Identities involving Partitions with bounded Parts

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April 27, 2010

Dedicated to my love, Yasaman

Abstract

In this paper, we briefly survey Euler works on identities in connection with his famous Pentagonal Number Theorem. We state a partial generalization of his theorem for partitions with no part exceeding an identified value k, along with some identities which link total partitions to partitions with distinct parts under the above constraint. We find both recurrence formulae and explicit forms for $\Delta_n(m)$, where $\Delta_n(m)$ is the number of partitions of m into an even number of distinct parts not exceeding n, minus the number of partitions of n into an odd number of distinct parts not exceeding n. In fact, Euler's Pentagonal Number Theorem asserts that for $m \leq n$, $\Delta_n(m)$ equals to ± 1 if m is a Pentagonal Number and is zero otherwise. Finally, we find two identities about the sum of bounded part partitions and their connection to prime factors of the bound integer.

1 Notations

We denote by $P(n)$ the number of partitions of n into summands where the summand order is unimportant; $P_k(n)$ the number of partitions of n into at most k parts, or by conjugation the number of partitions of n into addends not exceeding k. $Q(n)$ denotes the number of partitions of n into distinct parts and $Q_k(n)$ the number of partitions of n into distinct addends not exceeding k.

2 Introduction

From a well known identity due to Euler, we know that

$$\sum_{n=0}^{\infty} P(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} . \tag{1}$$

Euler empirically discovered that he could expand $\prod_{n=1}^{\infty}(1-q^n)$ as a power series:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} . \tag{2}$$

Many years later he managed to provide a proof of this identity. This formula is now known as Euler's Pentagonal Number Theorem. Like Euler did we could write that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} \sum_{n=0}^{\infty} P(n)q^n = 1 .$$

Comparing the coefficients of q^n on both sides of this last identity, we find the following recurrence relation in $P(n)$: $P(0) = 1$, and

$$P(n) - P(n-1) - P(n-2) + P(n-5) + P(n-7) - \cdots = 0, \ n > 0 . \tag{3}$$

Up to now, several recurrence equations has been found for $P(n)$; but no one has ever found a more efficient algorithm for computing $P(n)$. It computes a full table of values of $P(n)$ for $n \leq N$ in time $O(N^{3/2})$. The interested reader is referred to study (Skiena [5], p.77) and (Berndt [2], p.108) for the proofs of the two following recurrence equations in $P(n)$. (In the latter equation $\sigma_1(n)$ is the divisor function.)

$$\sum_{k=\lceil -\frac{\sqrt{24n+1}+1}{6}\rceil}^{\lfloor -\frac{\sqrt{24n+1}-1}{6}\rfloor} (-1)^k P\left(n - \frac{1}{2}k(3k+1)\right) = 0,$$

$$P(n) = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_1(n-k)P(k) .$$

The partition functions $P(n)$ and $Q(n)$ are closely connected together; so they have entangled properties and studying one of them will result in a better understanding of the other one. Among several equations which link $P(n)$ to $Q(n)$, we mention a few. A straight equation could be drawn from equation (1). We could write that

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{\prod_{n=1}^{\infty}(1+q^n)}{\prod_{n=1}^{\infty}(1-q^{2n})}$$

which leads us to the following identity:

$$\sum_{n=0}^{\infty} P(n)q^n = \sum_{k=0}^{\infty} P(k)q^{2k} \sum_{m=0}^{\infty} Q(m)q^m .$$

Comparing the coefficients of q^n at both sides, we acquire a recurrence identity:

$$P(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} Q(n-2k)P(k),$$

relating $P(n)$ to $Q(n)$.

There is another interesting equation which links partitions of an integer n into exactly k parts to partitions of n into exactly k distinct parts (Comtet [4], p.116). Let $P(n,k)$ and $Q(n,k)$ denote the number of ways of partitioning n into exactly k parts and exactly k distinct parts, respectively. To each partition $x_1 + x_2 + \cdots + x_k$ of $n - \binom{k}{2}$, where $x_1 \geq x_2 \geq \cdots \geq x_k$, we could correspond the partition $x_1 + (k-1) + x_2 + (k-2) + \cdots + x_k$ of n and vice versa. So there is a one to one correspondence between the sets of partitions of $n - \binom{k}{2}$ into exactly k parts, and partitions of n into exactly k distinct parts and we find that

$$Q(n,k) = P(n - \binom{k}{2}, k) .$$

3 New equations and Euler's Pentagonal Number Theorem for bounded part partitions

If we consider finite products in equation (1), by a similar argument we could find the following recurrence identity:

$$P_k(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} Q_k(n-2i)P_k(i) .$$

Now let us focus on the finite product case to find another identity which gives us $Q_k(n)$ in terms of $P_k(n)$. Using equation (1) we have

$$\prod_{k=1}^{n}(1+q^k) = \frac{\prod_{k=1}^{n}(1-q^{2k})}{\prod_{k=1}^{n}(1-q^k)}$$
$$= \sum_{m=0}^{\infty} P_n(m)q^m \prod_{k=1}^{n}(1-q^{2k}) .$$

We write that

$$\prod_{k=1}^{n}(1-q^{2k}) = \sum_{m=0}^{\theta} \Delta_n(m)q^{2m}, \text{ where } \theta = \frac{n(n+1)}{2}$$

to rewrite the above equation as

$$\prod_{k=1}^{n}(1+q^k) = \sum_{m=0}^{\infty} P_n(m)q^m \sum_{m=0}^{\theta} \Delta_n(m)q^{2m} .$$

Comparing the coefficients of q^m at both sides, we have two recurrence identities:

$$Q_n(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \Delta_n(i)P_n(m-2i), \text{ where } 1 \leq m \leq \theta, \tag{4}$$

and

$$\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \Delta_n(i)P_n(m-2i) = 0, \text{ where } \theta < m . \tag{5}$$

(we assume that $\Delta_n(i) = 0$ for $i > \theta$.)

It could be verified that $\Delta_k(n) = Q_{k,e}(n) - Q_{k,o}(n)$; where $Q_{k,e}(n)$ and $Q_{k,o}(n)$ are the numbers of partitions of n into an even and odd number of distinct parts not exceeding k, respectively.

Now we state Euler's Pentagonal Number Theorem formally. Let $\Delta(n) = Q_e(n) - Q_o(n)$, i.e. the number of partitions of n into an even number of distinct parts minus the number of partitions of n into an odd number of distinct parts. Then we have

$$\Delta(n) = \begin{cases} (-1)^j, & \text{if } n = \frac{j(3j\pm1)}{2} \\ 0, & \text{otherwise} \end{cases}$$

Remark 3.1 If $k \geq n$, obviously we have $P_k(n) = P(n)$, $\Delta_k(n) = \Delta(n)$, $Q_k(n) = Q(n)$. In (4) let $m = n$ to find that

$$Q(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \Delta(i)P(m-2i) .$$

Example

$$Q(10) = \sum_{i=0}^{5} \Delta(i)P_n(10-2i) = P(10) + (-1)^1 P(8) + (-1)^1 P(6) - 0 + 0 + 1 .$$

Remark 3.2 In (5) let $m = 2\theta$ to find the following identity:

$$\sum_{i=0}^{\theta} \Delta_n(i)P_n(m-2i) = 0 .$$

Now we find a kind of generalization for Euler's Pentagonal Number Theorem but before that, we have to state two lemmas:

Lemma 3.1 *In the expansion of*

$$\prod_{k=1}^{n}(1-q^k) = \sum_{i=0}^{\frac{n^2+n}{2}} \Delta_n(i)q^i$$

we have

$$\Delta_n\left(\frac{n^2+n}{2} - i\right) = (-1)^n \Delta_n(i) .$$

Proof. Let

$$P_n(q) = \prod_{k=1}^{n}(1-q^k) .$$

By factoring $-q^k$ from each term of the right side product we will see that

$$P_n(q) = (-1)^n q^{\frac{n^2+n}{2}} P_n(1/q) .$$

Comparing the coefficients of q^m at both sides gives us the result. □

Lemma 3.2 *Assume that a is a nonnegative integer and let*

$$T_1 = \left\lfloor \frac{\sqrt{24a+1}+1}{6} \right\rfloor, \quad T_2 = \left\lfloor \frac{\sqrt{24a+1}-1}{6} \right\rfloor$$

and

$$T(a) = \frac{1}{2}\left((-1)^{T_1} + (-1)^{T_2}\right).$$

Then we have

$$\sum_{k=0}^{a} \Delta(k) = T(a) \ .$$

Proof. We have

$$\sum_{k=0}^{a} \Delta(k) = \sum_{j=0}^{T_1} (-1)^j + \sum_{j=1}^{T_2} (-1)^j,$$

where T_1, T_2 are the greatest integers such that $T_1(3T_1 - 1)/2 \leq a$ and $T_2(3T_2 + 1)/2 \leq a$, respectively. The first sum equals to $\frac{1}{2}(1 + (-1)^{T_1})$ and the second sum equals to $\frac{1}{2}(-1 + (-1)^{T_2})$. Calculating T_1, T_2 and adding the two terms gives us the result. □

Recall that $\Delta_n(m)$ equals to the difference between the number of partitions of m into even number of distinct parts not exceeding n, and odd number of distinct parts not exceeding n; furthermore we assume that $T(a) = 0$, whenever $a < 0$.

Theorem 3.1 (Partial generalization of Euler's Theorem) *Let m, n be positive integers and $n > 1$. We have the equations*

$$\Delta_n(m) = \begin{cases} \Delta(m) + T(m - n - 1), & \text{for } 0 \leq m < 2n \\ (-1)^n \left\{ \Delta\left(\frac{n^2+n}{2} - m\right) + T\left(\frac{n^2-n-2}{2} - m\right) \right\}, & \text{for } a \leq m \leq b \end{cases}$$

where $a = \frac{(n-1)(n-2)}{2}, b = \frac{n(n+1)}{2}$.

Proof. The theorem assertion is evident for $m < n$; hence assume that $m \geq n$. Regarding the fact

$$\prod_{k=1}^{n} (1 - q^k) = (1 - q^n) \prod_{k=1}^{n-1} (1 - q^k)$$

and comparing the coefficients of q^n at both sides, we have $\Delta_n(m) = \Delta_{n-1}(m) - \Delta_{n-1}(m - n)$. With the presumption $n > (m + 1)/2$ we have:

$$\begin{aligned} \Delta_n(m) &= \Delta_{n-1}(m) - \Delta(m - n) \\ &= \Delta_{n-2}(m) - \Delta(m - n + 1) - \Delta(m - n) \\ &\ \vdots \\ &= \Delta_{\lfloor \frac{m+1}{2} \rfloor}(m) - \sum_{k=\lfloor \frac{m+3}{2} \rfloor}^{n} \Delta(m - k) \ . \end{aligned}$$

Now we let $m = n$ to find $\Delta_{\lfloor \frac{m+1}{2} \rfloor}(m)$:

$$\Delta_{\lfloor \frac{m+1}{2} \rfloor}(m) = \Delta(m) + \sum_{k=\lfloor \frac{m+3}{2} \rfloor}^{m} \Delta(m-k) \ .$$

Obviously, for $m = n$ we have $\Delta_n(m) = \Delta(m)$. If $m > n$ it follows that

$$\begin{aligned} \Delta_n(m) &= \Delta(m) + \sum_{k=n+1}^{m} \Delta(m-k) \\ &= \Delta(m) + \sum_{k=0}^{m-n-1} \Delta(k), \text{ where } n \geq \frac{m+1}{2} \ . \end{aligned}$$

Finally when we apply the above lemmas we find the result. □

Corollary 3.1 *If we apply the constraints*

$$0 \leq m \leq 2n-1 \quad \text{or} \quad \frac{(n-1)(n-2)}{2} \leq m \leq \frac{n(n+1)}{2},$$

then $\Delta_n(m)$ takes the values $0, \pm 1, \pm 2$.

The recurrence equation we applied in the above theorem, is in terms of both m, n. It may be more suitable to have an equation in terms of one variable. The following theorem gives us a recurrence equation in one variable.

Theorem 3.2 *Let $\sigma_n(m)$ denote sum of the divisors of an integer m that are less than or equal to n. Then we have the following recurrence equation:*

$$\Delta_n(m) = -\frac{1}{m} \sum_{k=0}^{m-1} \Delta_n(k) \sigma_n(m-k),$$

where $1 \leq m \leq n(n+1)/2$.

Proof. We start from the following identity:

$$\prod_{k=1}^{n} (1 - q^k) = \sum_{i=0}^{\theta} \Delta_n(m) q^m, \text{ where } \theta = \frac{n(n+1)}{2}.$$

We consider q as a real variable with the condition $|q| < 1$. First take the logarithm and then differentiate on both sides to find that

$$\sum_{1 \leq k \leq n} \frac{-kq^{k-1}}{1-q^k} = \frac{\sum_{m=1}^{\theta} m \Delta_n(m) q^{m-1}}{\sum_{m=0}^{\theta} \Delta_n(m) q^m} \ . \tag{6}$$

We could rewrite the left hand side in the form of a power series as follows:

$$\sum_{1\le k\le n} \frac{-kq^{k-1}}{1-q^k} = \sum_{1\le k\le n}\sum_{j\ge 0} -kq^{k-1}\cdot q^{kj}$$
$$= \sum_{1\le k\le n}\sum_{j\ge 1} -kq^{kj-1}\,.$$

For an integer $m \geq 0$ and every $1 \leq k \leq n$, the last sum produces the term $-kq^m$ whenever there is a positive integer j such that $kj - 1 = m$. This means that the coefficient of q^m in the last sum equals $\sigma_n(m+1)$. Hence we could write that

$$\sum_{1\le k\le n} \frac{-kq^{k-1}}{1-q^k} = -\sum_{m\ge 0} \sigma_n(m+1)q^m\,. \tag{7}$$

Combine equations (6) and (7) to find that

$$\sum_{m=1}^{\theta} m\Delta_n(m)q^{m-1} = -\sum_{m\ge 0}\sigma_n(m+1)q^m \sum_{m=0}^{\theta}\Delta_n(m)q^m\,.$$

Comparing the coefficients of q^m at both sides gives us the result. □

The generating function we found in equation (7), can be applied to find an explicit formula for $\Delta_n(m)$. The next theorem states this formula.

Theorem 3.3 *Let m, n be positive integers and $\sigma_n(r)$ denote sum of the divisors of an integer r that are less than or equal to n. Then we have the following identity:*

$$\sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{\substack{r_1,r_2,\cdots,r_k\ge 1\\ r_1+r_2+\cdots+r_k=m}} \prod_{i=1}^{k} \frac{\sigma_n(r_i)}{r_i} = \begin{cases} \Delta_n(m), & 1\le m\le \frac{n^2+n}{2} \\ 0, & \textit{otherwise} \end{cases}\,.$$

Proof. Let $F(q) = \sum_{m=0}^{\theta} \Delta_n(m)q^m$, where θ is just defined as in the above theorem. From equations (6) and (7) it follows that

$$\frac{F'(q)}{F(q)} = -\sum_{m\ge 0}\sigma_n(m+1)q^m\,.$$

Note that the power series on the right hand side is uniformly convergent for $|q| \leq 1 - \epsilon$. Remarking that $F(0) = 0$, integrate on both sides to find

$F(q)$:

$$F(q) = \exp\left(-\sum_{m\geq 0} \frac{\sigma_n(m+1)}{m+1} q^{m+1} \right) = \sum_{k\geq 0} \frac{\left(-\sum_{m\geq 1} \frac{\sigma_n(m)}{m} q^m\right)^k}{k!} . \quad (8)$$

It is easily seen that the power series on the right hand side has the conditions of term rearrangement. Thus for an integer $m > 0$ and every $0 < k \leq m$, the k-th term of the last sum produces the term

$$(-1)^k \sum_{\substack{r_1, r_2, \cdots, r_k \geq 1 \\ r_1+r_2+\cdots+r_k=m}} \prod_{i=1}^{k} \frac{\sigma_n(r_i)}{r_i} \cdot q^m .$$

So the total coefficient of q^m in the last sum equals to

$$\sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{\substack{r_1, r_2, \cdots, r_k \geq 1 \\ r_1+r_2+\cdots+r_k=m}} \prod_{i=1}^{k} \frac{\sigma_n(r_i)}{r_i} .$$

Comparing the coefficients of q^m at both sides of equation (8) concludes the proof. □

4 Bounded partitions and prime factors

In the remainder of the paper we consider

$$(-q;q)_n = \prod_{k=1}^{n} (1+q^k) = \sum_{m=0}^{\frac{n(n+1)}{2}} Q_n(m) q^m$$

and consider p as an odd prime factor of n. Let ω be a primitive p-th root of unity, i.e. $\omega = \cos(2\pi/p) + i \sin(2\pi/p)$. Let $Q(\omega)$ be the field extension of ω over $\mathbb{Q}$. First we calculate the amount of $(-\omega;\omega)_n$ in $Q(\omega)$. We have

$$(-\omega;\omega)_n = \prod_{k=1}^{n} (1+\omega^k) = \left(\prod_{k=0}^{p-1} (1+\omega^k) \right)^{n/p} .$$

Since ω is a primitive root of unity we have

$$P(z) = z^p - 1 = \prod_{k=0}^{p-1} (z - \omega^k) .$$

Let $z = -1$ in this equation to see that

$$\prod_{k=0}^{p-1}(1+\omega^k) = 2 .$$

Therefore we have $(-\omega;\omega)_n = 2^{n/p}$.

The polynomial

$$\frac{P(z)}{z-1} = 1 + z + z^2 + \cdots + z^{p-1}$$

is a minimal polynomial for ω over $\mathbb{Q}$. So we conclude that the set

$$A = \{1, \omega, \omega^2, \omega^3, \cdots, \omega^{p-2}\}$$

constitutes a $\mathbb{Q}$-basis for $Q(\omega)$ as a $\mathbb{Q}$-vector space. Thus

$$(-\omega;\omega)_n = 2^{n/p} = 2^{n/p}\cdot 1 + 0\cdot\omega + 0\cdot\omega^2 + \cdots + 0\cdot\omega^{p-2}$$

is the basis representation of $(-\omega;\omega)_n$ in $Q(\omega)$. On the other hand let us assume that

$$(-\omega;\omega)_n = \prod_{k=1}^{n}(1+\omega^k) = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-1}\omega^{p-1} .$$

In fact we have expanded the product and reduced the powers modulo p. It is clear that

$$a_i = \sum_{k=0}^{\alpha_i} Q_n(kp+i), \text{ where } \alpha_i = \lfloor n(n+1)/2p - i/p \rfloor . \tag{9}$$

The expansion above could be written in the form

$$(a_0 - a_{p-1})\cdot 1 + (a_1 - a_{p-1})\cdot\omega + (a_2 - a_{p-1})\cdot\omega^2 + \cdots + (a_{p-2} - a_{p-1})\cdot\omega^{p-2}$$

as a representation over basis A. Since the representation over a basis is unique, we have the following system of equations:

$$a_0 - a_{p-1} = 2^{n/p}, \quad a_i - a_{p-1} = 0,\ 1 \le i \le p-2 \quad \text{and} \quad \sum_{i=0}^{p-1} a_i = 2^n$$

which leads to the solution

$$a_0 = 2^{n/p} + \frac{2^n - 2^{n/p}}{p}, \quad a_i = \frac{2^n - 2^{n/p}}{p} \text{ for } 1 \le i \le p-1 .$$

So we have the following result:

Theorem 4.1 *Let n be a positive integer and p an odd prime factor of it. Then we have the following identities:*

$$\sum_{k=0}^{\alpha_0} Q_n(kp) = 2^{n/p} + \frac{2^n - 2^{n/p}}{p}, \quad \sum_{k=0}^{\alpha_i} Q_n(kp+i) = \frac{2^n - 2^{n/p}}{p} \text{ for } 1 \le i < p,$$

where $\alpha_i = \lfloor n(n+1)/2p - i/p \rfloor$.

4.1 Application

Case study 1. Let $X = \{1, 2, 3, \cdots, n\}$ and consider p as an odd prime factor of n. We are interested in the number of subsets of X for which sum of their members is congruent to i modulo p. In fact $Q_n(i), Q_n(p+i), Q_n(p+2i), \cdots$ are equal to the number of subsets of X for which the sum of their members are $i, p+i, p+2i, \cdots$, respectively. Looking at equation (9) makes it clear that each a_i in the expansion of

$$(-\omega;\omega)_n = \prod_{k=1}^{n}(1+\omega^k) = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-1}\omega^{p-1}$$

describes the number of subsets of X for which the sum of their members is congruent to i modulo p. So if we denote sum of the members of $S \subseteq X$ by $\sigma(S)$, then we have

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod p\} = \begin{cases} 2^{n/p} + \frac{2^n - 2^{n/p}}{p}, & i = 0 \\ \frac{2^n - 2^{n/p}}{p}, & i \neq 0 \end{cases}.$$

Case study 2. In the case $X = \{1, 2, 3, \cdots, n\}$, $n = pt + 1$ there would be a similar argument for the coefficients a_i of $(-\omega;\omega)_n$. But in this case we have

$$(-\omega;\omega)_n = \left(\prod_{k=0}^{p-1}(1+\omega^k)\right)^t (1+\omega) = 2^{(n-1)/p} + 2^{(n-1)/p}\omega .$$

So we have the following system of equations:

$$a_0 - a_{p-1} = 2^{(n-1)/p}, \qquad a_1 - a_{p-1} = 2^{(n-1)/p},$$

$$a_i - a_{p-1} = 0,\ 2 \le i \le p-2 \qquad \text{and} \quad \sum_{i=0}^{p-1} a_i = 2^n$$

with solution

$$a_0 = a_1 = \frac{2^n + (p-2)2^{(n-1)/p}}{p}, \quad a_i = \frac{2^n + 2^{(n+p-1)/p}}{p} \text{ for } 2 \le i \le p-1 .$$

And we conclude that

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod{p}\} = \begin{cases} \frac{2^n+(p-2)2^{(n-1)/p}}{p}, & i = 0, 1 \\ \frac{2^n+2^{(n+p-1)/p}}{p}, & i \neq 0, 1 \end{cases} .$$

Case study 3. In the case $X = \{1, 2, 3, \cdots, n\}$, $n = pt - 1$ we have

$$(-\omega; \omega)_n = \prod_{k=1}^{n}(1+\omega^k) = \left(\prod_{k=0}^{p-1}(1+\omega^k)\right)^{t-1} \prod_{k=1}^{p-1}(1+\omega^k) = 2^{(n-p+1)/p}$$

which by a similar argument finally gives us the following answer:

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod{p}\} = \begin{cases} 2^{(n-p+1)/p} + \frac{2^n-2^{(n-p+1)/p}}{p}, & i = 0 \\ \frac{2^n-2^{(n-p+1)/p}}{p}, & i \neq 0 \end{cases} .$$

Remark 4.1 The problem could be solved for the cases $n = pt \pm (p-1)/2$ in a similar way.

Now let us introduce the notation $P_{n,k}(m)$ which denotes the number of partitions of m into at most n parts with differences at most k, i.e. partitions like $b_1+b_2+\cdots+b_s$ of m where $b_i \leq b_{i+1}$, $s \leq n$, and $b_{i+1}-b_i \leq k$. By conjugation in the Ferrers diagram, $P_{n,k}(m)$ also stands for the number of partitions of m into parts for which each part has repeated at most k times and no part is greater than n.

The generating function for such kind of partitions is

$$T(q) = \prod_{j=1}^{n}\left(\sum_{i=0}^{k} q^{ij}\right) .$$

We want to generalize Theorem (4.1), so let us assume that $l = gcd(n, k) > 1$, and p be a prime factor of l. Let $\omega = \cos(2\pi/p) + i\sin(2\pi/p)$ be the root of unity. Then we have

$$T(\omega) = \prod_{j=1}^{n}\left(\sum_{i=0}^{k} \omega^{ij}\right) = \prod_{j=0}^{p-1}\left(\sum_{i=0}^{k} \omega^{ij}\right)^{n/p} .$$

Terms of the product inside parenthesis are all one, except the first term which is equal to $(k+1)^{n/p}$. Therefore we have $T(\omega) = (k+1)^{n/p}$. Just like Theorem (4.1) view it as a representation of $T(\omega)$ over the basis

$$A = \left\{1, \omega, \omega^2, \omega^3, \cdots, \omega^{p-2}\right\} .$$

On the other hand calculate the product which $T(\omega)$ consists of, and reduce the powers modulo p, i.e. write $T(\omega) = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-1}\omega^{p-1}$. A similar argument as in Theorem (4.1) leads us to the following equation:

$$(a_0-a_{p-1})+(a_1-a_{p-1})\omega+(a_2-a_{p-1})\omega^2+\cdots+(a_{p-2}-a_{p-1})\omega^{p-2} = (k+1)^{n/p}.$$

Therefore we have the system of equations

$$a_0-a_{p-1} = (k+1)^{\frac{n}{p}}, \quad a_i-a_{p-1} = 0,\ 1 \le i \le p-2 \quad \text{and} \quad \sum_{i=0}^{p-1} a_i = (k+1)^n$$

with solution

$$a_0 = (k+1)^{n/p} + \frac{(k+1)^n - (k+1)^{n/p}}{p}, \quad \text{and}$$

$$a_i = \frac{(k+1)^n - (k+1)^{n/p}}{p} \quad \text{for } 1 \le i \le p-1 .$$

So we conclude that the following result holds.

Theorem 4.2 *Let n, k be positive integers and let $l = gcd(n, k) > 1$ and p be a prime factor of l. Then we have the following identities:*

$$\sum_{j=0}^{\alpha_0} P_{n,k}(jp) = (k+1)^{n/p} + \frac{(k+1)^n - (k+1)^{n/p}}{p}, \quad \textit{and}$$

$$\sum_{j=0}^{\alpha_i} P_{n,k}(jp+i) = \frac{(k+1)^n - (k+1)^{n/p}}{p} \quad \textit{for } 1 \le i < p,$$

where $\alpha_i = \lfloor kn(n+1)/2p - i/p \rfloor$.

5 Concluding Remarks

If we use the notations $P_k^*(n)$ and $Q_k^*(n)$ for the partitions of n into exactly k parts and into exactly k distinct parts, the relation between $P_k^*(n)$ and $Q_k^*(n)$ to $P_k(n)$ and $Q_k(n)$ is $P_k^*(n) = P_k(n-k)$, $Q_k^*(n) = Q_{k-1}(n-k)$. So we may replace $P_k(n)$ and $Q_k(n)$ with $P_k^*(n)$ and $Q_k^*(n)$ to make new identities.

Unfortunately the idea applied to derive the generalization we found for Euler's Pentagonal Number Theorem does not work without further constraints. However, it would be possible to derive lower and upper bounds from the recursive equations we had.

In section (4), if we had considered a factor m of n that was not necessarily prime, then we would have had to deal with the Cyclotomic Polynomial

$$\Phi_n(X) = \prod_{\zeta \in U'_n} (X - \zeta),$$

where U'_n is the subset of primitive n-th roots of unity in the set of complex numbers. In this case there are difficulties with a basis representation; also we have more variables than equations. However, it still is possible to derive some new identities.

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