_{out}$. We obtain

$$\begin{aligned}
d(F^r) &= \frac{e(F^r_{\{E^1\}}) - \sum_{j=1}^{|E_1|} |\Delta^{1,j}_E|}{v(F^r_{\{E^1\}}) - \sum_{j=1}^{|E_1|}(|\Delta^{1,j}_V| + |\Delta^{1,j}_{V'}|)} \\
&= \frac{e(F^r_{\{E^1,E^2\}}) - \sum_{j=1}^{|E_1|} |\Delta^{1,j}_E| - \sum_{j=1}^{|E_2|} |\Delta^{2,j}_E|}{v(F^r_{\{E^1,E^2\}}) - \sum_{j=1}^{|E_1|}(|\Delta^{1,j}_V| + |\Delta^{1,j}_{V'}|) - \sum_{j=1}^{|E_2|}(|\Delta^{2,j}_V| + |\Delta^{2,j}_{V'}|)} \\
&= \frac{e(F^r_{\{E^1,E^2,\ldots,E^{r-1}\}}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E_i|} |\Delta^{i,j}_E|)}{v(F^r_{\{E^1,E^2,\ldots,E^{r-1}\}}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E^i|}(|\Delta^{i,j}_V| + |\Delta^{i,j}_{V'}|))}
\end{aligned}$$

$$= \frac{e(F^r_{out}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E_i|} |\Delta_E^{i,j}|)}{v(F^r_{out}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E^i|}(|\Delta_V^{i,j}| + |\Delta_{V'}^{i,j}|))}, \tag{4.5}$$

by using a similar method as in Lemma 2. By (4.3), we have $d(F^r_{out}) \geq k_r/(k_r - 1)$. By Proposition 1($a$) with (4.4) and (4.5), it follows

$$d(F^r) = \frac{e(F^r_{out}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E_i|} |\Delta_E^{i,j}|)}{v(F^r_{out}) - \sum_{i=1}^{r-1}(\sum_{j=1}^{|E^i|}(|\Delta_V^{i,j}| + |\Delta_{V'}^{i,j}|))} \geq \frac{e(F^r_{out})}{v(F^r_{out})} = d(F^r_{out}).$$

Hence $d(F^r) \geq d(F^r_{out}) \geq m((F^r)^*)$. □

More generally, the statement of Lemma 6 is essentially that members of the family $\mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$ that contain overlapped edges (vertices) are at least as dense as $(F^r)^*$.

By Lemmas 4-6, we can finish the proof of Theorem 2.

Proof of Theorem 2. By Lemma 6, $m(F^r) \geq d(F^r) \geq m((F^r)^*)$, thus $n^{2-1/m((F^r)^*)} \leq n^{2-1/m(F^r)}$. If $N \ll n^{2-1/m(F^r)}$, then $G(n, N)$ *a.a.s* contains no copy of F^r by Theorem 3. Hence $G(n, N)$ *a.a.s* contains no graph $F^r \in \mathcal{F}(C_{k_1}, C_{k_2}, \ldots, C_{k_r})$ when $N \ll n^{2-1/m((F^r)^*)} \leq n^{2-1/m(F^r)}$. That is, Painter can color the graph within N steps with a proper strategy in the $\mathcal{S}$-avoidance game. Hence, by Lemma 5, we have $N_0(\mathcal{S}, r, n) \geq n^{2-1/m((F^r)^*)} = n^{2-1/d((F^r)^*)} = n^{2-1/d(C_{k_1}, C_{k_2}, \ldots, C_{k_r})}$, where $\mathcal{S} = \{C_{k_1}, C_{k_2}, \ldots, C_{k_r}\}$. Finally, we will show

$$d(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) = \frac{\prod_{i=1}^r k_i}{\sum_{i=1}^r (k_i - 2)\prod_{j=1}^{i-1} k_j + 2}.$$

If $r = 2$, then the theorem follows directly by Theorem 4. If $r \geq 3$, then by induction

$$\begin{aligned} d(C_{k_1}, C_{k_2}, \ldots, C_{k_r}) &= \frac{k_r}{k_r + 1/m(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}}) - 2} \\ &\overset{\text{Lem.4}}{=} \frac{k_r}{k_r + 1/d(C_{k_1}, C_{k_2}, \ldots, C_{k_{r-1}}) - 2} \\ &\overset{\text{Ind.}}{=} \frac{k_r}{k_r + \dfrac{2 + \sum_{i=1}^{r-1}(k_i - 2)\prod_{j=1}^{i-1} k_j}{\prod_{i=1}^{r-1} k_i} - 2} \\ &= \frac{\prod_{i=1}^r k_i}{\sum_{i=1}^r (k_i - 2)\prod_{j=1}^{i-1} k_j + 2}. \end{aligned}$$

This completes the proof. □

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