THE RADIO NUMBER OF $C_n \square C_n$

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ABSTRACT. Radio labeling is a variation of Hale's channel assignment problem, in which one seeks to assign positive integers to the vertices of a graph G subject to certain constraints involving the distances between the vertices. Specifically, a radio labeling of a connected graph G is a function $c : V(G) \to \mathbb{Z}_+$ such that

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(G)$$

for every two distinct vertices u and v of G (where $d(u, v)$ is the distance between u and v). The span of a radio labeling is the maximum integer assigned to a vertex. The radio number of a graph G is the minimum span, taken over all radio labelings of G. This paper establishes the radio number of the Cartesian product of a cycle graph with itself (i.e. of $C_n \square C_n$.)

1. INTRODUCTION

Radio labeling is derived from the assignment of radio frequencies (channels) to a set of transmitters. The frequencies assigned depend on the geographical distance between the transmitters: the closer two transmitters are, the greater the potential for interference between their signals. Thus when the distance between two transmitters is small, the difference in the frequencies assigned must be relatively large, whereas two transmitters at a large distance may be assigned frequencies with a small difference.

The use of graphs to model the "channel assignment" problem was first proposed by Hale in 1980 [5]; Chartrand et al introduced the variation known as radio labeling in 2001 [2].

In the graph model of the channel assignment problem, the vertices correspond to the transmitters, and graph distance plays the role of geographical distance. We assume all graphs are connected and simple. The *distance* between two vertices u and v of a graph G, $d(u, v)$, is the length of a shortest path between u and v. The *diameter* of G, $\text{diam}(G)$, is the maximum

distance, taken over all pairs of vertices of G. A *radio labeling* of a graph G is then defined to be a function $c : V(G) \to \mathbb{Z}_+$ satisfying

$$d(u,v) + |c(u) - c(v)| \geq 1 + \text{diam}(G)$$

for all distinct pairs of vertices $u, v \in V(G)$. The *span* of a radio labeling c is the maximum integer assigned by c. The *radio number* of a graph G, $rn(G)$, is the minumum span, taken over all radio labelings of G[1].

We focus on Cartesian products of cycles. We remind the reader that the cycle graph of order n, C_n, may be represented with vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1\}$. The diameter of C_n is $\lfloor \frac{n}{2} \rfloor$.

The Cartesian product of two graphs G and H has vertex set $V(G \Box H) = V(G) \times V(H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\}$. The edges of $G \Box H$ consist of those pairs of vertices $\{(g,h), (g',h')\}$ satisfying $g = g'$ and h is adjacent to h' in H or $h = h'$ and g is adjacent to g' in G. We note that $C_n \Box C_n$ has n^2 vertices, and $\text{diam}(C_n \Box C_n) = 2 \lfloor \frac{n}{2} \rfloor$.

As Liu and Zhu write, "It is surprising that determining the radio number seems a difficult problem even for some basic families of graphs." [9] In fact, as of this writing, the only families of graphs for which the radio number is known are paths and cycles [9] and the squares of paths and cycles [8, 7]; wheels and gears [3], and some generalized prisms [10]. Meanwhile, bounds for the radio numbers of trees [6], ladders [4], and square grids [1] have been identified, while the radio number of cubes of the cycles C_n^3 for $n \leq 20$ and $n \equiv 0$, 2, or 4 (mod 6) is known [11].

The main result of this paper establishes the radio number of the Cartesian product of the n-cycle with itself:

Main Theorem. *Let n be a nonnegative integer. Then*

$$rn(C_n \Box C_n) = \begin{cases} \frac{n^2-2}{2}(k+2) + 2, & \text{for } n = 2k, \\ \frac{n^2-1}{2}(k+1) + 1, & \text{for } n = 2k+1. \end{cases}$$

This is the first fully determined radio number for a family of graphs that is itself a Cartesian product of graphs. As such, it provides evidence for use in considering an interesting question: how is the radio number of a graph product related to the radio numbers of the factors? It is also possible that the labeling algorithms used to establish an upper bound for the radio number may be adapted to serve the same purpose for other toroidal graphs.

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[1]We use the convention, established in [2], that the co-domain of a radio labeling is $\mathbb{Z}_+ = \{1, 2, \ldots\}$. Some authors use $\{0, 1, 2, \ldots\}$ as the co-domain; radio numbers specified using the non-negative integers as co-domain are one less than those determined using the positive integers.

We prove the main theorem in two steps. First we provide the lower bound for $rn(C_n \square C_n)$ in Section 2. In Section 3, we define a radio labeling of $C_n \square C_n$; the span of this labeling is equal to the lower bound, thus establishing the radio number of $C_n \square C_n$. Finally, we return to the question of the relationship of $rn(G \square H)$ to $rn(G)$ and $rn(H)$ in Section 4, by examining $rn(C_n \square C_n)$ and $rn(C_n)^2$, as well as $rn(K_m \square K_n)$ and $rn(K_m) \cdot rn(K_n)$.

2. Lower Bound

The lower bound for $rn(C_n \square C_n)$ is reached in three steps. First we examine the maximum possible sum of the pairwise distances between any three vertices of $C_n \square C_n$. We use this maximum sum to establish a minimum possible "gap" between the ith and $(i+2)^{nd}$ largest labels. Using 1 for the smallest label and taking the size of the gap into account then provides a lower bound for the span of any labeling.

We provide the details of this approach for $C_{2k} \square C_{2k}$ in Lemmas 2.1 and 2.2 and Theorem 2.3. As the logic of the proofs of the corresponding results for $C_{2k+1} \square C_{2k+1}$ is identical, we leave the details of Lemma 2.4, Lemma 2.5, and Theorem 2.6 to the reader.

Lemma 2.1. *Let $u, v, w \in V(C_{2k} \square C_{2k})$. Then $d(u,v) + d(v,w) + d(u,w) \leq 2\,diam(C_{2k} \square C_{2k})$.*

Proof. Express u, v, and w via their component vertices, i.e., as $u = (x_1, y_1)$, $v = (x_2, y_2)$, and $w = (x_3, y_3)$, where x_i and y_i, $i = 1, 2, 3$ are all vertices of C_{2k}. Then

$$d(u,v) + d(v,w) + d(u,w)$$
$$= d\big((x_1,y_1),(x_2,y_2)\big) + d\big((x_2,y_2),(x_3,y_3)\big) + d\big((x_1,y_1),(x_3,y_3)\big)$$
$$= d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) + d(y_1,y_2) + d(y_2,y_3) + d(y_1,y_3).$$

In taking shortest paths between x_1, x_2, and x_3 (all in C_{2k}), one never need take more steps than those necessary to completely traverse C_{2k}, i.e.

$$d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) \leq 2k.$$

The same is true of the sum of the pairwise distances between vertices y_1, y_2, and y_3. Thus

$$d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) + d(y_1,y_2) + d(y_2,y_3) + d(y_1,y_3) \leq 4k.$$

As $4k = 2\,diam(C_{2k} \square C_{2k})$, this establishes the lemma. $\qquad\square$

We use this maximum possible sum of the pairwise distances between three vertices of $C_{2k} \square C_{2k}$ together with the radio condition to determine the minimum distance between every other label (arranged in increasing order) in a radio labeling of $C_{2k} \square C_{2k}$.

Lemma 2.2. *Let c be a radio labeling of $C_{2k}\Box C_{2k}$. Then for any three vertices $u, v, w \in V(C_{2k}\Box C_{2k})$ satisfying $c(u) < c(v) < c(w)$, we have $c(w) - c(u) \geq k + 2$.*

Proof. Since $c(u)$, $c(v)$ and $c(w)$ are radio labels,

$$d(u,v) + |c(v) - c(u)| \geq 1 + \text{diam}(C_{2k}\Box C_{2k}),$$
$$d(v,w) + |c(w) - c(v)| \geq 1 + \text{diam}(C_{2k}\Box C_{2k}), \text{ and}$$
$$d(u,w) + |c(w) - c(u)| \geq 1 + \text{diam}(C_{2k}\Box C_{2k}).$$

Summing these inequalities yields

$$d(u,v) + d(v,w) + d(u,w) + c(v) - c(u) + c(w) - c(v) + c(w) - c(u)$$
$$\geq 3 + 3\,\text{diam}(C_{2k}\Box C_{2k}).$$

Furthermore, by Lemma 2.1, $d(u,w)+d(v,w)+d(u,w) \leq 2\,\text{diam}(C_{2k}\Box C_{2k})$, so we have

$$2\,\text{diam}(C_{2k}\Box C_{2k}) + 2c(w) - 2c(u) \geq 3 + 3\,\text{diam}(C_{2k}\Box C_{2k}).$$

As $\text{diam}(C_{2k}\Box C_{2k}) = 2k$, it follows that

$$2(2k) + 2c(w) - 2c(u) \geq 3 + 3(2k)$$
$$c(w) - c(u) \geq \frac{3 + 2k}{2} = \frac{3}{2} + k.$$

As $c(w) - c(u)$ is an integer, we may conclude that $c(w) - c(u) \geq 2 + k$. $\quad\square$

Knowledge of the size of the minimum gap allowable between the values of every other label makes it possible to calculate the minimum possible span of a radio labeling of $C_{2k}\Box C_{2k}$.

Theorem 2.3. *For $n = 2k$, $rn(C_n \Box C_n) \geq \frac{n^2-2}{2}(k+2) + 2$.*

Proof. Let c be a radio labeling of $C_{2k}\Box C_{2k}$. Rename the vertices of $C_{2k}\Box C_{2k}$ using the set $\{x_1, x_2, \ldots, x_{(2k)^2}\}$ so that $c(x_i) < c(x_j)$ whenever $i < j$. Consider the lowest possible values of $c(x_i)$ for each i. We have $c(x_1) \geq 1$ and $c(x_2) \geq 2$. From Lemma 2.2 we know $c(x_3) \geq c(x_1) + k + 2$, and in general,

$$c(x_i) \geq \begin{cases} 1 + \frac{i-1}{2}(k+2), & \text{when } i \text{ is odd} \\ 2 + \frac{i-2}{2}(k+2), & \text{when } i \text{ is even.} \end{cases}$$

Thus $rn(C_{2k}\Box C_{2k}) \geq \text{span}(c) = c(x_{(2k)^2}) \geq 2 + \frac{(2k)^2-2}{2}(k+2) = \frac{n^2-2}{2}(k+2) + 2$. $\quad\square$

A lower bound for the radio number of $C_{2k+1}\Box C_{2k+1}$ may be obtained in much the same way as the lower bound for $C_{2k}\Box C_{2k}$.

Lemma 2.4. *Let* $u, v, w \in V(C_{2k+1} \square C_{2k+1})$. *Then*

$$d(u, v) + d(v, w) + d(u, w) \leq 2 \, diam(C_{2k+1} \square C_{2k+1}) + 2.$$

Proof. As in the proof of Lemma 2.1, we write the vertices of $C_{2k+1} \square C_{2k+1}$ via their components: $u = (x_1, y_1)$, $v = (x_2, y_2)$, and $w = (x_3, y_3)$. Here, however, the sum $d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3)$ may be as much as $2k+1$ (i.e., once around the cycle). So

$$d(u,v) + d(v, w) + d(u, w)$$
$$= d\big((x_1, y_1), (x_2, y_2)\big) + d\big((x_2, y_2), (x_3, y_3)\big) + d\big((x_1, y_1), (x_3, y_3)\big)$$
$$= d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3) + d(y_1, y_2) + d(y_2, y_3) + d(y_1, y_3)$$
$$\leq 2(2k+1)$$
$$= 2 \, \text{diam}(C_{2k+1} \square C_{2k+1}) + 2.$$

$\square$

Lemma 2.5. *Let c be a radio labeling of $C_{2k+1} \square C_{2k+1}$. Then for any three vertices $u, v, w \in V(C_{2k+1} \square C_{2k+1})$ satisfying $c(u) < c(v) < c(w)$, we have $c(w) - c(u) \geq k + 1$.*

The proof of Lemma 2.5 is analogous to that of Lemma 2.2, with the substitution of $2 \, \text{diam}(C_{2k+1} \square C_{2k+1}) + 2 = 4k + 2$ for $2 \, \text{diam}(C_{2k} \square C_{2k}) = 4k$.

Theorem 2.6. *For $n = 2k + 1$, $rn(C_n \square C_n) \geq \frac{n^2-1}{2}(k+1) + 1$.*

Proof. The proof is analogous to that of Theorem 2.3, with the substitution of $k + 1$ (from Lemma 2.5) for $k + 2$ (from Lemma 2.2). Now, for any radio labeling c of $C_{2k+1} \square C_{2k+1}$, we have

$$c(x_i) \geq \begin{cases} 1 + \frac{i-1}{2}(k+1), & \text{when } i \text{ is odd} \\ 2 + \frac{i-2}{2}(k+1), & \text{when } i \text{ is even.} \end{cases}$$

As $C_{2k+1} \square C_{2k+1}$ has $(2k+1)^2$ vertices, we conclude $rn(C_{2k+1} \square C_{2k+1}) \geq \text{span}(c) = c(x_{(2k+1)^2}) \geq 1 + \frac{(2k+1)^2 - 1}{2}(k+1) = \frac{n^2 - 1}{2}(k+1) + 1$. $\square$

3. UPPER BOUND

Our general approach to establishing the upper bound for $rn(C_n \square C_n)$ consists of three steps. After some preliminaries, we define a position function $p : \{0, 1, \ldots, n^2 - 1\} \rightarrow V(C_n \square C_n)$ and argue that p is a bijection. Defining $x_i = p(i)$ allows us to rename the vertices of $C_n \square C_n$ in what will be a useful way. Next we give a labeling $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \rightarrow \mathbb{Z}_+$ for which $c(x_0) < c(x_1) < \cdots < c(x_{n^2-1})$. We then prove that c is a radio labeling of $C_n \square C_n$. (The fact that $c(x_i) < c(x_j)$ when $i < j$ simplifies the proof that c is a radio labeling.) It follows that $rn(C_n \square C_n) \leq \text{span}(c)$.

Recall that any radio labeling c of G must satisfy the radio condition

$$d(u,v) + |c(u) - c(v)| \geq 1 + \text{diam}(G)$$

for all distinct vertices $u, v \in V(G)$. Once $|c(u) - c(v)| \geq \text{diam}(G)$, the radio condition is satisfied for u, v and for any pair of vertices with label difference at least as big as $|c(u) - c(v)|$. The next remark states this fact precisely, and will be of use in limiting the number of vertex pairs for which it must be verified that specific labelings satisfy the radio condition.

Remark 3.1. *Let $c : \{x_0, x_1, \ldots, x_{|V(G)|-1}\} \rightarrow \mathbb{Z}_+$ be a labeling of $G = (V, E)$ satisfying $c(x_0) < c(x_1) < \cdots < c(x_{|V(G)|-1})$. If $c(x_l) - c(x_k) \geq diam(G)$ for some $k < l$, then c satisfies the radio condition for all pairs of vertices x_i, x_j with $i \leq k$ and $j \geq l$.*

In preparation for defining the position function, we employ a common means of representing C_n: $V(C_n) = \{v \in \mathbb{Z} \mid 0 \leq v \leq n - 1\}$, with $v, w \in V(C_n)$ adjacent exactly when $v \equiv w \pm 1 \pmod{n}$. This then gives the expected representation of $V(C_n \square C_n)$ as $\{(v, w) \mid 0 \leq v, w \leq n - 1\}$. Distance between vertices of $C_n \square C_n$ are calculated as in Remark 3.2.

Remark 3.2. *Let $(v_{i_1}, w_{i_2}), (v_{j_1}, w_{j_2}) \in V(C_n \square C_n)$. Then*

$$d\big((v_{i_1}, w_{i_2}),(v_{j_1}, w_{j_2})\big)$$
$$= \min\{|i_1 - j_1|, n - |i_1 - j_1|\} + \min\{|i_2 - j_2|, n - |i_2 - j_2|\}.$$

In Section 2 we establish a lower bound for $rn(C_n \square C_n)$ that depends on the parity of n. The upper bound also depends on the parity of n; the next two theorems establish this upper bound.

Note: All calculations on vertices in pair notation are performed modulo n.

Theorem 3.3. *Let $n = 2k$. Then $rn(C_n \square C_n) \leq \frac{n^2-2}{2}(k + 2) + 2$.*

Proof. Define $p : \{0, 1, \ldots, n^2 - 1\} \rightarrow \{(v, w) \mid 0 \leq v, w \leq n - 1\}$ by

$$p(i) = \begin{cases} (r, kr + s), & \text{when } i \equiv 0 \pmod{4}, \\ (r + k, kr + s + k), & \text{when } i \equiv 1 \pmod{4}, \\ (r, kr + s + k), & \text{when } i \equiv 2 \pmod{4}, \\ (r + k, kr + s), & \text{when } i \equiv 3 \pmod{4}, \end{cases}$$

$$\text{where } r = \left\lfloor \frac{i}{2n} \right\rfloor \text{ and } s = \left\lfloor \frac{i}{4} \right\rfloor \pmod{k}.$$

Claim: p is a bijection. Consider the following possibilities for the relationship of the indices i and j for $i \neq j$:

(1) $i \not\equiv j \pmod{4}$; i and j have opposite parity,
(2) $i \not\equiv j \pmod{4}$; i and j have the same parity,

12

(3) $i \equiv j \pmod 4$; $\lfloor \frac{i}{2n} \rfloor \neq \lfloor \frac{j}{2n} \rfloor$, and

(4) $i \equiv j \pmod 4$; $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$.

In the first case, the first components of $p(i)$ and $p(j)$ agree exactly when $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor + k$. But $\lfloor \frac{i}{2n} \rfloor \leq \lfloor \frac{n^2-1}{2n} \rfloor = \frac{n-2}{2} < k$, so this is impossible.

Suppose $i \not\equiv j \pmod 4$ and i and j have the same parity. If $\lfloor \frac{i}{2n} \rfloor \neq \lfloor \frac{j}{2n} \rfloor$ then the first components of $p(i)$ and $p(j)$ are not equal. Should $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$, we assume WLOG that $j > i$ and j is congruent to 1 or 2 modulo 4. Examine the second component of $p(j) - p(i)$:

$$\left(k \left\lfloor \frac{j}{2n} \right\rfloor + \left\lfloor \frac{j}{4} \right\rfloor \pmod k \right) + k \right) - \left(k \left\lfloor \frac{i}{2n} \right\rfloor + \left\lfloor \frac{i}{4} \right\rfloor \pmod k \right).$$

This reduces to $k + \left(\lfloor \frac{i}{4} \rfloor - \lfloor \frac{i}{4} \rfloor \right) \pmod k$, which is not 0. So $p(i) \neq p(j)$.

In the third case, the first components of $p(i)$ and $p(j)$ will be the same only when they are equivalent $\pmod n$. But the possible values for $r = \lfloor \frac{i}{2n} \rfloor$ never reach n, so we may rule out this eventuality. Finally, in the fourth case, note that the hypotheses imply that $p(i) = p(j)$ if and only if $s_i = \lfloor \frac{i}{4} \rfloor \pmod k = s_j = \lfloor \frac{j}{4} \rfloor \pmod k$. But $|i - j| < 2n = 4k$, so $\left| \lfloor \frac{i}{4} \rfloor - \lfloor \frac{j}{4} \rfloor \right| < k$, thus $s_1 \neq s_2$. Therefore we may conclude that p is a bijection.

Again, we rename the vertices of $C_n \square C_n$ by agreeing that $p(i) = x_i$. The labeling is given by $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \mathbb{Z}_+$ by

$$c(x_i) = \begin{cases} 1 + \frac{i}{2}(k+2), & \text{when } i \text{ is even}, \\ 2 + \frac{i-1}{2}(k+2), & \text{when } i \text{ is odd}. \end{cases}$$

As $c(x_{i+4}) - c(x_i) > 2k = \text{diam}(C_n \square C_n)$ for all $i = 0, 1, \ldots, n^2 - 5$, we again apply Remark 3.1 to limit the vertex pairs for which we must verify that c is a radio labeling. This verification consists of two subcases.

Subcase 1: Consider first pairs of vertices $\{x_i, x_j\}$ with $|i - j| \leq 3$ and $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$. For $i_2 = i_1 + 4m$ and $j_2 = j_1 + 4m$. where m is an integer, we have $d(x_{i_1}, x_{j_1}) = d(x_{i_2}, x_{j_2})$ and $|c(x_{i_1}) - c(x_{j_1})| = |c(x_{i_2}) - c(x_{j_2})|$, so consideration of the pairs for which the distances and the label differences are shown in the tables below suffices. The first table gives the distances between vertices; the second gives the label differences.

	x_1	x_2	x_3	x_4	x_5
x_0	$2k$	k	k		
x_1		k	k	$2k-1$	
x_2			$2k$	$k-1$	$k+1$
x_3				$k+1$	$k-1$

	x_1	x_2	x_3	x_4	x_5
x_0	$2k$	k	k		
x_1		k	k	$2k-1$	
x_2			$2k$	$k-1$	$k+1$
x_3				$k+1$	$k+1$

Summing the corresponding entries from each table shows that the radio condition is satisfied in all cases.

13

Subcase 2: It remains only to verify that the radio condition holds for vertices with index differences less than four and indices near a multiple of $2n$. Specifically, we must calculate $d(u,v)+|c(u)-c(v)|$ for all vertices $\{u,v\}$ of the form $\{x_{an-3}, x_{an}\}$, $\{x_{an-2}, x_{an}\}$, $\{x_{an-2}, x_{an+1}\}$, $\{x_{an-1}, x_{an}\}$, $\{x_{an-1}, x_{an+1}\}$, and $\{x_{an-1}, x_{an+2}\}$, where a is an even integer. Note that taking a even gives $an = a2k \equiv 0 \pmod 4$. Also, as $n \geq 2$, we know that

$$r = \left\lfloor \frac{i}{2n} \right\rfloor = \begin{cases} \left\lfloor \frac{an-t}{2n} \right\rfloor = \frac{a}{2} - 1, & \text{for } i = an - t \text{ and } t = 1,2,3 \\ \left\lfloor \frac{an+t}{2n} \right\rfloor = \frac{a}{2}, & \text{for } i = an + t \text{ and } t = 0,1,2. \end{cases}$$

Also, as $s = \left\lfloor \frac{i}{4} \pmod k \right\rfloor$, we know

$$s = \begin{cases} \frac{ak}{2} - 1 \pmod k = k - 1, & \text{for } i = an - t \text{ and } t = 1,2,3 \\ \frac{ak}{2} \pmod k = 0, & \text{for } i = an + t \text{ and } t = 0,1,2. \end{cases}$$

Accordingly, we calculate the position functions and the label values for each vertex of interest, providing the results in the table below.

index i	vertex x_i	label $c(x_i)$
$an - 3$	$\left(\frac{a}{2} - 1 + k, k\left(\frac{a}{2} - 1\right) - 1\right)$	$2 + \frac{an-4}{2}(k+2)$
$an - 2$	$\left(\frac{a}{2} - 1, k\left(\frac{a}{2} - 1\right) - 1\right)$	$1 + \frac{an-2}{2}(k+2)$
$an - 1$	$\left(\frac{a}{2} - 1 + k, \frac{ka}{2} - 1\right)$	$2 + \frac{an-2}{2}(k+2)$
an	$\left(\frac{a}{2}, \frac{ka}{2}\right)$	$1 + \frac{an}{2}(k+2)$
$an + 1$	$\left(\frac{a}{2} + k, \frac{ka}{2} + k\right)$	$2 + \frac{an}{2}(k+2)$
$an + 2$	$\left(\frac{a}{2}, \frac{ka}{2} + k\right)$	$1 + \frac{an+2}{2}(k+2)$

This allows the calculation of the distance between vertices and the difference of the labels for each vertex pair in question.

vertex pair	distance	label difference	distance plus label difference
x_{an-3}, x_{an}		$2k + 3$	$> 2k + 3$
x_{an-2}, x_{an}	k	$k + 2$	$2k + 2$
x_{an-1}, x_{an}	k	$k + 1$	$2k + 3$
x_{an-2}, x_{an+1}	k	$k + 3$	$2k + 3$
x_{an-1}, x_{an+1}	k	$k + 2$	$2k + 2$
x_{an-1}, x_{an+2}		$2k + 3$	$> 2k + 3$

The radio condition is satisfied in all cases.

Finally, we compute the span of this radio labeling:

$$span(c) = c(x_{n^2-1}) = 2 + \frac{n^2 - 2}{2}(k+2).$$

$\square$

The proof of Theorem 3.4 has a similar structure to that of Theorem 3.3, but the position and labeling functions depend on the parity of k.

Theorem 3.4. *Suppose $n = 2k+1$. Then $rn(C_n \Box C_n) \leq \frac{n^2-1}{2}(k+1) + 1$.*

Proof. For each of k odd and k even we provide a radio labeling with span $\frac{n^2-1}{2}(k+1) + 1$.

Case 1: k is odd.

Define $p : \{0, 1, \ldots, n^2 - 1\} \to \{(v, w) \mid 0 \leq v, w \leq n - 1\}$ by

$$p(i) = \left(ik, r + i\left(\frac{k+1}{2}\right)\right), \quad \text{where } r = \left\lfloor \frac{i}{n} \right\rfloor.$$

We wish to show that p is a bijection. Suppose that $p(i) = p(j)$ for some $i \neq j$. Examining the first components of $p(i)$ and $p(j)$, we see that $ik \pmod{n} = jk \pmod{n}$, i.e., $ik - jk \equiv 0 \pmod{n}$. As k and n are relatively prime, $i - j \equiv 0 \pmod{n}$. The second components of $p(i)$ and $p(j)$ must also be equivalent $\pmod{n}$: this gives

$$0 \equiv \left(\left\lfloor \frac{i}{n} \right\rfloor + i\left(\frac{k+1}{2}\right)\right) - \left(\left\lfloor \frac{j}{n} \right\rfloor + j\left(\frac{k+1}{2}\right)\right) \pmod{n}$$

$$= \left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor + (i - j)\left(\frac{k+1}{2}\right) \pmod{n}$$

$$= \left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor \pmod{n}.$$

But $i \neq j$ and $i \equiv j \pmod{n}$ imply $\left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor \not\equiv 0 \pmod{n}$. Thus $p(i) \neq p(j)$ for distinct i, j in the domain of p, and we may conclude that p is a bijection.

We now use the elements of the set $\{x_0, x_1, \ldots, x_{n^2-1}\}$ to rename the vertices of $C_n \Box C_n$ by agreeing that $p(i) = x_i$. Define the labeling $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \mathbb{Z}_+$ by

$$c(x_i) = 1 + i\left(\frac{k+1}{2}\right).$$

Claim: The labeling c is a radio labeling of $rn(C_n \Box C_n)$.

To establish our claim we must show that c satisfies the radio condition

$$d(u, v) + |c(u) - c(v)| \geq \text{diam}(G) + 1 = 2k + 1$$

for all distinct $u, v \in V(C_n \Box C_n)$. Note that $c(x_{i+4}) - c(x_i) \geq \text{diam}(G) = 2k$ for all $i = 0, \ldots n^2 - 4$, so Remark 3.1 indicates that we need only verify that c satisfies the radio condition for vertex pairs x_i, x_{i+j} with $j \leq 3$.

We will examine first pairs of vertices with fixed r, i.e., vertices with indices in $\{an, an + 1, \ldots, an + n - 1\}$ for $a = 0, 1, \ldots n - 1$. Subsequently we will show that the radio condition is satisfied for vertices of the form x_i,

x_{i+j} where $\lfloor \frac{i}{n} \rfloor \neq \lfloor \frac{i+j}{n} \rfloor$ and $j \leq 3$. We will handle the case $n = 3$ $(k = 1)$ separately.

Subcase 1: Take $x_i, x_{i+j} \in \{an, an+1, \ldots, an+n-1\}$ for $a = 0, 1, \ldots n-1$. Assume $k > 1$. The distance between x_i and x_{i+j} is given by examining the position function p and using Remark 3.2.

vertex pair	$d(x_i, x_{i+j})$	$c(x_{i+j}) - c(x_i)$	$d(x_i, x_{i+j}) + |c(x_j) - c(x_i)|$
x_i, x_{i+1}	$k + \frac{k+1}{2}$	$\frac{k+1}{2}$	$2k + 1$
x_i, x_{i+2}	$1 + k$	$k + 1$	$2k + 2$
x_i, x_{i+3}	$k - 1 + \frac{k-1}{2}$	$\frac{3k+3}{2}$	$3k$

Each sum in the last column is at least $2k + 1$ (given $k > 1$), so this completes the argument that the radio condition is satisfied by c for all vertex pairs specified in this subcase.

Subcase 2: Consider x_i, x_{i+j} with $j \leq 3$ and $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i+j}{n} \rfloor - 1$, again assuming $k > 1$. Compare the calculation of $d(x_i, x_{i+j})$ here with the analogous calculation in Subcase 1. The new condition introduced here, that $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i}{n} \rfloor - 1$, may change the distance by ± 1 (as r changes by 1 in the second component of $p(i + j)$). The previous verification that the radio condition holds thus suffices here for (x_i, x_{i+2}) and (x_i, x_{i+3}), as the sum of the distance and the label difference exceeded $2k + 1$. We recalculate $d(x_i, x_{i+1})$:

$$d(x_i, x_{i+1}) = d(x_{an-1}, x_{an})$$

$$= d\left(\left((an - 1)k, (a - 1) + (an - 1)\frac{k+1}{2} \right), \left(ank, a + (an)\frac{k+1}{2} \right) \right)$$

$$\equiv d\left(\left(-k, a - \frac{k+3}{2} \right), (0, a) \right)$$

$$= k + \frac{k+3}{2}.$$

The distance increases; the radio condition is satisfied.

The two subcases show that c is a radio labeling of $C_n \square C_n$ when $k > 1$ $(n > 3)$. However, c is also a radio labeling of $C_3 \square C_3$. To see this, let $n = 3$. Recall $\text{diam}(C_3 \square C_3) = 2$. Note that $p(i)$ and $p(i + 1)$ differ in both components for all $i = 0, 1, \ldots, 8$, thus $d(x_i, x_{i+1}) \geq 2$. When $|j - i| \geq 2$, we have $|c(x_j) - c(x_i)| \geq 2$. These two facts ensure that the radio condition is satisfied by all pairs of vertices of $C_3 \square C_3$.

This establishes the claim that c is a radio labeling of $C_n \square C_n$ (when $n = 2k + 1$ and k is odd). To calculate the span of c, we use the fact that c is an increasing function to note that

$$span(c) = c(x_{n^2-1}) = 1 + (n^2 - 1)\left(\frac{k+1}{2} \right) = \left(\frac{n^2 - 1}{2} \right)(k + 1) + 1.$$

Case 2: k is even.

As $C_1\square C_1$ has only one vertex, we label this vertex 1; the result follows.

Define D_i, the ith "diagonal" of $C_n\square C_n$, to be the set of all vertices $\{(v,w)\,|\,v-w\equiv i\pmod n)\}$. We define the position function p onto the vertices of diagonals $D_0, D_1, \ldots, D_{n-2}$ first. Define $p(0)=(0,0)$ and $p(1)=(k+1,k)$. Next define

$$p(i) = \begin{cases} p(i-2) + \left(\frac{k}{2},\frac{k}{2}\right), & i = 2,3,\ldots,2n-1, \\ p(i-2n) + (k+2,k), & i = 2n, 2n+1,\ldots,(n-1)n. \end{cases}$$

For $i = (n-1)n, (n-1)n+1, \ldots, n^2-2$, write $i = (n-1)n+4j+r$, where $r \in \{0,1,2,3\}$. The continuation of the definition of the position function maps these index values to vertices on diagonal D_{n-1}:

$$p(i+1) = \begin{cases} p(i) + (k-j, k-j), & r = 0,2, \\ p(i) + (\frac{k}{2}+1+j, \frac{k}{2}+1+j), & r = 1,3. \end{cases}$$

Claim: p is a bijection.

Consider $\{p(i)\,|\,i = 0,1,\ldots,2n-1\}$. Note $p(0)\in D_0$ and $p(1)\in D_1$. As adding $\left(\frac{k}{2},\frac{k}{2}\right)$ to a vertex on D_i yields a vertex on D_i, $\{p(i)\,|\,i = 0,1,\ldots,2n-1\} \subset D_0\cup D_1$. Furthermore, as $\frac{k}{2}$ and $n = 2k+1$ are relatively prime, $p(i)\neq p(j)$ for $1\le i < j \le 2n-1$. Thus $p|_{i=0,1,\ldots,2n-1}$ is a bijection onto $D_0\cup D_1$.

Next, observe that $\{p(i)\,|\,i = 2n, 2n+1,\ldots,4n-1\}$ shifts $\{p(i)\,|\,i = 0,1,\ldots,2n-1\}$ onto $D_2\cup D_3$ by adding $(k+2,k)$ to each vertex. Similarly, the assignments $\{p(i)\,|\,i = 4n, 4n+1,\ldots,6n-1\}$ are then onto $D_4\cup D_5$, etc. Thus $p|_{i=0,1,\ldots,(n-1)n-1}$ is a bijection onto the vertices of diagonals $D_0, D_1,\ldots,D_{n-2}$.

Finally, consider $p|_{i=(n-1)n,\ldots,n^2-1}$. As in the preceding paragraph, $p((n-1)n)$ is a shift of $p((n-3)n)$ onto diagonal D_{n-1}. Adding $(\star,\star)$ (where $\star$ is $k-j$ or $\frac{k}{2}+1+j$) then ensures that $p(i)\in D_{n-1}$ for $i = (n-1)n+1,\ldots,n^2-1$. As $D_{n-1} = \{(0,1),(1,2),\ldots,(n-1,0)\}$, the fact that $p|_{i=(n-1)n,\ldots,n^2-1}$ is a bijection may be established by considering only the first components of $\{p(i)\,|\,i = (n-1)n,\ldots,n^2-1\}$. Yet this is exactly the function $\tau : \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\}$ used by Liu and Zhu to specify an optimal radio labeling of C_{2k+1} for k even [9]. Accordingly, the proof that τ is a permutation also verifies that our adaptation, $p|_{i=(n-1)n,\ldots,n^2-1}$, is a bijection onto D_{n-1}. Thus p is a bijection onto the vertices of $C_n\square C_n$.

With the claim established, we may rename the vertices of $C_n\square C_n$ by specifying that $p(i) = x_i$. We then define a labeling function $c : V(C_n\square C_n) \to \mathbb{Z}_+$, using one definition for the vertices on diagonals $D_0, D_1, \ldots, D_{n-2}$ and a second for those on D_{n-1}. Define $c(x_0) = 1$, and for

$i = 0, 1, \ldots, (n-1)n - 1$, define

$$c(x_{i+1}) = \begin{cases} c(x_i) + 1, & i \text{ even,} \\ c(x_i) + k, & i \text{ odd.} \end{cases}$$

For the last $n - 1$ vertices, we again write $i = (n-1)n + 4j + r$ where $r \in \{0, 1, 2, 3\}$ (for $i = (n-1)n, \ldots, n^2 - 2$), and use this decomposition to define

$$c(x_{i+1}) = \begin{cases} c(x_i) + 2j + 1, & r = 0, 2, \\ c(x_i) + k - 2j, & r = 1, 3. \end{cases}$$

Claim: The labeling c is a radio labeling of $C_n \square C_n$.

Again we will show that c satisfies the radio condition for pairs of distinct vertices. Applying Remark 3.1 shows that, for vertex pairs on diagonals D_0 through D_{n-2}, we need only verify that c satisfies the radio condition for vertex pairs (x_i, x_{i-j}) with $j \leq 2$ when i is even and $j \leq 3$ when i is odd. On diagonal D_{n-1} it suffices to show c satisfies the radio condition for vertex pairs (x_i, x_{i-j}) with $j \leq 3$. We break this verification into subcases.

Subcase 1: Say $x_i, x_{i-j} \in D_s \cup D_{s+1}$ for some even s.
Using Remark 3.2 to calculate distances, we see

$$d(x_i, x_{i-1}) = \begin{cases} 2k, & i \text{ odd,} \\ k+1, & i \text{ even,} \end{cases}$$

for $i \in \{1, 2, \ldots, 2n - 1\}$, and $d(x_i, x_{i-2}) = k$ for $i \in \{2, 3, \ldots, 2n - 1\}$. For i odd, $i \in \{3, 4, \ldots, 2n - 1\}$, we have $d(x_i, x_{i-3}) = k + 1$. We summarize these distance calculations together with the corresponding label difference calculations below.

vertex pair	distance	label difference	distance + label diff.
x_i, x_{i-1} (i even)	$k+1$	k	$2k + 1$
x_i, x_{i-1} (i odd)	$2k$	1	$2k + 1$
x_i, x_{i-2}	k	$k+1$	$2k + 1$
x_i, x_{i-3} (i odd)	$k+1$	$k+2$	$2k + 3$

As the sum in the last column is at least $2k + 1$ for each vertex pair, c satisfies the radio condition for these vertex pairs. The extension of the position function in Part 2 of its definition via a constant shift ensures that c also satisfies the radio condition for any vertex pair on diagonals $D_s \cup D_{s+1}$ for s even.

Subcase 2: Consider x_i and x_{i+j} where $x_i \in D_s \cup D_{s+1}$ and $x_{i+j} \in D_{s+2} \cup D_{s+3}$ for some even s.
We examine the sum of vertex distance and label difference for the vertex pairs (x_{2n}, x_{2n-1}), (x_{2n}, x_{2n-2}), (x_{2n+1}, x_{2n-1}), and (x_{2n+1}, x_{2n-2}). Again taking advantage of the shift employed in the position function's definition, these sums then extend to all vertex pairs under consideration in this case.

To aid in calculating distances, we specify the vertices using the original notation: $x_{2n} = (k+2, k)$; $x_{2n-1} = \left(\frac{k}{2}+1, \frac{k}{2}\right)$; $x_{2n-2} = \left(\frac{3k}{2}+1, \frac{3k}{2}+1\right)$; $x_{2n+1} = (2, 2k)$. The next table gives distances between vertices and label differences. To help the reader follow the distance calculations, we give the contribution of the row indices to the sum first, followed by the contribution of the column indices, and then simplify. (See Remark 3.2.)

vertex pair	distance	label difference
x_{2n}, x_{2n-1}	$\left(\frac{k}{2}+1\right) + \frac{k}{2} = k+1$	k
x_{2n}, x_{2n-2}	$\left(\frac{k}{2}-1\right) + \left(\frac{k}{2}+1\right) = k$	$k+1$
x_{2n+1}, x_{2n-1}	$\left(\frac{k}{2}-1\right) + \left(\frac{k}{2}+1\right) = k$	$k+1$
x_{2n+1}, x_{2n-2}	$\left(\frac{k}{2}+2\right) + \left(\frac{k}{2}-1\right) = k+1$	$k+2$

As the sum of the distance between each pair plus the absolute difference of their label values always exceeds $2k$, the labeling c satisfies the radio condition for these vertex pairs.

Subcase 3: We consider here the two vertex pairs $(x_{(n-1)n-1}, x_{(n-1)n+1})$ and $(x_{(n-1)n-2}, x_{(n-1)n+1})$. (These vertex pairs have their first vertex in D_{n-2} and their second in D_{n-1}. As the first vertex in D_{n-1}, $x_{(n-1)n}$, follows the labeling pattern of the first vertex in all evenly indexed diagonals, we do not need to consider any pair containing it here.) The pair notation for the vertices under consideration is $x_{(n-1)n-2} = (2k-1, 1)$; $x_{(n-1)n-1} = (k-1, k+1)$, and $x_{(n-1)n+1} = \left(\frac{k}{2}-1, \frac{k}{2}\right)$. This gives the values in the following table.

vertex pair	distance	label difference
$x_{(n-1)n-1}, x_{(n-1)n+1}$	$\frac{k}{2} + \left(\frac{k}{2}+1\right) = k+1$	$k+1$
$x_{(n-1)n-2}, x_{(n-1)n+1}$	$\left(\frac{k}{2}+1\right) + \left(\frac{k}{2}-1\right) = k$	$k+2$

Both distance plus label difference sums are at least $2k+1$, as required.

Subcase 4: Say $x_i, x_{i+l} \in D_{n-1}$, with $l \leq 3$.
Recall that to define c for these vertices, we write $i = (n-1)n + 4j + r$ for $r \in \{0, 1, 2, 3\}$. The table below giving the results of the necessary calculations for each vertex pair follows from the definitions of p and c.

vertex pair(s)	distance	label difference
x_{4j}, x_{4j+1}; x_{4j+2}, x_{4j+3}	$2k - 2j$	$1 + 2j$
x_{4j}, x_{4j+2}; x_{4j+1}, x_{4j+3}; $x_{4j+2}, x_{4(j+1)}$	k	$1 + k$
x_{4j}, x_{4j+3}	$k - 2j$	$k + 2 + 2j$
x_{4j+1}, x_{4j+2}; $x_{4j+3}, x_{4(j+1)}$	$k + 2 + 2j$	$k - 2j$
$x_{4j+1}, x_{4(j+1)}$; $x_{4j+3}, x_{4(j+1)+2}$	$2 + 2j$	$2k + 1 - 2j$
$x_{4j+3}, x_{4(j+1)+1}$	$k + 2$	$k + 3$

All sums of the last two columns (distance plus label difference) are at least $2k + 1$, so once again we see that c satisfies the radio condition for these vertex pairs. This proves the claim.

Having established that c is a radio labeling on $C_n \Box C_n$, it remains only to compute the span of c to obtain an upper bound for $rn(C_n \Box C_n)$. We take advantage of the fact that $c(x_{i+2}) = c(x_i) + k + 1$ for all even i: this gives $c(x_{n^2-1}) = c(x_0) + \frac{n^2-1}{2}(k+1)$. As $c(x_{n^2-1}) = span(c)$, we conclude that $rn(C_n \Box C_n) \leq \frac{n^2-1}{2}(k+1) + 1$.

$\square$

As each upper bound for the radio number of the Cartesian product of a cycle with itself is equal to the corresponding lower bound, we have the radio numbers themselves. That is, Theorems 2.3, 2.6, 3.4, and 3.3 establish our main theorem, Theorem 1.

4. ADDITIONAL COMMENTS

It would be interesting to determine the general relationship between the radio number of a Cartesian product of graphs and the radio numbers of the factor graphs (i.e.,between $rn(G \Box H)$, $rn(G)$, and $rn(H)$). For instance, $rn(K_m) = m$ and $rn(K_m \Box K_n) = mn = |V(K_m \Box K_n)|$. While this might lead to the hope that $rn(G \Box H) = rn(G)rn(H)$, the result in this paper together with Liu and Zhu's result on the radio number of cycles [9] shows that this is not the case:

$$rn(C_{2k}) = k^2 + k + 2, \text{ and}$$

$$rn(C_{2k+1}) = \begin{cases} k^2 + k + 1, & \text{when } k \text{ is even,} \\ k^2 + 2k + 1, & \text{when } k \text{ is odd,} \end{cases}$$

whereas

$$rn(C_{2k} \Box C_{2k}) = \left(\frac{(2k)^2 - 2}{2}\right)(k + 2) + 2, \text{ and}$$

$$rn(C_{2k+1} \Box C_{2k+1}) = \left(\frac{(2k+1)^2 - 1}{2}\right)(k + 1) + 1.$$

From this, we see that $rn(C_n \Box C_n)$ is markedly less than $rn(C_n)^2$. At this point, not enough is known about radio numbers of Cartesian products to venture a conjecture regarding the relationship of $rn(G \Box H)$ to $rn(G)$ and $rn(H)$.

5. ACKNOWLEDGEMENTS

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