q-HARMONIC SUM IDENTITIES WITH MULTI-BINOMIAL COEFFICIENT

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ABSTRACT. By the partial fraction decomposition method, we establish a q-harmonic sum identity with multi-binomial coefficient, from which we can derive a fair number of harmonic number identities.

In 1998, Ahlgren et al [1] proved the following beautiful identity

$$\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{n}^2\Big\{1+2k\mathrm{H}_{n+k}+2k\mathrm{H}_{n-k}-4k\mathrm{H}_k\Big\}=0, \tag{1}$$

where $\mathrm{H}_0=0, \mathrm{H}_n=\sum_{k=1}^{n}\frac{1}{k}$ for $n\in N$. This identity was used by Ahlgren and Ono [2] in proving the Apéry number supercongruence. In [4], Chu gave a simple proof by means of the partial fraction decomposition. Using the same method, Zheng [10] gave its q-generalization. Furthermore, McCarthy [7] derived its two binomial coefficient-generalizations, which are a key ingredient in the proofs of numerous supercongruences [8, 9]. Recently, Mansour et al [6] presented the q-analog of McCarthy's result by means of q-partial fractions.

In this paper, inspired by the work in [6, 10], we consider a q-harmonic sum identity with multi-binomial coefficient, from which we can derive a number of binomial coefficient-harmonic sum identities.

Here, we adopt the standard notation, respectively,

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n=\prod_{k=0}^{n-1}(1-xq^k) \quad \text{for} \quad n\in\mathbb{N},$$

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$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{1+n-k};q)_k}{(q;q)_k} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, D_x = \frac{d}{dx}$$

for the q-shifted factorial, Gaussian binomial coefficient and the derivative operator with respect to x. We further define two generalized q-harmonic numbers [3] by

$$H_0^{(\ell)} = 0 \quad \text{and} \quad H_n^{(\ell)} = \sum_{k=1}^{n} \frac{1}{(1-q^k)^\ell} \quad \text{for} \quad \ell,\, n \in \mathbb{N};$$

$$\tilde{H}_0^{(\ell)} = 0 \quad \text{and} \quad \tilde{H}_n^{(\ell)} = \sum_{k=1}^{n} \frac{q^{k\ell}}{(1-q^k)^\ell} \quad \text{for} \quad \ell,\, n \in \mathbb{N}.$$

When $m = 1$, we shall write $H_n := H_n^{(1)}$ and $\tilde{H}_n := \tilde{H}_n^{(1)}$.

In order to proceed smoothly, we fix the following rational function

$$h(x) = \prod_{i=1}^{\lambda} \frac{x^{n_i}(q/x;q)_{n_i}}{(q;q)_{n_i}};$$

$$\tilde{h}(x) = \prod_{i=1}^{\lambda} \frac{(q;q)_{n_i}}{(x;q)_{n_i+1}} \times (1-xq^k)^\lambda, 0 \le k \le n_1;$$

$$\bar{h}(x) = \prod_{i=1}^{\lambda} \frac{(q;q)_{n_i}}{(x;q)_{n_i+1}} \times (1-xq^k)^{\lambda-j}, n_j < k \le n_{j+1}, j = 1, \cdots, \lambda-1,$$

where λ, $n_1, \cdots, n_\lambda \in N$ with $0 \le \theta \le \lambda - 1$ and $n_1 \le n_2 \cdots \le n_\lambda$. Applying the n-times logarithmic derivatives to them, we define further functions related to q-harmonic numbers

$$\mathcal{H}_\ell(x) = \sum_{t=1}^{\lambda}\sum_{i=1}^{n_t} \frac{1}{(x-q^i)^\ell} \Rightarrow \mathcal{H}_\ell(q^{-k}) = q^{k\ell}\Big\{\sum_{t=1}^{\lambda} H_{n_t+k}^{(\ell)} - \lambda H_k^{(\ell)}\Big\};$$

$$\tilde{\mathcal{H}}_\ell(x) = \sum_{t=1}^{\lambda}\sum_{\substack{i=0\\ i\neq k}}^{n_t} \frac{q^{i\ell}}{(1-xq^i)^\ell} \Rightarrow \tilde{\mathcal{H}}_\ell(q^{-k}) = q^{k\ell}\Big\{\sum_{t=1}^{\lambda} \tilde{H}_{n_t-k}^{(\ell)} + (-1)^\ell \lambda H_k^{(\ell)}\Big\};$$

$$\bar{\mathcal{H}}_\ell(x) = \sum_{t=1}^{j}\sum_{i=0}^{n_t} \frac{q^{i\ell}}{(1-xq^i)^\ell} + \sum_{t=j+1}^{\lambda}\sum_{\substack{i=0\\ i\neq k}}^{n_t} \frac{q^{i\ell}}{(1-xq^i)^\ell};$$

$$\Rightarrow \bar{\mathcal{H}}_\ell(q^{-k}) = q^{k\ell}\Big\{\sum_{t=j+1}^{\lambda} \tilde{H}_{n_t-k}^{(\ell)} + (-1)^\ell\big(\lambda H_k^{(\ell)} - \sum_{t=1}^{j} H_{k-n_t-1}^{(\ell)}\big)\Big\}.$$

Now we are ready to state our main result.

Theorem 1. *Let λ, θ and $n_1, \cdots, n_\lambda$ be natural numbers with $0 \leq \theta \leq \lambda - 1$ and $n_1 \leq n_2 \cdots \leq n_\lambda$. There holds*

$$\begin{aligned}
&\prod_{i=1}^{\lambda} \frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}} \times (1-x)^\theta \\
&= \sum_{k=0}^{n_1} (-1)^{k\lambda} q^{\lambda\binom{k+1}{2}} \prod_{i=1}^{\lambda} \left[\begin{bmatrix} n_i \\ k \end{bmatrix} \begin{bmatrix} n_i + k \\ k \end{bmatrix} \right] \sum_{\ell=0}^{\lambda-1} \frac{(-1)^\ell q^{-k\ell} \Omega_\ell(\lambda, \theta, q^{-k})}{\ell!(1-xq^k)^{\lambda-\ell}} \\
&+ \sum_{j=1}^{\lambda-1} \sum_{k=n_j+1}^{n_{j+1}} (-1)^{\sum_{s=1}^{j} n_s + (\lambda-j)k} q^{(\lambda-j)\binom{k+1}{2} - \sum_{s=1}^{j}\binom{n_s+1}{2} - k\sum_{i=j+1}^{\lambda} n_s} \\
&\times \frac{\prod_{i=j+1}^{\lambda} \begin{bmatrix} n_i \\ k \end{bmatrix} \prod_{i=1}^{\lambda} \begin{bmatrix} n_i + k \\ k \end{bmatrix}}{(1-q^{-k})^j \prod_{i=1}^{j} \begin{bmatrix} k-1 \\ n_i \end{bmatrix}} \sum_{\ell=j}^{\lambda-1} \frac{(-1)^{\ell-j} q^{-k(\ell-j)} \tilde{\Omega}_{\ell-j}(\lambda, \theta, q^{-k})}{(\ell-j)!(1-xq^k)^{\lambda-\ell}},
\end{aligned}$$

where the Ω, $\tilde{\Omega}$-coefficients are determined by

$$\Omega_\ell(\lambda, \theta, x) := \frac{\mathcal{D}_x^\ell \{h(x)\tilde{h}(x)(1-x)^\theta\}}{h(x)\tilde{h}(x)} \tag{2a}$$

$$= \ell!(1-x)^{\theta-\ell} \sum_{\sigma(\ell)} \prod_{i=1}^{\ell} \frac{\{(1-x)^i[\tilde{\mathcal{H}}_i(x) - (-1)^i \mathcal{H}_i(x)] - \theta\}^{m_i}}{m_i! i^{m_i}}, \tag{2b}$$

$$\tilde{\Omega}_\ell(\lambda, \theta, x) := \frac{\mathcal{D}_x^\ell \{h(x)\bar{h}(x)(1-x)^\theta\}}{h(x)\bar{h}(x)} \tag{2c}$$

$$= \ell!(1-x)^{\theta-\ell} \sum_{\sigma(\ell)} \prod_{i=1}^{\ell} \frac{\{(1-x)^i[\bar{\mathcal{H}}_i(x) - (-1)^i \mathcal{H}_i(x)] - \theta\}^{m_i}}{m_i! i^{m_i}}; \tag{2d}$$

and the multiple sum runs over $\sigma(\ell)$such that $\sum_{i=1}^{\ell} i m_i = \ell$.

Proof. Using partial fraction decomposition, we may write

$$\begin{aligned}
\prod_{i=1}^{\lambda} \frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}} (1-x)^\theta &= \sum_{\ell=0}^{\lambda-1} \sum_{k=0}^{n_1} \frac{A(k,\ell)}{(1-xq^k)^{\lambda-\ell}} \\
&+ \sum_{j=1}^{\lambda-1} \sum_{k=n_j+1}^{n_{j+1}} \sum_{\ell=j}^{\lambda-1} \frac{B(j,k,\ell)}{(1-xq^k)^{\lambda-\ell}},
\end{aligned}$$

where the coefficients $A(k,\ell)$, $B(j,k,\ell)$ can be isolated. Noting

$$h(q^{-k}) = q^{-k\sum_{s=1}^{\lambda} n_s} \prod_{i=1}^{\lambda} \begin{bmatrix} n_i + k \\ k \end{bmatrix}, \bar{h}(q^{-k}) = (-1)^{k\lambda} q^{\lambda\binom{k+1}{2}} \prod_{i=1}^{\lambda} \begin{bmatrix} n_i \\ k \end{bmatrix},$$

$$\bar{h}(q^{-k}) = (-1)^{\sum_{s=1}^{j} n_s + (\lambda-j)k} q^{(\lambda-j)\binom{k+1}{2} + \sum_{s=1}^{j}(kn_s - \binom{n_s+1}{2}))} \frac{\prod_{i=j+1}^{\lambda} \begin{bmatrix} n_i \\ k \end{bmatrix}}{(1-q^{-k})^j \prod_{i=1}^{j} \begin{bmatrix} k-1 \\ n_i \end{bmatrix}},$$

we need only to check that

$$A(k,\ell) = (-1)^{\ell} q^{-k\ell} h(q^{-k}) \tilde{h}(q^{-k}) \frac{\Omega_\ell(\lambda,\theta,q^{-k})}{\ell!}; \tag{3a}$$

$$B(j,k,\ell) = (-1)^{\ell-j} q^{-k(\ell-j)} h(q^{-k}) \bar{h}(q^{-k}) \frac{\tilde{\Omega}_{\ell-j}(\lambda,\theta,q^{-k})}{(\ell-j)!}. \tag{3b}$$

We prove them by the induction principle. We prove (3a) first. For $\ell = 0$, noting $\Omega_0(\lambda,\theta,x) = (1-x)^\theta$, we have

$$A(k,0) = \lim_{x \to q^{-k}} h(x)\tilde{h}(x)(1-x)^\theta = h(q^{-k})\tilde{h}(q^{-k}) \times \Omega_0(\lambda,\theta,q^{-k}).$$

Next for $\ell = 1$, by L'Hôspital rule, we get

$$A(k,1) = \lim_{x \to q^{-k}} (1-xq^k)^{\lambda-1} \left\{ \prod_{i=1}^{\lambda} \frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}} (1-x)^\theta - \frac{A(k,0)}{(1-xq^k)^\lambda} \right\}$$

$$= \lim_{x \to q^{-k}} \frac{h(x)\tilde{h}(x)(1-x)^\theta - A(k,0)}{1-xq^k} = -q^{-k} \lim_{x \to q^{-k}} \mathcal{D}_x\{h(x)\tilde{h}(x)(1-x)^\theta\}$$

$$= -q^{-k} h(q^{-k})\tilde{h}(q^{-k}) \times \Omega_1(\lambda,\theta,q^{-k}).$$

Suppose $A(k,\ell) = (-1)^\ell q^{-k\ell} h(q^{-k})\tilde{h}(q^{-k}) \frac{\Omega_\ell(\lambda,\theta,q^{-k})}{\ell!}$ is true for $\ell = 0, \cdots, m-1$ with $m < \lambda$. Then we verify it also for $\ell = m$. Applying the L'Hôspital rule for m-times, we have

$$A(k,m)$$

$$= \lim_{x \to q^{-k}} (1-xq^k)^{\lambda-m} \left\{ \prod_{i=1}^{\lambda} \frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}} (1-x)^\theta - \sum_{\ell=0}^{m-1} \frac{A(k,\ell)}{(1-xq^k)^{\lambda-\ell}} \right\}$$

$$= \lim_{x \to q^{-k}} \frac{1}{(1-xq^k)^m} \left\{ h(x)\tilde{h}(x)(1-x)^\theta - \sum_{\ell=0}^{m-1} A(k,\ell) \times (x+k)^\ell \right\}$$

$$= (-1)^m q^{-km} \lim_{x \to q^{-k}} h(x)\tilde{h}(x) \frac{\mathcal{D}_x^m\{h(x)\tilde{h}(x)(1-x)^\theta\}}{m! h(x)\tilde{h}(x)}$$

$$= (-1)^m q^{-km} h(q^{-k})\tilde{h}(q^{-k}) \times \frac{\Omega_m(\lambda,\theta,q^{-k})}{m!}.$$

Similarly, we prove (3b). When $n_j < k \le n_{j+1}, j = 1, \cdots, \lambda-1$, for $\ell = j$, we have $\tilde{\Omega}_0(\lambda,\theta,x) = (1-x)^\theta$, therefore,

$$B(j,k,j) = \lim_{x \to q^{-k}} h(x)\bar{h}(x)(1-x)^\theta = h(q^{-k})\bar{h}(q^{-k}) \times \tilde{\Omega}_0(\lambda,\theta,q^{-k}).$$

Next for $\ell = j+1$,

$$
\begin{aligned}
& B(j,k,j+1) \\
= & \lim_{x\to q^{-k}} (1-xq^k)^{\lambda-j-1}\left\{\prod_{i=1}^{\lambda}\frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}}(1-x)^\theta - \frac{B(j,k,j)}{(1-xq^k)^{\lambda-j}}\right\} \\
= & \lim_{x\to q^{-k}} \frac{h(x)\bar{h}(x)(1-x)^\theta - B(j,k,j)}{1-xq^k} \\
= & -q^{-k}\lim_{x\to q^{-k}} \mathcal{D}_x\{h(x)\bar{h}(x)(1-x)^\theta\} \\
= & -q^{-k}h(q^{-k})\bar{h}(q^{-k})\times\tilde{\Omega}_1(\lambda,\theta,q^{-k}).
\end{aligned}
$$

Suppose $B(j,k,\ell) = (-1)^{\ell-j}q^{-k(\ell-j)}h(q^{-k})\bar{h}(q^{-k})\frac{\tilde{\Omega}_{\ell-j}(\lambda,\theta,q^{-k})}{(\ell-j)!}$ is true for $\ell = j, j+1, \cdots, m-1$ with $m<\lambda$. Then we verify it for $\ell = m$. Applying the L'Hôspital rule for m-times, we derive

$$
\begin{aligned}
& B(j,k,m) \\
= & \lim_{x\to q^{-k}} (1-xq^k)^{\lambda-m}\left\{\prod_{i=1}^{\lambda}\frac{x^{n_i}(q/x;q)_{n_i}}{(x;q)_{n_i+1}}(1-x)^\theta - \sum_{\ell=j}^{m-1}\frac{B(j,k,\ell)}{(1-xq^k)^{\lambda-\ell}}\right\} \\
= & \lim_{x\to q^{-k}} \frac{1}{(1-xq^k)^{m-j}}\left\{h(x)\bar{h}(x)(1-x)^\theta - \sum_{\ell=0}^{m-j-1} B(j,k,\ell+j)(1-xq^k)^\ell\right\} \\
= & (-1)^{m-j}q^{-k(m-j)}\lim_{x\to q^{-k}} h(x)\bar{h}(x)\frac{\mathcal{D}_x^{m-j}\{h(x)\bar{h}(x)(1-x)^\theta\}}{(m-j)!h(x)\bar{h}(x)} \\
= & (-1)^{m-j}q^{-k(m-j)}h(q^{-k})\bar{h}(q^{-k})\frac{\tilde{\Omega}_{m-j}(\lambda,\theta,q^{-k})}{(m-j)!}.
\end{aligned}
$$

We just need to show that these coefficients can be calculated explicitly through equation (2a-2d). Specifying the function in Faà di Bruno formula [5, P. 139] with $\phi(y) = e^y$ and $f(x) = \ln\{h(x)\bar{h}(x)(1-x)^\theta\}$, $\bar{f}(x) = \ln\{h(x)\bar{h}(x)(1-x)^\theta\}$, we derive their derivatives

$$
\begin{aligned}
\frac{D_y^m\phi(y)}{\phi(y)} &= 1, \\
D_x^k f(x) &= (k-1)!(1-x)^{-k}\{(1-x)^k[\bar{\mathcal{H}}_k(x) - (-1)^k\mathcal{H}_k(x)] - \theta\}, \\
D_x^k \bar{f}(x) &= (k-1)!(1-x)^{-k}\{(1-x)^k[\bar{\mathcal{H}}_k(x) - (-1)^k\mathcal{H}_k(x)] - \theta\},
\end{aligned}
$$

as well as the partial Bell polynomials

$$
B_{m,\ell}(f) = \ell!(1-x)^{-\ell}\sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\{(1-x)^i[\bar{\mathcal{H}}_i(x) - (-1)^i\mathcal{H}_i(x)] - \theta\}^{m_i}}{m_i!i^{m_i}},
$$

$$B_{m,\ell}(\tilde{f}) = \ell!(1-x)^{-\ell}\sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\{(1-x)^i[\bar{\mathcal{H}}_i(x)-(-1)^i\mathcal{H}_i(x)]-\theta\}^{m_i}}{m_i!\,i^{m_i}},$$

which lead us to (2a-2d). We complete the proof.

Below, we display several examples as applications of Theorem 1.

When $\lambda = 2$, Theorem 1 reduces to the following result.

Corollary 2 ([6, Thm 2.2]: $\theta = 1$). *Let n, m, θ be natural numbers with $0 \le \theta \le 1$ and $n \le m$. There holds*

$$\frac{x^{m+n}(q/x;q)_n(q/x;q)_m(1-x)^\theta}{(x;q)_{n+1}(x;q)_{m+1}}$$
$$= \sum_{k=0}^{n} q^{2\binom{k+1}{2}-(m+n)k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m+k \\ k \end{bmatrix}\Big\{\frac{(1-q^{-k})^\theta}{(1-xq^k)^2}$$
$$+ \frac{(1-q^{-k})^{\theta-1}}{1-xq^k}[q^{-k}\theta+(1-q^{-k})(4H_k-H_{n+k}-H_{m+k}-\tilde{H}_{n-k}-\tilde{H}_{m-k})]\Big\}$$
$$+ \sum_{k=n+1}^{m}(-1)^{k+n}q^{\binom{k+1}{2}-mk-\binom{n+1}{2}}\frac{\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m+k \\ k \end{bmatrix}}{\begin{bmatrix} k-1 \\ n \end{bmatrix}}\frac{(1-q^{-k})^{\theta-1}}{1-xq^k}.$$

When $m = n$ and $\theta = 1$, the last identity reduces to [10, Thm 1].

Multiplying by x on the both sides and letting $x \to \infty$ in Corollary 2, we have

Corollary 3 (q-analog of [7, Thm 2]: $\theta = 1$).

$$\sum_{k=0}^{n} q^{k^2-(m+n)k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m+k \\ k \end{bmatrix}(1-q^{-k})^{\theta-1}$$
$$\times\ [q^{-k}\theta+(1-q^{-k})(4H_k-H_{n+k}-H_{m+k}-\tilde{H}_{n-k}-\tilde{H}_{m-k})]\Big\}$$
$$+ \sum_{k=n+1}^{m}(-1)^{k+n}q^{\binom{k}{2}-mk-\binom{n+1}{2}}\frac{\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} m+k \\ k \end{bmatrix}}{\begin{bmatrix} k-1 \\ n \end{bmatrix}}(1-q^{-k})^{\theta-1}$$
$$= \begin{cases} 0, & \theta = 0, \\ (-1)^{m+n}q^{-\binom{n+1}{2}-\binom{m+1}{2}}, & \theta = 1. \end{cases}$$

Letting $x \to 1$ and using the L'Hôspital rule in Corollary 2, we derive

Corollary 4 ($\theta = 1$).

$$\sum_{k=0}^{n} q^{k^2-(m+n)k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} \begin{pmatrix} 4H_k - H_{n+k} - H_{m+k} \\ -\tilde{H}_{n-k} - \tilde{H}_{m-k} \end{pmatrix}$$

$$+ \sum_{k=n+1}^{m} (-1)^{k+n} q^{\binom{k+1}{2}-mk-\binom{n+1}{2}} \frac{\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix}}{\begin{bmatrix} k-1 \\ n \end{bmatrix} (1-q^k)} = 0;$$

$$\sum_{k=0}^{n} q^{k^2-2nk} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 (2H_k - H_{n+k} - \tilde{H}_{n-k}) = 0.$$

Corollary 5 ($\theta = 0$).

$$\sum_{k=1}^{n} q^{2\binom{k+1}{2}-(m+n)k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix}$$

$$\times \left\{ \frac{H_{n+k} + H_{m+k} + \tilde{H}_{n-k} + \tilde{H}_{m-k} - 4H_k}{1-q^k} - \frac{1}{(1-q^k)^2} \right\}$$

$$+ \sum_{k=n+1}^{m} (-1)^{k+n} q^{\binom{k+1}{2}+k-mk-\binom{n+1}{2}} \frac{\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix}}{\begin{bmatrix} k-1 \\ n \end{bmatrix} (1-q^k)^2}$$

$$= \frac{1}{2} \Big[H_n^{(2)} + H_m^{(2)} - \tilde{H}_n^{(2)} - \tilde{H}_m^{(2)} - (H_n + H_m + \tilde{H}_n + \tilde{H}_m)^2 \Big];$$

$$\sum_{k=1}^{n} q^{2\binom{k+1}{2}-2nk} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \left\{ \frac{2H_{n+k} + 2\tilde{H}_{n-k} - 4H_k}{1-q^k} - \frac{1}{(1-q^k)^2} \right\}$$

$$= H_n^{(2)} - \tilde{H}_n^{(2)} - 2(H_n + \tilde{H}_n)^2.$$

When $\lambda = 3$, Theorem 1 reduces to the following result.

Corollary 6. *Let n, m, ℓ, θ be natural numbers with $0 \le \theta \le 2$ and $n \le m \le \ell$. There holds*

$$\frac{x^{m+n+\ell} (q/x;q)_n (q/x;q)_m (q/x;q)_\ell (1-x)^\theta}{(x;q)_{n+1} (x;q)_{m+1} (x;q)_{\ell+1}}$$

$$= \sum_{k=0}^{n} (-1)^k q^{3\binom{k+1}{2}-(m+n+\ell)k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} \ell \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} m+k \\ k \end{bmatrix} \begin{bmatrix} \ell+k \\ k \end{bmatrix}$$

$$\times \left\{ \frac{(1-q^{-k})^\theta}{(1-xq^k)^3} + \frac{(1-q^{-k})^{\theta-1}}{(1-xq^k)^2} \begin{pmatrix} q^{-k}\theta + (1-q^{-k})(6H_k - H_{n+k} - H_{m+k} - H_{\ell+k} \\ -\tilde{H}_{n-k} - \tilde{H}_{m-k} - \tilde{H}_{\ell-k})) \end{pmatrix} \right\}$$

$$
\begin{aligned}
&+ \frac{(1-q^{-k})^{\theta-2}}{2(1-xq^{k})}\begin{pmatrix}[q^{-k}\theta+(1-q^{-k})(6H_k-H_{n+k}-H_{m+k}-H_{\ell+k}-\tilde{H}_{n-k}-\tilde{H}_{m-k}-\tilde{H}_{\ell-k})]^2\\ -q^{-2k}\theta+(1-q^{-k})^2(6H_k^{(2)}-H_{n+k}^{(2)}-H_{m+k}^{(2)}-H_{\ell+k}^{(2)}+\tilde{H}_{n-k}^{(2)}+\tilde{H}_{m-k}^{(2)}+\tilde{H}_{\ell-k}^{(2)})\end{pmatrix}\Bigg\}\\
&+ \sum_{k=n+1}^{m}(-1)^n q^{2\binom{k+1}{2}-(m+\ell)k-\binom{n+1}{2}}\frac{\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}\ell\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}m+k\\k\end{bmatrix}\begin{bmatrix}\ell+k\\k\end{bmatrix}}{\begin{bmatrix}k-1\\n\end{bmatrix}}\\
&\times\left\{\frac{(1-q^{-k})^{\theta-1}}{(1-xq^{k})^2}+\frac{(1-q^{-k})^{\theta-2}}{1-xq^{k}}\begin{pmatrix}q^{-k}\theta+(1-q^{-k})(6H_k-H_{n+k}-H_{m+k}-H_{\ell+k}\\ -\tilde{H}_{k-n-1}-\tilde{H}_{m-k}-\tilde{H}_{\ell-k}))\end{pmatrix}\right\}\\
&+ \sum_{k=m+1}^{\ell}(-1)^{k+n+m}q^{\binom{k+1}{2}-\ell k-\binom{n+1}{2}-\binom{m+1}{2}}\frac{\begin{bmatrix}\ell\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}m+k\\k\end{bmatrix}\begin{bmatrix}\ell+k\\k\end{bmatrix}}{\begin{bmatrix}k-1\\n\end{bmatrix}\begin{bmatrix}k-1\\m\end{bmatrix}\frac{1-xq^k}{(1-q^{-k})^{\theta-2}}}.
\end{aligned}
$$

Multiplying by x on the both sides and letting $x \to \infty$ in Corollary 6, we have

Corollary 7.

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n}(-1)^k q^{3\binom{k+1}{2}-(m+n+\ell+1)k}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}\ell\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}m+k\\k\end{bmatrix}\begin{bmatrix}\ell+k\\k\end{bmatrix}\\
&\times (1-q^{-k})^{\theta-2}\begin{pmatrix}[q^{-k}\theta+(1-q^{-k})(6H_k-H_{n+k}-H_{m+k}-H_{\ell+k}-\tilde{H}_{n-k}-\tilde{H}_{m-k}-\tilde{H}_{\ell-k})]^2\\ -q^{-2k}\theta+(1-q^{-k})^2(6H_k^{(2)}-H_{n+k}^{(2)}-H_{m+k}^{(2)}-H_{\ell+k}^{(2)}+\tilde{H}_{n-k}^{(2)}+\tilde{H}_{m-k}^{(2)}+\tilde{H}_{\ell-k}^{(2)})\end{pmatrix}\\
&+ \sum_{k=n+1}^{m}(-1)^n q^{k^2-(m+\ell)k-\binom{n+1}{2}}\frac{\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}\ell\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}m+k\\k\end{bmatrix}\begin{bmatrix}\ell+k\\k\end{bmatrix}}{\begin{bmatrix}k-1\\n\end{bmatrix}}\\
&\times (1-q^{-k})^{\theta-2}[q^{-k}\theta+(1-q^{-k})(6H_k-H_{n+k}-H_{m+k}-H_{\ell+k}-\tilde{H}_{k-n-1}-\tilde{H}_{m-k}-\tilde{H}_{\ell-k})]\\
&+ \sum_{k=m+1}^{\ell}(-1)^{k+n+m}q^{\binom{k}{2}-\ell k-\binom{n+1}{2}-\binom{m+1}{2}}\frac{\begin{bmatrix}\ell\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}m+k\\k\end{bmatrix}\begin{bmatrix}\ell+k\\k\end{bmatrix}}{\begin{bmatrix}k-1\\n\end{bmatrix}\begin{bmatrix}k-1\\m\end{bmatrix}(1-q^{-k})^{2-\theta}}\\
&= \begin{cases} 0, & \theta = 0,1,\\ (-1)^{n+m+\ell}q^{-\binom{n+1}{2}-\binom{m+1}{2}-\binom{\ell+1}{2}}, & \theta = 2.\end{cases}
\end{aligned}
$$

When $n = m = \ell$, the last identity yields

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{3\binom{k+1}{2}-(3n+1)k}\begin{bmatrix}n\\k\end{bmatrix}^3\begin{bmatrix}n+k\\k\end{bmatrix}^3(1-q^{-k})^{\theta-2}\\
&\times\begin{pmatrix}\left[q^{-k}\theta+3(1-q^{-k})(2H_k-\tilde{H}_{n-k}-H_{n+k})\right]^2\\ +3(1-q^{-k})^2(2H_k^{(2)}+\tilde{H}_{n-k}^{(2)}-H_{n+k}^{(2)})-q^{-2k}\theta\end{pmatrix} = \begin{cases} 0, & \theta = 0,1,\\ (-1)^n 2q^{-3\binom{n+1}{2}}, & \theta = 2.\end{cases}
\end{aligned}
$$

Letting $m = n = \ell$, $\theta = 2$, $x \to 1$ in Corollary 6 and using the L'Hôspital rule, we have

Corollary 8.

$$\sum_{k=0}^{n}(-1)^k q^{3\binom{k+1}{2}-(3n+2)k}\begin{bmatrix} n \\ k \end{bmatrix}^3\begin{bmatrix} n+k \\ k \end{bmatrix}^3\Big[2(2H_k-\bar{H}_{n-k}-H_{n+k})$$

$$-\ 3(1-q^k)(2H_k-\bar{H}_{n-k}-H_{n+k})^2-(1-q^k)(2H_k^{(2)}+\bar{H}^{(2)}_{n-k}-H^{(2)}_{n+k})\Big]=0$$

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