On Fall Colorings of Graphs

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Abstract

A fall k-coloring of a graph G is a proper k-coloring of G such that each vertex of G sees all k colors on its closed neighborhood. We denote Fall(G) the set of all positive integers k for which G has a fall k-coloring. In this paper, we study fall colorings of lexicographic product of graphs and categorical product of graphs. Also, we show that for each graph G, Fall($M(G)$) = $\emptyset$ where $M(G)$ is the Mycielskian of the graph G. Finally, we prove that for each bipartite graph G, Fall(G^c) $\subseteq$ $\{\chi(G^c)\}$ and it is polynomial time to decide whether Fall(G^c) = $\{\chi(G^c)\}$ or not.

Keywords: fall coloring, lexicographic product, categorical product.

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1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$ and $[k] := \{i \mid i \in \mathbb{N},\ 1 \leq i \leq k\}$. A k-coloring (proper k-coloring) of G is a function $f : V \to [k]$ such that for each i, $1 \leq i \leq k$, $f^{-1}(i)$ is an independent set. We say that G is k-colorable whenever G admits a k-coloring f, in this case, we denote $f^{-1}(i)$ by V_i and call it a color class (of f) corresponding to color i, $1 \leq i \leq k$. The minimum integer k for which G has a k-coloring, is called the chromatic number of G and is denoted by $\chi(G)$.

Let G be a graph, f be a k-coloring of G and v be a vertex of G. The vertex v is called colorful (or color-dominating or b-dominating) if each color $1 \leq i \leq k$ appears on the closed neighborhood of v

$(f(N[v]) = [k])$. The k-coloring f is said to be a fall k-coloring (of G) if each vertex of G is colorful. There are graphs G for which G has no fall k-coloring for any positive integer k. For example, C_5 (a cycle with 5 vertices) and every graph with at least one edge and one isolated vertex, have no fall k-colorings for any positive integer k. The notation Fall(G) stands for the set of all positive integers k for which G has a fall k-coloring. Whenever Fall(G) $\neq \emptyset$, we call min(Fall(G)) and max(Fall(G)), fall chromatic number of G and fall achromatic number of G and denote them by $\chi_f(G)$ and $\psi_f(G)$, respectively. The concept of fall coloring introduced in 2000 in [1].

2 Fall colorings of lexicographic product of graphs

Let G and H be graphs. The lexicographic product of G and H, denoted by $G[H]$, is defined the graph with vertex set $V(G) \times V(H)$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\}|\ x_1, x_2 \in V(G)$ and $y_1, y_2 \in V(H)$ and $[\ (\{x_1, x_2\} \in E(G))$ or $(x_1 = x_2, \{y_1, y_2\} \in E(H))\ ]\}$. For each $x \in V(G)$, the induced subgraph of $G[H]$ on $\{x\} \times V(H)$ is denoted by H_x.

Note that $G[H]$ and $H[G]$ are not necessarily isomorphic. For example, let $G := K_2$ and H be the complement of G. $G[H]$ has 4 edges and $H[G]$ has 2 edges and therefore, they are not isomorphic. But lexicographic product of graphs is associative up to isomorphism (For arbitrary graphs G_1, G_2 and G_3, $(G_1[G_2])[G_3]$ and $G_1[G_2[G_3]]$ are isomorphic.).

Theorem 1. *Let G and H be graphs and $k \in$ Fall($G[H]$) and f be a fall k-coloring of $G[H]$. Then, for each $x \in V(G)$, $S_x := f(V(H_x))$ forms a fall $|S_x|$-coloring of H_x.*

Proof. Let $x \in V(G)$ and (x, y) be an arbitrary vertex of H_x and its color be α. Then, for each $\beta \in S_x \setminus \{\alpha\}$, there exists a vertex (a, b) of $G[H]$ adjacent with (x, y) which is colored β. Obviously $a = x$, otherwise, since $\beta \in S_x$, there exists a vertex $(x, z) \in V(H_x)$ colored β. (x, y) is adjacent with (a, b) and $x \neq a$, so $\{x, a\} \in E(G)$ and therefore, (x, z) and (a, b) are adjacent in $G[H]$ and both of

them are colored β, which is a contradiction. Therefore, $a = x$ and $(a, b) \in V(H_x)$. Hence, S_x forms a fall $|S_x|$-coloring of H_x. ∎

Corollary 1. *Let G and H be graphs. If* Fall$(G[H]) \neq \emptyset$, *then,* Fall$(H) \neq \emptyset$.

Corollary 2. *Let G and H be graphs such that* Fall$(G[H]) \neq \emptyset$. *Then,* Fall$(H) \neq \emptyset$ *and for each fall k-coloring f of $G[H]$ and each $x \in V(G)$, $\chi_f(H) \leq |f(V(H_x))| \leq \psi_f(H)$.*

There are pairs of graphs (G, H) for which Fall$(G) = \emptyset$ but Fall$(G[H]) \neq \emptyset$. For example, Fall$(C_5) = \emptyset$ but $C_5[K_2]$ has a fall 5-coloring. First let's label the vertices of $C_5[K_2]$ lexicographically: $1 := (1,1)$, $2 := (1,2)$, $3 := (2,1), \ldots$, $10 := (5,2)$. Here is a fall 5-coloring f of $C_5[K_2]$: $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$, $f(5) = 1$, $f(6) = 5$, $f(7) = 2$, $f(8) = 4$, $f(9) = 5$, $f(10) = 3$. Also, there are pairs of graphs (G, H) for which Fall$(G) = \emptyset$ and Fall$(H) \neq \emptyset$ and Fall$(G[H]) = \emptyset$. For example, Fall$(C_5) = \emptyset$ and Fall$(K_1) \neq \emptyset$ and Fall$(C_5[K_1]) = $ Fall$(C_5) = \emptyset$. The next theorem shows that if Fall$(G) \neq \emptyset$ and Fall$(H) \neq \emptyset$, then, Fall$(G[H]) \neq \emptyset$.

Theorem 2. *Let G and H be graphs for which* Fall$(G) \neq \emptyset$ *and* Fall$(H) \neq \emptyset$. *Then,* $\{\sum_{i=1}^{s} k_i |\ s \in$ Fall$(G), \forall i, 1 \leq i \leq s : k_i \in$ Fall$(H)\} \subseteq$ Fall$(G[H])$.

Proof. Let $s \in$ Fall(G) and $g : V(G) \rightarrow [s]$ be a fall s-coloring of G and for each $1 \leq i \leq s$, $k_i \in$ Fall(H) and h_i be a fall k_i-coloring of H. Let's color each vertex (x, y) of $G[H]$ by color $(g(x), h_{g(x)}(y))$. Indeed, let's consider the function $f : V(G[H]) \rightarrow S := \{(g(x), h_{g(x)}(y)) \mid (x, y) \in V(G) \times V(H)\}$ which assigns to each (x, y) of $G[H]$, $(g(x), h_{g(x)}(y))$. For each adjacent vertices (x, y) and (a, b) in $G[H]$, $\{x, a\} \in E(G)$ or $(x = a$ and $\{y, b\} \in E(H))$. So, $g(x) \neq g(a)$ or $(g(x) = g(a)$ and $h_{g(x)}(y) \neq h_{g(a)}(b))$. Therefore, $(g(x), h_{g(x)}(y)) \neq (g(a), h_{g(a)}(b))$. This shows that f is a $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$ such that uses exactly $\sum_{i=1}^{s} k_i$ colors. Now let's show that f is a fall $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$. For each $(x, y) \in V(G[H])$ and each $(\alpha, \beta) \in S \setminus \{(g(x), h_{g(x)}(y))\}$, there is a vertex (u, v) of $G[H]$ colored (α, β), or equivalently, $(g(u), h_{g(u)}(v)) = (\alpha, \beta)$. Now, there are two cases:

Case I) $g(x) = g(u)$: In this case, $h_{g(x)} = h_{g(u)}$ and $h_{g(x)}(y) \neq h_{g(u)}(v)$. Since $h_{g(x)}$ is a fall $k_{g(x)}$-coloring of H, there exists a vertex $z \in V(H)$ such that $\{z, y\} \in E(H)$ and $h_{g(x)}(z) = h_{g(u)}(v)$. The vertex (x, z) of $G[H]$ is adjacent with (x, y) and its color is $f((x, z)) = (g(x), h_{g(x)}(z)) = (g(u), h_{g(u)}(v)) = (\alpha, \beta)$.

Case II) $g(x) \neq g(u)$: Since g is a fall s-coloring of G, there exists a vertex $l \in V(G)$ such that $\{x, l\} \in E(G)$ and $g(l) = g(u)$. So, $h_{g(u)}(v) = h_{g(l)}(v)$. The vertex (l, v) is adjacent with (x, y) in $G[H]$ and $f((l, v)) = (g(l), h_{g(l)}(v)) = (g(u), h_{g(u)}(v)) = (\alpha, \beta)$.

Hence, f is a fall $(\sum_{i=1}^{s} k_i)$-coloring of $G[H]$. Therefore, $\{\sum_{i=1}^{s} k_i |$ $s \in \text{Fall}(G), \forall i, 1 \leq i \leq s \colon k_i \in \text{Fall}(H)\} \subseteq \text{Fall}(G[H])$. ■

Corollary 3. *Let G and H be graphs for which* $\text{Fall}(G) \neq \emptyset$ *and* $\text{Fall}(H) \neq \emptyset$. *Then,* $\chi_f(G[H]) \leq \chi_f(G)\chi_f(H) \leq \psi_f(G)\psi_f(H) \leq \psi_f(G[H])$.

$\chi_f(G[H])$ and $\chi_f(G)\chi_f(H)$ are not necessarily equal. For example, $\chi_f(C_9) = 3$ and $\chi_f(K_2) = 2$. Therefore, $\chi_f(C_9)\chi_f(K_2) = 6$, but $\chi_f(C_9[K_2]) \leq 5$, first let's label the vertices of $C_9[K_2]$ lexicographically: 1:=(1,1), 2:=(1,2), 3:=(2,1), ..., 18:=(9,2). Here is a fall 5-coloring f of $C_9[K_2]$: $f(1) = 1$, $f(2) = 4$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, $f(6) = 1$, $f(7) = 4$, $f(8) = 2$, $f(9) = 3$, $f(10) = 1$, $f(11) = 5$, $f(12) = 2$, $f(13) = 4$, $f(14) = 3, f(15) = 1$, $f(16) = 2$, $f(17) = 5$ and $f(18) = 3$. Also, $\psi_f(G)\psi_f(H)$ and $\psi_f(G[H])$ are not necessarily equal. For example, $\psi_f(C_8) = 2$ and $\psi_f(K_2) = 2$ and therefore, $\psi_f(C_8)\psi_f(K_2) = 4$. But $\psi_f(C_8[K_2]) \geq 5$. First let's label the vertices of $C_8[K_2]$ lexicographically: $1 := (1, 1)$, $2 := (1, 2)$, $3 := (2, 1), \ldots$, $16 := (8, 2)$. Here is a fall 5-coloring f of $C_8[K_2]$: $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$, $f(5) = 5$, $f(6) = 1$, $f(7) = 2$, $f(8) = 3$, $f(9) = 4$, $f(10) = 1$, $f(11) = 5$, $f(12) = 2$, $f(13) = 3$, $f(14) = 1$, $f(15) = 5$ and $f(16) = 4$.

Theorem 2 says that if G and H are graphs for which $\text{Fall}(G) \neq \emptyset$ and $\text{Fall}(H) \neq \emptyset$, Then, $\{\sum_{i=1}^{s} k_i | s \in \text{Fall}(G), \forall i, 1 \leq i \leq s \colon k_i \in \text{Fall}(H)\} \subseteq \text{Fall}(G[H])$. Since $5 \in \text{Fall}(C_9[K_2])$ and $5 \notin \{\sum_{i=1}^{s} k_i \mid s \in \text{Fall}(C_9), \forall i, 1 \leq i \leq s \colon k_i \in \text{Fall}(K_2)\}$, $\text{Fall}(G[H])$ and $\{\sum_{i=1}^{s} k_i | s \in \text{Fall}(G), \forall i, 1 \leq i \leq s \colon k_i \in \text{Fall}(H)\}$ are not necessarily equal.

Theorem 3. *There are pairs of graphs (G, H) for which* $\text{Fall}(G) \neq$

$\emptyset$ and Fall$(H) \neq \emptyset$ and the following strictly inequalities hold.
$$\chi_f(G[H]) < \chi_f(G)\chi_f(H) < \psi_f(G)\psi_f(H) < \psi_f(G[H]).$$

Proof. Let $G := C_6 \bigvee C_8 \bigvee C_9$ (the join of C_6 and C_8 and C_9) and $H := K_2$. Since $(C_6 \bigvee C_8 \bigvee C_9)[K_2]$ and $(C_6[K_2]) \bigvee (C_8[K_2]) \bigvee (C_9[K_2])$ are isomorphic, $\chi_f((C_6 \bigvee C_8 \bigvee C_9)[K_2]) = \chi_f(C_6[K_2]) + \chi_f(C_8[K_2]) + \chi_f(C_9[K_2]) \leq 4+4+5 = 13$ and $\psi_f((C_6 \bigvee C_8 \bigvee C_9)[K_2]) = \psi_f(C_6[K_2]) + \psi_f(C_8[K_2]) + \psi_f(C_9[K_2]) \geq 6+5+6 = 17$. Also, $\chi_f(C_6 \bigvee C_8 \bigvee C_9) = 7$ and $\psi_f(C_6 \bigvee C_8 \bigvee C_9) = 8$ and $\chi_f(K_2) = \psi_f(K_2) = 2$, as desired.

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Theorem 4. *For each $\varepsilon > 0$, There exists a pair of graphs (S,T) for which $\min\{\psi_f(S[T]) - \psi_f(S)\psi_f(T), \psi_f(S)\psi_f(T) - \chi_f(S)\chi_f(T), \chi_f(S)\chi_f(T) - \chi_f(S[T])\} \geq \varepsilon$.*

Proof. With no loss of generality, we can assume that ε is a natural number. Let $G := C_6 \bigvee C_8 \bigvee C_9$ and $S := K_\varepsilon[G]$ and $T := K_2$. Since $S[T]$ and $K_\varepsilon[G[T]]$ are isomorphic and $\chi_f(K_\varepsilon[G[T]]) = \varepsilon\chi_f(G[T])$ and $\psi_f(K_\varepsilon[G[T]]) = \varepsilon\psi_f(G[T])$, the theorem implies. ■

One can easily observe that if G and H are graphs such that Fall$(G[H]) \neq \emptyset$, then, $\chi_f(G[H]) \geq \omega(G)\chi_f(H)$. The next clear proposition introduces a sufficient condition for equality.

Proposition 1. *Let G and H be graphs such that Fall$(G) \neq \emptyset$ and Fall$(H) \neq \emptyset$ and $\chi_f(G) = \omega(G)$. Then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

Corollary 4. *If G is a tree or a complete graph or C_{2k} (for some $k \in \mathbb{N} \setminus \{1\}$) and H is a graph such that Fall$(H) \neq \emptyset$, then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

Theorem 1 says that in every fall k-coloring of $G[H]$ and each $x \in V(G)$, the number of colors appear on $V(H_x)$ is at most $\psi_f(H)$. Hence, $\psi_f(G[H]) \leq (\delta(G)+1)\psi_f(H)$. The following clear proposition introduces a sufficient condition for equality.

Proposition 2. *Let G and H be graphs for which Fall$(G) \neq \emptyset$ and Fall$(H) \neq \emptyset$ and $\psi_f(G) = \delta(G) + 1$. Then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

Corollary 5. *If G is a tree or a complete graph or C_{3k} (for some $k \in \mathbb{N}$) and H is a graph such that $\mathrm{Fall}(H) \neq \emptyset$, then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

3 Type-II graph homomorphisms and lexicographic product of graphs

Now we study a type of graph homomorphisms that is related to fall colorings of graphs.

Definition 1. Let G and H be graphs. A function $f : V(G) \to V(H)$ is called a type-II graph homomorphism from G to H if f satisfies the following two conditions.

1) $\{u, v\} \in E(G) \Rightarrow \{f(u), f(v)\} \in E(H)$.
2) $\{u_1, v_1\} \in E(H) \Rightarrow \forall v \in f^{-1}(v_1) : \exists u \in f^{-1}(u_1)$ s.t $\{u, v\} \in E(G)$. ♠

Type-II graph homomorphisms introduced by Laskar and Lyle in 2009 in [3]. They showed that for any graph G, $k \in \mathrm{Fall}(G)$ iff there exists a type-II graph homomorphism from G to K_k. Note that every type-II graph homomorphism from a graph G to a complete graph, is surjective. If f_1 is a type-II graph homomorphism from G to H and f_2 is a type-II graph homomorphism from H to I, then, $f_2 \circ f_1$ is a type-II graph homomorphism from G to I. Also, if there exists a type-II graph homomorphism from G to H and $k \in \mathrm{Fall}(H)$, then, $k \in \mathrm{Fall}(G)$. If there exists a type-II graph homomorphism from G_1 to G_2 and a type-II graph homomorphism from H_1 to H_2, then, there exists a type-II graph homomorphism from $G_1 \square H_1$ to $G_2 \square H_2$. We prove a similar theorem for lexicographic product of graphs.

Theorem 5. *Let G_1, G_2, H_1 and H_2 be graphs and f_1 be a type-II graph homomorphism from G_1 to G_2 and f_2 be a surjective type-II graph homomorphism from H_1 to H_2. Then, there exists a type-II graph homomorphism f_3 from $G_1[H_1]$ to $G_2[H_2]$.*

Proof. Let $f_3 : V(G_1[H_1]) \to V(G_2[H_2])$ be defined the function which assigns to each $(g, h) \in V(G_1[H_1])$, $f_3((g, h)) = (f_1(g), f_2(h))$.

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For each $\{(x_1, y_1), (x_2, y_2)\} \in E(G_1[H_1])$, $\{x_1, x_2\} \in E(G_1)$ or $(x_1 = x_2$ and $\{y_1, y_2\} \in E(H_1))$. Therefore, $\{f_1(x_1), f_1(x_2)\} \in E(G_2)$ or $(f_1(x_1) = f_1(x_2)$ and $\{f_2(y_1), f_2(y_2)\} \in E(H_2))$. This shows that $\{(f_1(x_1), f_2(y_1)), (f_1(x_2), f_2(y_2))\} \in E(G_2[H_2])$ and therefore, Property 1 holds. Now for each $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} \in E(G_2[H_2])$ and each $(u_1, v_1) \in f_3^{-1}((\alpha_1, \beta_1))$, there are two cases:

Case I) $\{\alpha_1, \alpha_2\} \in E(G_2)$: Since f_1 is a type-II graph homomorphism and $u_1 \in f_1^{-1}(\alpha_1)$, there exists $u_2 \in f_1^{-1}(\alpha_2)$ such that $\{u_1, u_2\} \in E(G_1)$. Surjectivity of f_2 implies that there exists $v_2 \in f_2^{-1}(\beta_2)$. Therefore, $(u_2, v_2) \in f_3^{-1}((\alpha_2, \beta_2))$ and $\{(u_1, v_1), (u_2, v_2)\} \in E(G_1[H_1])$ and accordingly, Property 2 holds.

Case II) $\alpha_1 = \alpha_2$ and $\{\beta_1, \beta_2\} \in E(H_2)$: In this case, $u_1 \in f_1^{-1}(\alpha_2)$ and since f_2 is a type-II graph homomorphism and $v_1 \in f_2^{-1}(\beta_1)$, there exists $v_2 \in f_2^{-1}(\beta_2)$ such that $\{v_1, v_2\} \in E(H_1)$. Hence, $(u_1, v_2) \in f_3^{-1}((\alpha_2, \beta_2))$ and $\{(u_1, v_1), (u_1, v_2)\} \in E(G_1[H_1])$ and therefore, Property 2 holds. Thus, f_3 is a type-II graph homomorphism. ∎

Corollary 6. *If G and H are graphs such that $r_1 \in \text{Fall}(G)$ and $r_2 \in \text{Fall}(H)$, then,*

$$\chi_f(G[H]) \leq \chi_f(G[K_{r_2}]) \leq \chi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(G[K_{r_2}]) \leq \psi_f(G[H]).$$

4 Fall colorings of categorical product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The graph $G_1 \times G_2 := (V_1 \times V_2, \{\{(x_1, y_1), (x_2, y_2)\}| \{x_1, x_2\} \in E(G_1)$ and $\{y_1, y_2\} \in E(G_2)\})$ is called the categorical product of G and H.

Categorical product of graphs is commutative and associative up to isomorphism (For each arbitrary graphs G_1, G_2 and G_3, $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic, also, $(G_1 \times G_2) \times G_3$ and $G_1 \times (G_2 \times G_3)$ are isomorphic.). For arbitrary graphs G and H, if $E(G) = \emptyset$ or $E(H) = \emptyset$, then, $E(G \times H) = \emptyset$ and therefore, $G \times H$ has only a fall 1-coloring and $\text{Fall}(G \times H) = \{1\}$. Thus, we focus on nonempty edge set graphs, unless stated otherwise. Firstly, note that $\text{Fall}(G := (\{a, b, c, d\}, \{\{a, b\}, \{b, c\}, \{c, a\}, \{d, a\}\})) = \emptyset$ and $\text{Fall}(G \times G) = \emptyset$. Secondly, note that $\text{Fall}(C_5 := (\{0, 1, 2, 3, 4\}, \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\}\})) = \emptyset$, but the function $f : V(C_5 \times C_5) \to [5]$ which

assigns to each (i, j) of $V(C_5 \times C_5)$, $f((i, j)) :=$(the arithmetic residue of $i + 2j$ modulo 5)+1 where the last $+$ is the natural summation in $\mathbb{Z}$, is a fall 5-coloring of $C_5 \times C_5$, and therefore, Fall$(C_5 \times C_5) \neq \emptyset$. The next theorem implies that if Fall$(G) \neq \emptyset$ or Fall$(H) \neq \emptyset$, then, Fall$(G \times H) \neq \emptyset$.

Theorem 6. *Let $n \in \mathbb{N}$ and $G_1, \ldots, G_n$ be arbitrary graphs.*
For every i, $1 \leq i \leq n$, we have: Fall$(G_i) \subseteq$ Fall$(\times_{i=1}^n G_i)$.

Proof. Since categorical product of graphs is commutative and associative up to isomorphism, it suffices to prove that Fall$(G_1) \subseteq$ Fall$(G_1 \times G_2)$. If Fall$(G_1) = \emptyset$, the theorem holds trivially. For each $k \in$ Fall(G_1), there exists a fall k-coloring f of G_1. Now, the function $g : V(G_1 \times G_2) \to [k]$ which assigns to each $(u, v) \in V(G_1 \times G_2)$, $g((u, v)) = f(u)$ is a fall k-coloring of $G_1 \times G_2$ and therefore, $k \in$ Fall$(G_1 \times G_2)$. Hence, Fall$(G_1) \subseteq$ Fall$(G_1 \times G_2)$. ∎

Corollary 7. *For each $n \in \mathbb{N}$ and arbitrary graphs $G_1, \ldots, G_n$ such that for each $i \in [n]$, Fall$(G_i) \neq \emptyset$, the following inequalities hold.*
$$\chi_f(\times_{i=1}^n G_i) \leq \min\{\chi_f(G_i)| \ i \in [n]\} \leq \max\{\psi_f(G_i)| \ i \in [n]\} \leq \psi_f(\times_{i=1}^n G_i).$$

Theorem 7. *Let G_1, G_2, H_1 and H_2 be graphs and f_1 be a type-II graph homomorphism from G_1 to G_2 and f_2 be a type-II graph homomorphism from H_1 to H_2. Then, there exists a type-II graph homomorphism f_3 from $G_1 \times H_1$ to $G_2 \times H_2$.*

Proof. Let $f_3 : V(G_1 \times H_1) \to V(G_2 \times H_2)$ be defined the function which assigns to each $(g, h) \in V(G_1 \times H_1)$, $f_3((g, h)) = (f_1(g), f_2(h))$. For each $\{(x_1, y_1), (x_2, y_2)\} \in E(G_1 \times H_1)$, $\{x_1, x_2\} \in E(G_1)$ and $\{y_1, y_2\} \in E(H_1)$. Therefore, $\{f_1(x_1), f_1(x_2)\} \in E(G_2)$ and $\{f_2(y_1), f_2(y_2)\} \in E(H_2)$. Hence, $\{f_3((x_1, y_1)), f_3((x_2, y_2))\} \in E(G_2 \times H_2)$ and therefore, Property 1 of type-II graph homomorphisms holds. Now for each $\{(a, b), (c, d)\} \in E(G_2 \times H_2)$ and each $(\alpha, \beta) \in f_3^{-1}((c, d))$, $\alpha \in f_1^{-1}(c)$ and $\beta \in f_2^{-1}(d)$. So, there exist $x \in f_1^{-1}(a)$ and $y \in f_2^{-1}(b)$ such that $\{x, \alpha\} \in E(G_1)$ and $\{y, \beta\} \in E(H_1)$, hence, $(x, y) \in f_3^{-1}((a, b))$ and $\{(x, y), (\alpha, \beta)\} \in E(G_1 \times H_1)$. So, Property 2 of type-II graph homomorphisms holds, too. Consequently, f_3 is a type-II graph homomorphism. ∎

We know that if f is a type-II graph homomorphism from G to H and $k \in \mathrm{Fall}(H)$, then, $k \in \mathrm{Fall}(G)$. Also, for each graph G and each natural number k, $k \in \mathrm{Fall}(G)$ iff there exists a type-II graph homomorphism from G to K_k. Therefore, the previous theorem implies the following corollary.

Corollary 8. *Let $n \in \mathbb{N}$ and for each $i \in [n]$, G_i be a graph and $k_i \in \mathrm{Fall}(G_i)$. Then, there exists a type-II graph homomorphism from $\times_{i=1}^n G_i$ to $\times_{i=1}^n K_{k_i}$ and $\mathrm{Fall}(\times_{i=1}^n K_{k_i}) \subseteq \mathrm{Fall}(\times_{i=1}^n G_i)$. Also, $\chi_f(\times_{i=1}^n G_i) \leq \chi_f(\times_{i=1}^n K_{k_i}) \leq \psi_f(\times_{i=1}^n K_{k_i}) \leq \psi_f(\times_{i=1}^n G_i)$. These inequalities can easily extend to more inequalities in general. For example, in the case $n = 2$, $\chi_f(G_1 \times G_2) \leq \begin{cases} \chi_f(K_{k_1} \times G_2) \\ \chi_f(G_1 \times K_{k_2}) \end{cases} \leq$ $\chi_f(K_{k_1} \times K_{k_2}) \leq \psi_f(K_{k_1} \times K_{k_2}) \leq \begin{cases} \psi_f(K_{k_1} \times G_2) \\ \psi_f(G_1 \times K_{k_2}) \end{cases} \leq \psi_f(G_1 \times G_2)$.*

Dunbar, et al. in [1] showed that for each $m, n \in \mathbb{N} \setminus \{1\}$, $\mathrm{Fall}(K_m \times K_n) = \{m, n\}$. They also showed that if $n \in \mathbb{N} \setminus \{1\}$ and for each $i \in [n]$, $r_i \in \mathbb{N} \setminus \{1\}$, then, $\{r_1, ..., r_n\} \subseteq \mathrm{Fall}(\times_{i=1}^n K_{r_i})$. They constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$ and asked for conditions for a finite and with more than two elements set $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$ for which $S \subsetneq \mathrm{Fall}(\times_{i=1}^n K_{r_i})$.

Theorem 8. *Let $n \geq 3$, $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$, $r_1 < r_2 < ... < r_n$ and S contain at least one even integer. Then, $S \subsetneq \mathrm{Fall}(\times_{i=1}^n K_{r_i})$, besides, $\mathrm{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n.*

Proof. There are five cases.

Case I) $r_1 = 2$: In this case, let $t \in \{r_1, ..., r_n\} \setminus \{r_1, r_n\}$. Consider $K_2 \times K_t \times K_{r_n}$. Let σ be a disarrangement of $[t]$ (a permutation σ of $[t]$ such that for each $i \in [t]$, $\sigma(i) \neq i$). Obviously, $\{\{(1, i, 1), (1, \sigma(i), 2), (2, i, 2), (2, \sigma(i), 1)\} | \ 1 \leq i \leq t\} \bigcup \{\{(x, y, z) | (x, y, z) \in K_2 \times K_t \times K_{r_n}, \ z = i\} | 3 \leq i \leq r_n\}$ is the set of color classes of a fall $(r_n + t - 2)$-coloring of $K_2 \times K_t \times K_{r_n}$. But $r_n + t - 2 > r_n$ and therefore, in this case, $\mathrm{Fall}(K_2 \times K_t \times K_{r_n})$ contains an integer greater than r_n.

Case II) $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that one of them is r_n and the other is r_s that $s \in \{1, ..., n-1\}$: Let $r_j \in \{r_1, ..., r_n\} \setminus \{r_s, r_n\}$. Consider $K_{r_s} \times K_{r_j} \times K_{r_n}$

and a disarrangement σ of $[r_j]$. For each t, $1 \leq t \leq r_j$, color the vertices $(1,t,1),(1,\sigma(t),2),(2,t,2)$ and $(2,\sigma(t),1)$ with the color t and color each other vertex (x,y,z) with the color $\lfloor\frac{x-1}{2}\rfloor(\frac{r_j r_n}{2})+\lfloor\frac{z-1}{2}\rfloor r_j+$ the color of $(x-2\lfloor\frac{x-1}{2}\rfloor,y,z-2\lfloor\frac{z-1}{2}\rfloor)$. This is a fall $\frac{r_s r_j r_n}{4}$-coloring of $K_{r_s} \times K_{r_j} \times K_{r_n}$. Since $2 < r_1$, $\frac{r_s r_j r_n}{4} > \max\{r_s, r_j, r_n\}$. Hence, Theorem 6 implies that $\text{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n.

Case III) $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that none of them is r_n : Similar to the Case II, $\text{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n.

Case IV) $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and r_n is even : In this case, consider $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ and a disarrangement σ of $[r_{n-1}]$. For each t, $1 \leq t \leq r_{n-1}$, color the vertices $(1,t,1),(1,\sigma(t),2),(2,t,2)$ and $(2,\sigma(t),1)$ with the color t and color each other vertex (x,y,z) of $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\lfloor\frac{x-1}{2}\rfloor(\frac{r_{n-1}r_n}{2})+\lfloor\frac{z-1}{2}\rfloor r_{n-1}+$ the color of $(x-2\lfloor\frac{x-1}{2}\rfloor,y,z-2\lfloor\frac{z-1}{2}\rfloor)$. Also, color each vertex (r_{n-2},y,z) of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\frac{(r_{n-2}-1)r_{n-1}r_n}{4}+1$. Therefore, a fall $(\frac{(r_{n-2}-1)r_{n-1}r_n}{4}+1)$-coloring of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ and also of $\times_{i=1}^n K_{r_i}$ yields. But, $\frac{(r_{n-2}-1)r_{n-1}r_n}{4}+1 > r_n$. Thus, $\text{Fall}(K_2 \times K_t \times K_{r_n})$ and therefore $\text{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n.

Case V) $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and r_n is odd : In this case, similar to the Case IV, $\text{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n.

Accordingly, in all cases, $\{r_1, ..., r_n\} \subsetneq \text{Fall}(\times_{i=1}^n K_{r_i})$. Besides, $\text{Fall}(\times_{i=1}^n K_{r_i})$ contains an integer greater than r_n. ∎

Even though Dunbar, et al. in [1] constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$, this theorem also shows that in the Corollary 7, the inequality $\max\{\psi_f(G_i)| \ i \in [n]\} \leq \psi_f(\times_{i=1}^n G_i)$ can be strict in many cases. But we conjecture that the inequality $\chi_f(\times_{i=1}^n G_i) \leq \min\{\chi_f(G_i)| \ i \in [n]\}$ is always an equality.

Conjecture 1. *For each $n \in \mathbb{N}$ and for each arbitrary graphs $G_1, ..., G_n$ such that for each $i \in [n]$, $\text{Fall}(G_i) \neq \emptyset$, the following equality holds.*

$$\chi_f(\times_{i=1}^n G_i) = \min\{\chi_f(G_i)| \ i \in [n]\}.$$

5 Fall colorings of some graph constructions

In this section, we study fall colorings of union of graphs and My-cielskian of graphs and then, we show that fall k-colorings of a graph are some proper k-colorings of some graphs in a specified set.

5.1 Fall colorings of union of graphs

Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, G_i be a graph. The graph $(\bigcup_{i=1}^{n} (\{i\} \times V(G_i)), \bigcup_{i=1}^{n} \{\{(i,x), (i,y)\} | \{x,y\} \in E(G_i)\})$ is called the union graph of $G_1, ..., G_n$ and is denoted by $\biguplus_{i=1}^{n} G_i$.

The following obvious proposition describes fall colorings of union of graphs.

Proposition 3. *Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, G_i be a graph. Then, the following three statements hold.*
1) If $\text{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, *then, for each $1 \leq i \leq n$, $\text{Fall}(G_i) \neq \emptyset$.*
2) $\text{Fall}(\biguplus_{i=1}^{n} G_i) = \bigcap_{i=1}^{n} \text{Fall}(G_i)$.
3) If $\text{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, *then, $\chi_f(\biguplus_{i=1}^{n} G_i) = \min \bigcap_{i=1}^{n} \text{Fall}(G_i)$ and $\psi_f(\biguplus_{i=1}^{n} G_i) = \max \bigcap_{i=1}^{n} \text{Fall}(G_i)$.*

Since any graph G is isomorphic to any union graph of all its connected components, the following corollary yields immediately.

Corollary 9. *Let G be a graph and G_i, $1 \leq i \leq n$, be all its connected components. Then, the following three statements hold.*
1) If $\text{Fall}(G) \neq \emptyset$, *then, for each $1 \leq i \leq n$, $\text{Fall}(G_i) \neq \emptyset$.*
2) $\text{Fall}(G) = \bigcap_{i=1}^{n} \text{Fall}(G_i)$.
3) If $\text{Fall}(G) \neq \emptyset$, *then, $\chi_f(G) = \min \bigcap_{i=1}^{n} \text{Fall}(G_i)$ and $\psi_f(G) = \max \bigcap_{i=1}^{n} \text{Fall}(G_i)$.*

5.2 Fall Colorings of Mycielskian of graphs

Let $G := (\{x_1, \ldots, x_n\}, E(G))$ be a graph. The Mycielskian of G (denoted by $M(G)$) is a graph with $2n+1$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ with edge set $E(G) \bigcup \{\{y_i, x_j\} | i, j \in [n], \{x_i, x_j\} \in E(G)\} \bigcup \{\{z, y_i\} | i \in [n]\}$.

For example, $M(K_2)$ is C_5. We know that $\text{Fall}(M(K_2)) = \text{Fall}(C_5) = \emptyset$. Now we prove that for each graph G, $\text{Fall}(M(G)) = \emptyset$.

Theorem 9. *For each graph G, $\mathrm{Fall}(M(G)) = \emptyset$.*

Proof. If $E(G) = \emptyset$, then, $M(G)$ has at least one isolated vertex and also at least one edge. Therefore, $\mathrm{Fall}(M(G)) = \emptyset$. Now we prove the theorem for the case $E(G) \neq \emptyset$. If $E(G) \neq \emptyset$ and $\mathrm{Fall}(M(G)) \neq \emptyset$, then, there exists a fall k-coloring f of $M(G)$ for some $k \in \mathbb{N}$. Since $E(G) \neq \emptyset$, there exists an integer $i_0 \in [n]$ such that $f(x_{i_0}) \neq f(z)$ and since for each $j \in [n]$, $f(y_j) \neq f(z)$ and f is a fall k-coloring, there exists $i_1 \in [n]$ such that $x_{i_1} \in N_G(x_{i_0})$ and $f(x_{i_1}) = f(z)$. Since for each $i \in [n]$ with $f(x_i) \neq f(z)$, $N(y_i) \setminus \{z\} \subseteq N(x_i)$, so $f(x_i) \in \{f(y_i), f(z)\}$, on the other hand, $f(x_i) \neq f(z)$, hence, $f(x_i) = f(y_i)$. This immediately shows that each color of $[k]$ appears on the neighborhood of y_{i_1}, which is a contradiction. Hence, $\mathrm{Fall}(M(G)) = \emptyset$.

∎

5.3 Restriction of fall t-colorings of a graph into some proper t-colorings of some graphs in a specified set

Now we prove that fall k-colorings of a graph are some proper k-colorings of some graphs in a specified set.

Let G be a graph and $1 \leq t \leq \delta(G)+1$ be a fixed natural number. For each $v \in V(G)$, choose $t-1$ arbitrary elements of $N_G(v)$ and join these $t-1$ vertices to each other and name the new graph H. Let $\widehat{G_t}$ be the set of all graphs H constructed like this.

Proposition 4. *For each $1 \leq t \leq \delta(G) + 1$, $t \in \mathrm{Fall}(G)$ iff $t \in \{\chi(H)|\ H \in \widehat{G_t}\}$. Specially, $\mathrm{Fall}(G) = \bigcup_{i=1}^{\delta(G)+1}(\{\chi(H)|\ H \in \widehat{G_i}\} \cap \{i\})$.*

Proof. Let $1 \leq t \leq \delta(G) + 1$. If $t \in \{\chi(H)|\ H \in \widehat{G_t}\}$, then, there exists a graph H in $\widehat{G_t}$ such that $\chi(H) = t$ and there exists a t-coloring f of H. This coloring f of H, is obviously a fall t-coloring of G and therefore, $t \in \mathrm{Fall}(G)$. Conversely, if $t \in \mathrm{Fall}(G)$, then, there exists a fall t-coloring g of G. For each $v \in V(G)$, there exist $t-1$ elements of $N_G(v)$ such that the set of their colors and the color of v is equal to $[t]$, join all of them to each other to construct a new graph T in $\widehat{G_t}$. The fall t-coloring g of G is obviously a t-coloring

of T, also $\omega(T) \geq t$, thus, $\chi(T) = t$ and $t \in \{\chi(H) | H \in \widehat{G_t}\}$. The second part of the proposition follows immediately.

∎

Restricting this proposition into r-regular graphs and $t = r + 1$, yields a beautiful proposition of [2] but in different terminologies.

Proposition 5. *For each r-regular graph G, $r + 1 \in \mathrm{Fall}(G)$ iff $\chi(G^{(2)}) = r + 1$, where $G^{(2)} = (V(G), \{\{x, y\} | x, y \in V(G), x \neq y, d_G(x, y) \leq 2\})$.*

6 Fall colorings of complement of bipartite graphs

Complement of bipartite graphs are very interesting graphs, because in each proper k-coloring, the cardinality of each color class is at most 2. The following theorem characterizes all fall colorings of this type of graphs.

Theorem 10. *Let G be a bipartite graph. Then, $\mathrm{Fall}(G^c) \subseteq \{\chi(G^c)\}$. Besides, it is polynomial time to decide whether $\mathrm{Fall}(G^c) = \{\chi(G^c)\}$ or not.*

Proof. Let A and B be the partite sets of G. If $\mathrm{Fall}(G^c) \neq \emptyset$, then, there exists some k belong to $\mathrm{Fall}(G^c)$. Suppose that f is a fall k-coloring of G^c. Obviously, each color class of f is either of the form $\{x\}$ or of the form $\{y, z\}$ such that $y \in A$ and $z \in B$. A color class is of the form $\{x\}$ iff x is an isolated vertex of the graph G. Therefore, the set of color classes of f is the union of $\{\{x\} | x \in V(G), deg_G(x) = 0\}$ and the set of edges of a perfect matching of the induced subgraph of G on $\{x | x \in V(G), deg_G(x) > 0\}$, also, $k = |V(G)| - \frac{1}{2}|\{x | x \in V(G), deg_G(x) > 0\}|$. Therefore, $\mathrm{Fall}(G^c) \subseteq \{|V(G)| - \frac{1}{2}|\{x | x \in V(G), deg_G(x) > 0\}|\}$. Besides, if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of G on $\{x | x \in V(G), deg_G(x) > 0\}$ has a perfect matching, in this case, obviously, $\chi(G^c) = |V(G)| - \frac{1}{2}|\{x | x \in V(G), deg_G(x) > 0\}|$, and consequently, $\mathrm{Fall}(G^c) = \{\chi(G^c)\}$. Therefore, for each bipartite graph G, $\mathrm{Fall}(G^c) \subseteq \{\chi(G^c)\}$. We know that if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of G on

$\{x|\ x \in V(G),\ deg_G(x) > 0\}$ has a perfect matching. Conversely, if the induced subgraph of G on $\{x|\ x \in V(G),\ deg_G(x) > 0\}$ has a perfect matching, then, the union of $\{\{x\}|\ x \in V(G),\ deg_G(x) = 0\}$ and the edge set of each perfect matching of the induced subgraph of G on $\{x|\ x \in V(G),\ deg_G(x) > 0\}$ is the set of color classes of a fall $(|V(G)| - \frac{1}{2}|\{x|\ x \in V(G),\ deg_G(x) > 0\}|)$-coloring of G^c and therefore, $\text{Fall}(G^c) \neq \emptyset$. Accordingly, $\text{Fall}(G^c) = \{\chi(G^c)\}$ iff $\text{Fall}(G^c) \neq \emptyset$ iff the induced subgraph of G on $\{x|\ x \in V(G),\ deg_G(x) > 0\}$ has a perfect matching. Since the problem of determining whether or not the induced subgraph of G on $\{x|\ x \in V(G),\ deg_G(x) > 0\}$ has a perfect matching, is a polynomial time problem, thus, it is polynomial time to decide whether $\text{Fall}(G^c) = \{\chi(G^c)\}$ or not. ∎

Acknowledgements

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