Super edge-magic total labeling of subdivided stars

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Abstract. An *edge-magic total labeling* of a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \cdots, |V(G) \cup E(G)|\}$ with the property that, there is an integer constant c such that $\lambda(x) + \lambda(x, y) + \lambda(y) = c$ for any $(x, y) \in E(G)$. If $\lambda(V(G)) = \{1, 2, \cdots, |V(G|\}$ then edge-magic total labeling is called *super edge-magic total labeling.* In this paper, we formulate super edge-magic total labeling on subdivision of stars $K_{1,p}$, for $p \geq 5$.

Keywords : Super edge-magic total labeling, subdivision of stars.

1 Introduction

All graphs in this paper are finite, simple, planar and undirected. The graph G has the vertex-set $V(G)$ and edge-set $E(G)$. A general reference for graph-theoretic ideas can be seen in [12].

A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, we focus on one type of labeling called *edge-magic total labeling (EMTL).* An *edge-magic total labeling* of a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \cdots, |V(G) \cup E(G)|\}$ with the property that, there is an integer constant c such that $\lambda(x) + \lambda(x, y) + \lambda(y) = c$ for any $(x, y) \in E(G)$. An edge-magic total labeling λ of graph G is called *super edge-magic total labeling (SEMTL)* if $\lambda(V(G)) = \{1, 2, \cdots, |V(G|\}$.

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The subject of edge-magic total labeling of graphs has its origin in the work of Kotzig and Rosa [8, 9], on what they called magic valuations of graphs. The notion of super edge-magic total labeling was introduced by Enomoto et al. in [3] and they proposed following conjecture:

Conjecture 1 *[3] Every tree admits a super edge-magic total labeling.*

In the effort of attacking this conjecture, many authors have considered super edge-magic total labeling for some particular classes of trees for example [1, 6, 11]. Lee and Shah [10] have verified this conjecture by a computer search for trees on at most 17 vertices. Earlier, in [8] Kotzig and Rosa proved that every caterpillar is super edge-magic total. However, conjecture 1 still remains open. A star is a particular type of tree. Super edge-magic total labeling for subdivision of star $K_{1,3}$ was studied by Baskoro et al. [2]. In [7] Javaid et al. furnished super edge-magic total labeling on subdivision of $K_{1,4}$ and w-tree. However, super edge-magic total labeling for subdivision of star $K_{1,p}$, for $p \geq 5$ is still open. In this paper we find super-edge magic total labelings on subdivision of star $K_{1,p}$, for $p \geq 5$.

In the following section we present super edge-magic total labelings on subdivision of $K_{1,p}$.

2 Main Results

For $n_i \geq 1$ and $p \geq 5$, let $G \cong T(n_1, n_2, ..., n_p)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the $i-$th edge of the star $K_{1,p}$, where $1 \leq i \leq p$. Thus, the graph $T\underbrace{(1, 1, ..., 1)}_{p-time}$ is a star $K_{1,p}$.

Before giving our main results, let us consider the following lemma found in [4] that gives a necessary and sufficient condition for a graph to be super edge-magic total.

Lemma 1. *A graph G with v vertices and e edges is super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \rightarrow \{1, 2, \cdots, v\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of e consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G with magic constant $a = v + e + s$, where $s = min(S)$ and*

$$\begin{aligned} S &= \{\lambda(x) + \lambda(y) | xy \in E(G)\} \\ &= \{a - (v + 1), a - (v + 2), \cdots, a - (v + e)\}. \end{aligned}$$

Theorem 1. *For any odd $n \geq 3$, $G \cong T(n,n,n-1,n,2n-1)$ admits super edge-magic total labeling with magic constant $a = 15n$.*

Proof.
Let us denote the vertices and edges of G, as follows:

$$V(G) = \{c\} \cup \{x_{ij} \mid 1 \leq i \leq 5 \; ; \; 1 \leq j \leq n_i\},$$

$$E(G) = \{c\, x_i \mid 1 \leq i \leq 5\} \cup \{x_{ij}x_{ij+1} \mid 1 \leq i \leq 5 \; ; \; 1 \leq j \leq n_i - 1\}.$$

If v=$|V(G)|$ and e=$|E(G)|$ then

$$v = 6n - 1,$$

$$e = 6n - 2.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, ..., v + e\}$ as follows:

$$\lambda(c) = 4n + 1.$$

For odd j,

$$\lambda(u) = \begin{cases} \frac{j+1}{2}, & \text{for } u = x_{1j} \\ n + 1 - \frac{j-1}{2}, & \text{for } u = x_{2j} \\ \frac{j+1}{2} + n + 1, & \text{for } u = x_{3j} \\ \frac{n-j}{2} + \frac{3(n+1)}{2}, & \text{for } u = x_{4j} \\ \frac{2n-j-1}{2} + 2(n+1), & \text{for } u = x_{5j} \end{cases}$$

For even j,

$$\lambda(u) = \begin{cases} \frac{j}{2} + 3n + 1, & \text{for } u = x_{1j} \\ \frac{n-j-1}{2} + 3n + 1 + \frac{n+1}{2}, & \text{for } u = x_{2j} \\ \frac{j}{2} + 4n + 1, & \text{for } u = x_{3j} \\ \frac{n-j-1}{2} + 4n + 1 + \frac{n+1}{2}, & \text{for } u = x_{4j} \\ \frac{2n-j-2}{2} + 5n + 1, & \text{for } u = x_{5j} \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = 3n+3, 3n+4, \cdots, 8n+2$. Therefore, by Lemma 1 λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = 15n$. □

Theorem 2. *For any odd $n \geq 3$, $G \cong T(n, n, n-1, n, 2n-1, 4n-3)$ admits super edge-magic total labeling with magic constant $a = 25n - 7$.*

Proof.
Let us denote the vertices and edges of G, as follows:

$$V(G) = \{c\} \cup \{x_{ij} \mid 1 \leq i \leq 6\ ;\ 1 \leq j \leq n_i\},$$

$$E(G) = \{c\, x_i \mid 1 \leq i \leq 6\} \cup \{x_{ij}x_{ij+1} \mid 1 \leq i \leq 6\ ;\ 1 \leq j \leq n_i - 1\}.$$

If v=$|V(G)|$ and e=$|E(G)|$ then

$$v = 10n - 4,$$

$$e = 10n - 5.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, ..., v + e\}$ as follows:

$$\lambda(c) = 6n.$$

For odd j,

$$\lambda(u) = \begin{cases} \frac{j+1}{2}, & \text{for}\ \ u = x_{1j} \\ n + 1 - \frac{j-1}{2}, & \text{for}\ \ u = x_{2j} \\ \frac{j+1}{2} + n + 1, & \text{for}\ \ u = x_{3j} \\ \frac{n-j}{2} + \frac{3(n+1)}{2}, & \text{for}\ \ u = x_{4j} \\ \frac{2n-j-1}{2} + 2(n+1), & \text{for}\ \ u = x_{5j} \\ \frac{4n-j-3}{2} + 3n + 2, & \text{for}\ \ u = x_{6j} \end{cases}$$

For even j,

$$\lambda(u) = \begin{cases} \frac{j}{2} + 5n, & \text{for } u = x_{1j} \\ \frac{n-j-1}{2} + 5n + \frac{n+1}{2}, & \text{for } u = x_{2j} \\ \frac{j}{2} + 6n, & \text{for } u = x_{3j} \\ \frac{n-j-1}{2} + 6n + \frac{n+1}{2}, & \text{for } u = x_{4j} \\ \frac{2n-j-2}{2} + 7n, & \text{for } u = x_{5j} \\ \frac{4n-j-4}{2} + 8n - 1, & \text{for } u = x_{6j} \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = 5n+2, 5n+3, \cdots, 11n+1$. Therefore, by Lemma 1 λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = 25n - 7$. □

Theorem 3. *For any odd $n \geq 3$ and $p \geq 5$, $G \cong T(n, n, n-1, n, n_5, ..., n_p)$ admits super edge-magic total labeling, where $n_p = n + \frac{(n-1)(p-3)(p-4)}{2}$.*

Proof.
Let us denote the vertices and edges of G, as follows:

$$V(G) = \{c\} \cup \{x_{ij} \mid 1 \leq i \leq p \ ; \ 1 \leq j \leq n_i\},$$

$$E(G) = \{c\, x_i \mid 1 \leq i \leq p\} \cup \{x_{ij}x_{ij+1} \mid 1 \leq i \leq p \ ; \ 1 \leq j \leq n_i - 1\}.$$

If v=$|V(G)|$ and e=$|E(G)|$ then

$$v = pn + (n-1)\sum_{i=1}^{p-4} i(p-i-3),$$

$$e = pn + (n-1)\sum_{i=1}^{p-4} i(p-i-3) - 1.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, ..., v+e\}$ as follows:

$$\lambda(c) = 3n + \frac{1}{2}[(p-4)(n+1) + (n-1)\sum_{i=1}^{p-4} i(p-i-3)] + 1.$$

For odd j,

$$\lambda(u)=\begin{cases}\frac{j+1}{2}, & \text{for } u=x_{1j}\\ n+1-\frac{j-1}{2}, & \text{for } u=x_{2j}\\ \frac{j+1}{2}+n+1, & \text{for } u=x_{3j}\\ \frac{n-j}{2}+\frac{3(n+1)}{2}, & \text{for } u=x_{4j}\end{cases}$$

For odd j and $5\le k\le p$,

$$\lambda(x_{kj})=\frac{n_k-j}{2}+2(n+1)+\frac{1}{2}[(k-5)(n+1)+(n-1)\sum_{i=1}^{k-5}i(k-i-4)].$$

Let $\alpha=(2n+1)+\frac{1}{2}[(p-4)(n+1)+(n-1)\sum_{i=1}^{p-4}i(p-i-3)]$ and $n_k=n+\frac{(n-1)(k-3)(k-4)}{2}$.

For even j,

$$\lambda(u)=\begin{cases}\frac{j}{2}+\alpha, & \text{for } u=x_{1j}\\ \frac{n-j-1}{2}+\alpha+\frac{n+1}{2}, & \text{for } u=x_{2j}\\ \frac{j}{2}+\alpha+n, & \text{for } u=x_{3j}\\ \frac{n-j-1}{2}+\alpha+n+\frac{n+1}{2}, & \text{for } u=x_{4j}.\end{cases}$$

For even j and $5\le k\le p$,

$$\lambda(x_{kj})=\frac{n_k-1-j}{2}+2n+\alpha+\frac{1}{2}[(k-5)(n-1)+(n-1)\sum_{i=1}^{k-5}i(k-i-4)].$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+pn+1$. Therefore, by Lemma

1 λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = \alpha + 2pn + 2(n-1)\sum_{i=1}^{p-4} i(p-i-3) + 1$. □

3 Conclusion

In this paper, we have shown that a subclass of trees, namely subdivided stars $G \cong T(n, n, n-1, n, n_5, ..., n_p)$, admits super edge-magic total labeling only for odd n, $n_p = n + \frac{(n-1)(p-3)(p-4)}{2}$ and $p \geq 5$. For the remaining cases, problem is still open. Therefore, for further research we propose following open problems.

Open Problem 1 *$G \cong T(n, n, ..., n)$ admits super edge-magic total labelings, for any positive integer $n \geq 3$.*

Open Problem 2 *For $n_i \geq 3$, $G \cong T(n_1, n_2, ..., n_p)$ is super edge-magic total labelings with $p \geq 5$ and $1 \leq i \leq p$.*

References

1. A. Ahmad, K. Ali, and E. T. Baskoro, On super edge-magic total labeling of a forest of banana trees, *Utilitas Math.*, to appear.
2. E. T. Baskoro, A. A. G. Ngurah, and R. Simanjuntak, On (super) edge-magic total labeling of subdivision of $K_{1,3}$, *SUT J. Math.*, 43 (2007), 127-136.
3. H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringle, Super edge-magic graphs, *SUT J. Math.*, 34 (1980), 105-109.
4. R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labeling among other classes of labeling, *Discrete Math.*, 231 (2001), 153-168.
5. J. A. Gallian, A dynamic survey of graph labeling, *J. Combin.*, January 2009.
6. M. Hussain, E. T. Baskoro, and Slamin, On super edge-magic total labeling of banana trees, *Utilitas Math.*, 79 (2009), 243-251.
7. M. Javaid, M. Hussain, K. Ali, and H. Shaker, Super edge-magic total labeling on subdivision of trees, *Utilitas Math.*, to appear.
8. A. Kotzig, and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, 13 (1970), 451-461.
9. A. Kotzig, and A. Rosa, Magic valuation of complete graphs, *Centre de Recherches Mathematiques, Universite de Montreal*, (1972), CRM-175.
10. S. M. Lee, and Q. X. Shah, All trees with at most 17 vertices are super edge-magic, *16th MCCCC Conference, Carbondale*, University Southern Illinois, November 2002.
11. V. Swaminathan, and P. Jeyanthi, Super edge-magic strength of fire crackers, banana trees and unicyclic graphs, *Discrete Math.*, 306 (2006), 1624-1636.
12. D. B. West, An Introduction to Graph Theory (*Prentice-Hall, 1996*).