Structural Form of a Minimal Critical Set for a Latin Square Representing the Elementary Abelian 2-group of Order 8

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Abstract

In this paper, we consider the problem of determining the structure of a minimal critical set in a latin square L representing the elementary abelian 2-group of order 8.

AMS Subject Classification (1991) : 05B15

Keywords and Phrases : Latin square representing the elementary abelian 2-group, critical sets, minimal critical sets.

1 Introduction

In recent years many researchers have dealt with the study of existence and construction of critical sets which consist of the minimum amount of information needed to recreate combinatorial structures uniquely (cf. Nelder [11], Smetanuik [14], Curran and van Rees [3], Cooper, Donovan

*Visiting faculty, Missouri University of Science and Technology, Rolla, MO 65409-0020, U.S.A and Seberry [1], Cooper, McDonough and Mavron [2], Donovan, Cooper, Nott and Seberry [5], Donovan and Cooper [6], Fu, Fu and Rodger [8], Donovan and Howse [7], Donovan [4], and SahaRay, Adhikari and Seberry [12, 13]).

Despite more than 30 years of research, many gaps exist in the public knowledge about critical sets in latin squares. There is not a lot known about the critical sets in latin squares in general. Donovan [4] has provided a reference list for the existence of critical sets in latin squares of order less than or equal to 10 and exhibited a critical set from a latin square of each order. Donovan, Cooper, Nott and Seberry [5] have dealt with critical sets in some specific classes of latin squares. Among various types, they considered the latin square L_v representing an elementary abelian 2-group C_2^v of order $n = 2^v$ and obtained a lower bound on the size of the critical set. In particular, they claimed that for a latin square representing the elementary abelian 2-group of order 8, the lower bound on the size of the critical set is 24. Khodkar [10] improved this lower bound to 25 and presented a critical set of this size. The precise pattern of the critical set is not discussed. In this paper, we further investigate this problem and explicitly determine the unique structural form of a minimal critical set, specifying the number of triples from each row of the latin square in the set.

2 Preliminary Definitions and Notations

A latin square L of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs precisely once in each row and in each column. In what follows N is assumed to be $\{1, 2, \ldots, n\}$. Following Donovan, Cooper, Nott and Seberry [5], we represent a latin square L of order n by a set of ordered triples $\{(i, j; k) |$ element k occurs in the position (i, j), $i, j, k \in N\}$.

A *partial latin square* P of order n is an $n \times n$ array with entries chosen from N such that each element of N occurs at most once in each row and in each column of P. Then $|P|$ is said to be the size of the partial latin square and the set of positions $S_P = \{(i, j) | (i, j; k) \in P, \exists\ k \in N\}$ is said to determine the shape of P. Let P and P' be two partial latin squares of the same order, with the same size and shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position in P. A *latin trade* I is a partial latin square for which there exists another partial latin square I' of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. Thus in L, if we replace I by I', the properties of a latin square still hold.

If L contains a $s \times s$ subarray S and if S is a latin square of order s, then we say that S is a *latin subsquare* of L. A *cycle* is a latin subsquare of order 2, also known as an intercalate.

A *uniquely completable set* (UC set) $\mathcal{U}$ of triples is such that it yields only one latin square L of order n which has element k in the position (i, j) for each $(i, j; k) \in \mathcal{U}$.

Definition 2.1 *A set C is said to be a critical set if*

1. *C is a UC set, and*

2. *no proper subset of C satisfies 1.*

A *minimal critical set* is a critical set of the smallest possible size.

Let r_x (c_x) denote the row x (column x) of L, $x = 1, 2, \ldots, n$. Following the notation introduced in Donovan, Cooper, Nott and Seberry [5], let $C - r_x$ ($C - c_x$) denote a critical set with triples chosen from $n-1$ rows (columns), distinct from the row x (column x) of L.

We display below three latin squares L_i, $i = 1, 2, 3$ of order 2^i respectively, representing the elementary abelian 2 group. Our discussion centers around these special structured latin squares.

L_1 :

1	2
2	1

; L_2 :

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

; L_3 :

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

In the following, we assume $n = 8$ and hence $N = \{1, 2, \ldots, 8\}$. We denote by $I_{ii'}^{kk'}$, the cycle formed by the elements k and k' from the ith and i'th rows of L_3, where $k, k', i, i' \in N$ and $k \neq k'$ and $i \neq i'$. For example, for $i = 1$, $i' = 3$, $k = 2$ and $k' = 4$,

$$I_{13}^{24} = \{(1, 2; 2), (1, 4; 4), (3, 4; 2), (3, 2; 4)\}.$$

The subsets of N, denoted by A_j and $\bar{A}_j = N \setminus A_j$, $j = 1, 2, \ldots, 7$, defined below play a key role in our subsequent discussion, where

$$\left.\begin{array}{lll} A_1 = \{1,2,3,4\}, & A_2 = \{1,2,5,6\}, & A_3 = \{1,2,7,8\}, \\ A_4 = \{1,3,5,7\}, & A_5 = \{1,3,6,8\}, & A_6 = \{1,4,5,8\} \text{ and} \\ A_7 = \{1,4,6,7\}. & & \end{array}\right\} \quad (2.1)$$

Corresponding to the subsets A_i and A_j of N, we define $S_{A_iA_j}$ to be a latin subsquare of L_3 formed by the rows r_x and columns c_y, where $x \in A_i$ and $y \in A_j$.

We now wish to draw the reader's attention to the following remarks which are crucial in the construction of a critical set in L_3.

Remark 2.2 *There exist precisely four cycles, corresponding to any two rows of L_3. For example, corresponding to the 1st and 3rd rows of L_3, there exist four cycles I_{13}^{13}, I_{13}^{24}, I_{13}^{57}, I_{13}^{68}.*

Remark 2.3 *Corresponding to each A_i as given in (2.1), $i = 1, 2, \ldots, 7$, there exist four latin subsquares $S_{A_iA_i}$, $S_{A_i\bar{A}_i}$, $S_{\bar{A}_iA_i}$ and $S_{\bar{A}_i\bar{A}_i}$, each isomorphic to L_2, embedded in L_3.*

To elucidate the structure of a minimal critical set in L_3, we use below some definitions and theorems from Donovan, Cooper, Nott and Seberry [5] and Khodkar [10].

Definition 2.4 *[5] Two latin squares L and M of order n are said to be isotopic or equivalent if there exists an ordered triple (α, β, γ) of permutations such that α, β, γ map the rows, columns and elements respectively of L onto M. That is, if $(i, j; k) \in L$, then $(i\alpha, j\beta; k\gamma) \in M$.*

Isotopism of two critical sets A and B can be defined along the same line.

Theorem 2.5 *[5] Let L be a latin square of order n with a critical set C. Let (α, β, γ) be an isotopism from the critical set C onto $\bar{C}$. Then $\bar{C}$ is a critical set in a latin square $\bar{L}$ and $\bar{L}$ of order n is isotopic to L.*

Theorem 2.6 *[5] Let L_2 be the latin square representing the elementary Abelian 2-group of order 2^2. Let C be a minimal critical set in L_2. Then $|C| = 5$.*

Corollary 2.7 *Any critical set in L_3 must contain at least 5 triples chosen from each of the four latin subsquares $S_{A_iA_i}$, $S_{A_i\bar{A}_i}$, $S_{\bar{A}_iA_i}$ and $S_{\bar{A}_i\bar{A}_i}$, where A_i, $i = 1, 2, \ldots, 7$ are as given in (2.1).*

Theorem 2.8 *[5] Let L_v be a latin square representing the elementary Abelian 2-group C_2^v of order $n = 2^v$. Let $C - r_x$ be a critical set in L_v. Then $|C - r_x| \geq 2^{v-1}(2^v - 1)$.*

Theorem 2.9 *[5] Let L be a latin square, C a critical set in L and S a latin subsquare of order 2 in L. Then $|C \cap S| \geq 1$.*

Theorem 2.10 *[10] Let L_3 be a latin square representing the elementary Abelian 2-group of order $n = 2^3$. Let A be a critical set in L_3. Then $|A| \geq 25$.*

3 Main Result:

Khodkar [10] obtained a minimal critical set of size 25 in L_3. However, the precise structural form of a minimal critical set has not been discussed so far in the literature. In this section we determine the unique structural form of the minimal critical set in L_3 and in the process of construction, an alternative proof to the lower bound on the size of the critical set is also presented.

Theorem 3.1 *Let C be a minimal critical set in L_3. Let $n' = (n_1, n_2, \ldots, n_8)$, where n_i is the number of the triples from the ith row of L_3 to be included in C. Then $n' = (3,2,4,4,3,3,3,3)$ up to some permutation.*

Proof : It is to be noted that interchanging columns with rows or entries with rows leaves the structure of L_3 unaltered. Thus in view of the existence of a minimal critical set of size 25, we assume that the minimal critical set in L_3 contains at least one triple chosen from each row, each column and each element of N, otherwise, by Theorem 2.8 the size of the minimal critical set would be at least 28. We now discuss the structural form of a minimal critical set C in L_3. In this connection, we define n_i to be the number of triples from the ith row of L_3 in C, $i = 1, 2, \ldots, 8$. The different possibilities of n_i in C are dealt below.

$$\left.\begin{array}{l}\textbf{Case (a): } n_i = 1 \text{ for at least one } i = 1,\ldots,8.\\ \textbf{Case (b): } n_i \geq 2,\ \ \forall i = 1,\ldots,8 \text{ and } n_i = 2 \text{ for at least one}\\ \qquad\qquad i = 1,\ldots,8.\\ \textbf{Case (c): } n_i \geq 3\ \ \forall i = 1,\ldots,8.\end{array}\right\} \quad (3.1)$$

Case (a):
Let $n_1 = 1$ and without loss of generality, we assume that from r_1 only $(1,1;1) \in C$. In view of Remark 2.2, there exist four disjoint cycles corresponding to r_1 and r_l, $l = 2,3,\ldots,8$. Hence using Theorem 2.9, it is clear that each r_l, $l = 2,3,\ldots,8$ must contribute at least three triples to C, one from each of the cycles other than that consisting of $(1,1;1)$. Thus considering the subsquares

$$S_{A_1A_1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\ \hline 2 & 1 & 4 & 3\\ \hline 3 & 4 & 1 & 2\\ \hline 4 & 3 & 2 & 1\\ \hline\end{array} \qquad S_{A_1\bar{A}_1} = \begin{array}{|c|c|c|c|}\hline 5 & 6 & 7 & 8\\ \hline 6 & 5 & 8 & 7\\ \hline 7 & 8 & 5 & 6\\ \hline 8 & 7 & 6 & 5\\ \hline\end{array},$$

it follows that C must contain at least two triples chosen from each of the rows r_2, r_3 and r_4 of $S_{A_1\bar{A}_1}$ to ensure inclusion of at least one triple from each of the cycles I_{12}^{56}, I_{12}^{78}, I_{13}^{57}, I_{13}^{68}, I_{14}^{58}, I_{14}^{67}. Again from Corollary

2.7, it follows that C must contain at least 5 triples from $S_{A_1A_1}$. Thus as $n_1 = 1$ and $(1,1;1) \in C$, it follows that C must contain at least four triples collectively from the rows r_2, r_3 and r_4 of $S_{A_1A_1}$ and by the pigeon hole principle at least two triples from one of the rows, say r_2. Hence combining the triples from $S_{A_1\bar{A}_1}$, C must contain at least four triples from r_2 and at least three triples from each of the rows r_3 and r_4. Thus $n_2 \geq 4$, $n_3 \geq 3$ and $n_4 \geq 3$. Similar arguments considering each pair of subsquares $S_{A_jA_j}$ and $S_{A_j\bar{A}_j}$, yield that $n_i \geq 4$ for at least one i in A_j and $n_{i'} \geq 3$, $i \neq 1$, $i' \neq i$, $i' \in A_j$, $j = 4, 5, 6, 7$. Hence the size of C attains the minimum only if $n_i = 4$ for exactly one $i \in A_j$, $i \neq 1$, $j = 4, 5, 6, 7$ leading to the possibilities of either $n_3 = n_4 = 4$ or $n_5 = n_6 = 4$ or $n_7 = n_8 = 4$ as $A_4 \cap A_5 = \{1,3\}$, $A_6 \cap A_7 = \{1,4\}$; $A_4 \cap A_6 = \{1,5\}$, $A_5 \cap A_7 = \{1,6\}$; $A_4 \cap A_7 = \{1,7\}$ and $A_5 \cap A_6 = \{1,8\}$. Summing up the above arguments it follows that under the assumption of $n_1 = 1$, $|C|$ attains the minimum only if one of the following cases hold.

$$\left.\begin{array}{ll} \textit{Case (i)}: & n_2 = n_3 = n_4 = 4,\ n_5 = n_6 = n_7 = n_8 = 3; \\ \textit{Case (ii)}: & n_2 = n_5 = n_6 = 4,\ n_3 = n_4 = n_7 = n_8 = 3; \\ \textit{Case (iii)}: & n_2 = n_7 = n_8 = 4,\ n_3 = n_4 = n_5 = n_6 = 3. \end{array}\right\} \quad (3.2)$$

In each of these cases of (3.2), $|C| = 1 + 4 \cdot 3 + 3 \cdot 4 = 25$. Without loss of generality, we assume that *Case*(i) of (3.2) holds. As $n_1 = 1$ and $n_5 = 3$, in view of Theorem 2.9, considering four disjoint cycles formed by r_1 and r_5, it follows that C must contain exactly one triple from each of I_{15}^{26}, I_{15}^{37} and I_{15}^{48}, leading to eight possible choices of three triples from r_5 which can be classified into essentially two distinct cases:

$$\left.\begin{array}{ll} \textit{Case (1)}: & \text{three triples from } r_5 \text{ of } S_{\bar{A}_1A_1}(S_{\bar{A}_1\bar{A}_1}) \\ & \text{with no triples from } r_5 \text{ of } S_{\bar{A}_1\bar{A}_1}(S_{\bar{A}_1A_1}); \\ \textit{Case (2)}: & \text{two triples from } S_{\bar{A}_1A_1}(S_{\bar{A}_1\bar{A}_1}) \\ & \text{with one triple from} S_{\bar{A}_1\bar{A}_1}(S_{\bar{A}_1A_1}). \end{array}\right\} \quad (3.3)$$

Under *Case* (1) of (3.3), satisfying the conditions $n_1 = 1$, $n_2 = n_3 = n_4 = 4$, $n_5 = n_6 = n_7 = n_8 = 3$, we verified using computer search that all feasible choices of a partial latin square of size 25 lead to more than one completion of the latin square. Hence any partial latin square thus formed is not uniquely completable and therefore is not a critical set. Under *Case* (2) of (3.3), without loss of generality, we assume that (5,3;7), (5,4;8) and (5,6;2) $\in C$. Now considering all cycles arising from the rows r_5 and r_6, it can be verified that at least one of (6,7;4) and (6,8;3) $\in C$ and to satisfy $n_6 = 3$, there are only three possibilities of three triples from r_6, viz. $\{(6,7;4),(6,2;5),(6,3;8)\}$ or $\{(6,8;3),(6,2;5),(6,4;7)\}$ or $\{(6,7;4),(6,8;3),(6,2;5)\}$. Assuming that $\{(6,7;4),(6,8;3),(6,2;5)\} \subset C$, with the given choices of triples in the rows r_1 and r_5, it follows from Remark 2.2 and Theorem 2.9, that the possible choices of triples in C must be either $\{(7,2;8),(7,4;6),(7,5;3)\}$ from r_7 combined with $\{(8,3;6),(8,5;4),(8,7;2)\}$ from r_8 or $\{(7,4;6),(7,5;3),(7,8;2)\}$ from r_7 combined with $\{(8,2;7),(8,3;6),(8,5;4)\}$ from r_8. For each of these choices, it is checked that all the possibilities of four triples from r_2 in C violate Corollary 2.7 as the requirement of at least five triples in C from the subsquare $S_{A_3A_3}$ is not satisfied. Thus if $\{(6,7;4),(6,8;3),(6,2;5)\} \subset C$, there does not exist any critical set C of size 25. A similar conclusion holds true if we assume that $\{(6,7;4),(6,2;5),(6,3;8)\} \subset C$ or $\{(6,8;3),(6,2;5),(6,4;7)\} \subset C$. Using exhaustive computer search we arrive at the same conclusion under *Case* (*ii*) and *Case* (*iii*) of (3.2).

Case (b):
Without loss of generality, we consider the case that C contains only (1,1;1) and (1,2;2) from r_1 and at least two triples from every other row of L_3. In view of Theorem 2.9, it follows that C contains at least three triples from r_2 to represent at least one triple from each of the cycles I_{12}^{34}, I_{12}^{56} and I_{12}^{78}. Thus $n_2 \geq 3$. Moreover, considering cycles I_{13}^{57} and I_{13}^{68}, r_3 should contribute at least two triples from $S_{A_1\bar{A}_1}$. Similarly, considering cycles I_{14}^{58} and I_{14}^{67} formed by the rows r_1 and r_4, C should have at least two triples from r_4 of $S_{A_1\bar{A}_1}$. Moreover, there should be at least one triple from each of the cycles I_{34}^{34} and I_{34}^{12} from $S_{A_1A_1}$ in C, which results in at least six triples from the set of rows $\{r_3,\ r_4\}$ in C. Similar arguments apply to the set of rows $\{r_5,\ r_6\}$ and also to the set of rows $\{r_7,\ r_8\}$. Thus $n_3 + n_4 \geq 6$, $n_5 + n_6 \geq 6$ and $n_7 + n_8 \geq 6$. Hence, besides r_1 there can be at most one row from each of the sets $\{r_3,\ r_4\}$, $\{r_5,\ r_6\}$ and $\{r_7,\ r_8\}$ with exactly two triples in C. Let $\delta \in \{0,1,2,3\}$ be the number of rows beside r_1 contributing exactly two triples to C. In the following, starting with the highest value of δ, we argue that $|C| \geq 26$, $\forall\delta$.
$\boldsymbol{\delta = 3}$ **:**
Under this condition, one of the following eight cases must happen.

$$\left.\begin{array}{llll} (i) & n_1 = n_3 = n_5 = n_7 = 2, & (ii) & n_1 = n_3 = n_6 = n_8 = 2, \\ (iii) & n_1 = n_4 = n_5 = n_8 = 2, & (iv) & n_1 = n_4 = n_6 = n_7 = 2, \\ (v) & n_1 = n_3 = n_5 = n_8 = 2, & (vi) & n_1 = n_3 = n_6 = n_7 = 2, \\ (vii) & n_1 = n_4 = n_5 = n_7 = 2, & (viii) & n_1 = n_4 = n_6 = n_8 = 2. \end{array}\right\} \quad (3.4)$$

Dealing with these cases of (3.4) one by one we find that under (*i*), there are exactly eight triples in C from $S_{A_4A_4} \cup S_{A_4\bar{A}_4}$ violating Corollary 2.7. Similar arguments hold for (*ii*) – (*iv*). Now dealing with (*v*), we proceed sequentially to select triples from r_3, r_5 and r_8 in order. It turns out that for the choice of triples $\{(3,6;8),\ (3,7;5)\}$ or $\{(3,5;7),\ (3,8;6)\}$ from r_3, the choice of two triples from r_5 is fixed. Then every choice of two triples from r_8 violates Theorem 2.9. Each of the other possible choices of two triples from r_3 viz. $\{(3,8;6),\ (3,7;5)\}$ and $\{(3,5;7),\ (3,6;8)\}$ lead to a valid choice of two triples from r_8, which yields $|C| \geq 26$. We only explain this situation assuming that $\{(3,8;6),\ (3,7;5)\} \subset C$. In this case, there are fix choices of two triples from r_5 and r_8 sequentially. Recall that $n_3+n_4 \geq 6$, $n_5+n_6 \geq 6$ and $n_7+n_8 \geq 6$. Thus the choice of eight triples from r_1, r_3, r_5 and r_8 in order leads to $n_6 \geq 4$ and $n_7 \geq 4$. But all the feasible choices satisfying $n_6 = 4$ ($n_7 = 4$) violate Corollary 2.7 considering $S_{A_5A_5}$ or $S_{A_5\bar{A}_5}$ ($S_{A_4A_4}$ or $S_{A_4\bar{A}_4}$). Thus $n_6 \geq 5$ and $n_7 \geq 5$ independently. Again considering the cycles formed by r_2 and r_3, there should be at least one triple from each of the cycles I_{23}^{23} and I_{23}^{14} in C. Since $n_3 = 2$, these two triples must be from r_2, leading to $n_2 \geq 4$. Thus under Case (v), the conditions that $n_1 = 2$, $n_2 \geq 4$, $n_3+n_4 \geq 6$, $n_5+n_6 \geq 7$ and $n_7+n_8 \geq 7$ must be satisfied leading to $|C| \geq 26$. Similar arguments hold for Cases (vi) – $(viii)$.

$\boldsymbol{\delta = 2:}$

To begin with, let us assume that besides r_1 both r_3 and r_5, have two triples in C, namely $(3,5;7)$, $(3,8;6)$ and hence $(5,4;8)$, $(5,7;3) \in C$. Recall that $n_2 \geq 3$ considering the cycles formed by the rows r_1 and r_2 and in view of the fact that only $(1,1;1)$ and $(1,2;2) \in C$. Again as $n_3 = 2$ and only $(3,5;7)$ and $(3,8;6) \in C$, the requirement of the representation of at least one triple from each of the cycles I_{23}^{23} and I_{23}^{14} leads to $n_2 \geq 4$. Now considering the two subsquares $S_{A_4A_4}$ and $S_{A_4\bar{A}_4}$, it follows from Corollary 2.7 that C must contain at least 10 triples from the set of rows $\{r_1,r_3,r_5,r_7\}$ i.e., $n_1+n_3+n_5+n_7 \geq 10$. This implies that $n_7 \geq 4$ as $n_1 = n_3 = n_5 = 2$. Now we consider the triples that need to be chosen from r_8 in C. It follows that at least one triple from each of the cycles I_{18}^{36}, I_{18}^{45}, I_{38}^{38}, I_{38}^{25}, I_{58}^{67} and I_{58}^{14} should be in C leading to $n_8 \geq 4$. Thus $n_1 = 2$, $n_2 \geq 4$, $n_3+n_4 \geq 6$, $n_5+n_6 \geq 6$ and $n_7+n_8 \geq 8$, implying thereby that $|C| \geq 26$ and hence C cannot be a minimal critical set.

$\boldsymbol{\delta = 1:}$

We assume that besides r_1 only r_3 contributes exactly two triples to C, whereas every other row contributes at least three triples to C. It can be verified using Theorem 2.9 that the set of two triples from r_3 can be one of the following 4 choices viz. $\{(3,5;7),(3,8;6)\}$; $\{(3,5;7),(3,6;8)\}$; $\{(3,7;5),(3,6;8)\}$; $\{(3,7;5),(3,8;6)\}$. We argue below assuming that $\{(3,5;7),(3,8;6)\} \subset C$. The other choices can be dealt with similarly. As argued in the previous paragraphs, in the case of inclusion of $(1,1;1)$ and $(1,2;2)$ in C, $n_3+n_4 \geq 6$, $n_5+n_6 \geq 6$ and $n_7+n_8 \geq 6$ still hold. Moreover, $n_3 = 2$ with the given choices of triples forces $n_2 \geq 4$ considering cycles formed by $\{r_1,r_2\}$ and $\{r_2,r_3\}$. Similarly $n_4 \geq 4$ when cycles formed by $\{r_1,r_4\}$ and $\{r_3,r_4\}$ are considered. Under the requirement of $n_i \geq 3$, $i \in \{5,6,7,8\}$, $|C|$ can attain the minimum size of 25 only if $n_i = 3$ for at least three

i's, $i \in \{5,6,7,8\}$. Without loss of generality, we assume that $n_5 = 3$. The consideration of cycles formed by the sets of rows $\{r_5, r_1\}$ and $\{r_5, r_3\}$ separately, yields that there are in all eight choices of three triples from r_5 as given below

$$\left.\begin{array}{ll} C_1 = \{(5,6;2),(5,7;3),(5,8;4)\}, & C_2 = \{(5,4;8),(5,7;3),(5,8;4)\}, \\ C_3 = \{(5,1;5),(5,3;7),(5,4;8)\}, & C_4 = \{(5,3;7),(5,4;8),(5,7;3)\}, \\ C_5 = \{(5,1;5),(5,4;8),(5,7;3)\}, & C_6 = \{(5,4;8),(5,5;1),(5,7;3)\}, \\ C_7 = \{(5,4;8),(5,6;2),(5,7;3)\}, & C_8 = \{(5,2;6),(5,4;8),(5,7;3)\}. \end{array}\right\} \quad (3.5)$$

For each of the choices except the choice C_2 in (3.5) above, we sequentially proceed to choose the minimum number of triples from r_7, r_8, r_6, r_4, r_2 in order. Using Corollary 2.7 or Theorem 2.9 depending on the situation in each case, it turns out that $|C| \geq 26$. Our argument below explains this fact corresponding to the choice of C_1 from r_5. Under the given choices of triples from r_1, r_3 and r_5, at least 4 triples from r_7 must be chosen, as otherwise, the choice of exactly three triples from r_7 violates Corollary 2.7 with respect to $S_{A_4A_4}$. Hence $n_7 \geq 4$. If $n_7 \geq 5$, then $n_7 + n_8 \geq 8$, which yields $|C| \geq 26$. In the case of $n_7 = 4$, exhaustive computer search shows that, either $n_8 \geq 4$ or $n_6 \geq 4$. In each of these cases $n_5+n_6+n_7+n_8 \geq 14$ leading to $|C| \geq 26$. Similar arguments hold true for other choices of $C_i, i \neq 1,2$ from r_5. For the choice of C_2 in r_5, using Remark 2.7 it can be argued that the size of C is at least 26 always, except when C consists of the triples $(1,1;1),(1,2;2)$ from r_1; $(3,5;7),(3,8;6)$ from r_3; $(5,4;8),(5,7;3),(5,8;4)$ from r_5; $(7,3;5),(7,6;4),(7,7;1)$ from r_7; $(8,2;7),(8,4;5),(8,6;3)$ from r_8; $(6,1;6),(6,6;1),(6,7;4),(6,8;3)$ from r_6; $(4,2;3),(4,3;2),(4,4;8),(4,5;7)$ from r_4; and $(2,2;2),(2,3;4),(2,5;6),(2,7;8)$ from r_2. In this case of $|C| = 25$, apparently there is no contradiction via Corollary 2.7 and/or Theorem 2.9 as has been observed earlier, but interestingly, this particular choice of C can be completed to a Latin square L' different from L_3 as displayed below, establishing that C is not even a UC.

$L' =$

1	2	3	6	4	8	7	5
2	1	4	3	6	5	8	7
3	4	8	1	7	2	5	6
5	3	2	4	8	7	6	1
7	5	1	8	2	6	3	4
6	8	7	2	5	1	4	3
8	6	5	7	3	4	1	2
4	7	6	5	1	3	2	8

$\delta = 0$:

Under this condition, C contains precisely two triples from exactly one row of L, and at least three triples from the remaining rows. In view of Theorem 2.5, without loss of generality, we assume that (1,1;1), (1,2;2), (1,3;3), (2,3;6) and (2,7;8) $\in C$. Then considering the cycles formed by the sets of rows $\{r_3, r_1\}$, $\{r_3, r_2\}$, $\{r_4, r_1\}$ and $\{r_4, r_2\}$ separately , it turns out that r_3 and r_4 should contribute at least 4 triples to C. Now under these conditions, we searched for a critical set C with $|C| = 25$ and $n' = (3, 2, 4, 4, 3, 3, 3, 3)$. We present below one such critical set among many.

$C =$

1	2	3					
				6		8	
3		1			8	5	
	3	2		8	7		
					2	3	4
6			7		1		
		5	6				2
	7		5	4			

Case (c):
Under the condition that $n_i \geq 3, \forall i = 1, \dots 8$, there can exist a critical set C with $|C| = 25$, only if C contains exactly 3 triples from 7 rows and exactly 4 triples from the remaining row of L_3. In view of Theorem 2.5 without loss of generality we can assume that each of the rows r_1, r_2, r_3, r_4 has three triples in C. Exhaustive computer search with all possible choices of three triples from the first four rows of L_3 reveals that this requires choosing at least 4 triples from two of the remaining rows r_5, r_6, r_7, r_8, increasing the size of the critical set to 26. Thus the minimum size of 25 for a critical set in L_3 is attainable only when $n' = (3, 2, 4, 4, 3, 3, 3, 3)$ up to some permutation.

Acknowledgment: We would like to thank the referee for his constructive suggestions and comments that resulted in a much improved version of the presentation of this article. We also wish to gratefully acknowledge the help provided by Mr Neel Choudhury and Mr Chitradeep Dutta Roy for writing computer programs used in this investigation.

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