On Oriented Graphs with Certain Extension Properties

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Abstract

Let Γ be an oriented graph. We denote the in-neighborhood and out-neighborhood of a vertex v in Γ by $\Gamma^-(v)$ and $\Gamma^+(v)$ respectively. We say Γ has *Property A* if, for each arc (u,v) in Γ, each of the graphs induced by $\Gamma^+(u)\cap\Gamma^+(v)$, $\Gamma^+(u)\cap\Gamma^-(v)$, $\Gamma^-(u)\cap\Gamma^+(v)$ and $\Gamma^-(u)\cap\Gamma^-(v)$ contains a directed cycle, and Γ has *Property B* if each arc (u,v) in Γ extends to a 3-path (t,u), (u,v), (v,w), such that $|\Gamma^+(t)\cap\Gamma^-(u)|\geq 5$ and $|\Gamma^+(v)\cap\Gamma^-(w)|\geq 5$.

We show that the only oriented graphs of order at most 17, which have both properties A and B are the Tromp graph T_{16} and the graph T_{16}^+ obtained by duplicating a vertex of T_{16}.

We apply this result to prove the existence of an oriented planar graph with oriented chromatic number at least 18.

1 Introduction

This paper is concerned with *oriented* graphs, that is digraphs without loops or opposite arcs; if Γ is an oriented graph, then we write $\Gamma=(V,A)$, where V and A are respectively the vertex set and arc set of Γ. We denote the out-neighborhood (resp. in-neighborhood) of a vertex v in Γ by $\Gamma^{+1}(v)$ (resp. $\Gamma^{-1}(v)$). We usually denote these sets by $\Gamma^+(v)$ and $\Gamma^-(v)$ respectively.

We will investigate oriented graph which have the following extension properties.

Definition:- An oriented graph Γ has *Property A* if for each arc (u,v) in (Γ),

$$\text{Each of the graphs induced by } \Gamma^+(u)\cap\Gamma^+(v),\ \Gamma^+(u)\cap\Gamma^-(v),\ \Gamma^-(u)\cap\Gamma^+(v) \text{ and } \Gamma^-(u)\cap\Gamma^-(v) \text{ contains a directed cycle,} \quad (1)$$

*AMS 2000 Subject classification: Primary 05C15, Secondary 05C25. Key words and phrases: oriented graph, planar graph, homomorphism, homomorphism bound, Paley tournament, Tromp graph.

and *Property B* if,

Each arc $(u, v) \in \Gamma$, extends to a 3-path (t, u), (u, v), (v, w), such that $|\Gamma^+(t) \cap \Gamma^-(u)| \geq 5$ and $|\Gamma^+(v) \cap \Gamma^-(w)| \geq 5$ (2)

Our main result is:

Theorem 1 *Up to isomorphism, the only oriented graphs of order at most 17, which have both properties A and B are the Tromp graph* T_{16} *and the graph* T_{16}^+ *obtained by duplicating a vertex of* T_{16}.

We will define the graphs T_{16} and T_{16}^+ in section 2.

This work originated from some problems concerning homomorphisms of oriented planar graphs. A *homomorphism* from $G_1 = (V_1, A_1)$ to $G_2 = (V_2, A_2)$ is a function $\phi : V_1 \rightarrow V_2$ such that, if $(u, v) \in A_1$, then $(\phi(u), \phi(v)) \in A_2$. An oriented graph H is a *homomorphism bound* for a class $\mathcal{C}$ of oriented graphs–in short a $\mathcal{C}$*-bound*–if H admits a homomorphism from every graph in $\mathcal{C}$. By analogy with the unoriented case, we define the *oriented chromatic number* $\chi_o(G)$ of an oriented graph G, as the smallest order of a homomorphism bound for $\{G\}$. For a class $\mathcal{C}$, we then define $\chi_o(\mathcal{C})$ as the supremum of $\chi_o(G)$ for $G \in \mathcal{C}$ (this may of course be ∞). Hell and Nešetřil's book [1] section 6.4 gives some further background on homomorphisms and homomorphism bounds for oriented graphs.

We will be concerned with $\mathcal{P}$-bounds, where $\mathcal{P}$ is the class of oriented planar graphs. An obvious question concerning $\mathcal{P}$-bounds is: how small can they be (assuming that any exist at all)? This questions can be seen as an oriented analog of the four color problem (In the category of *un*oriented graphs the 4CT is just the statement that K_4 is a homomorphism bound for the planar graphs). Raspaud and Sopena [4] have constructed a $\mathcal{P}$-bound of order 80, and this remains the smallest known. Thus $\chi_o(\mathcal{P}) \leq 80$. In the other direction, it is quite easy to show that $\chi_o(\mathcal{P}) \geq 15$, and this bound has been improved in turn to 16 by Sopena [6] and 17 by the author [2].

Trivially, any extension of a $\mathcal{P}$-bound is also a $\mathcal{P}$-bound, so it is natural to focus on *minimal* such graphs, that is oriented graphs which are $\mathcal{P}$-bounds, but whose proper subgraphs are not. The next result relates $\mathcal{P}$-bounds to the extension properties A and B.

Theorem 2 *Every minimal* $\mathcal{P}$*-bound* Γ *has properties A and B.*

We prove this result in Section 6, together with the following

Lemma 3 ([2]) *Neither* T_{16} *nor* T_{16}^+ *bound* $\mathcal{P}$.

Interestingly, T_{16} *does* bound the oriented planar graphs of girth 5 or more [3], as well as the oriented graphs (planar or not) of treewidth 3 or less [5]. That T_{16} is not a $\mathcal{P}$-bound is proved in [2], but the proof given here is much easier.

Finally, Theorems 1, 2 and Lemma 3 combine to give an improved lower bound for $\chi_o(\mathcal{P})$.

Theorem 4 *Every $\mathcal{P}$-bound has order at least 18. Thus* $\chi_o(\mathcal{P}) \geq 18$.

(The second statement follows readily from the first. For each oriented graph H of order less than 18, there is an oriented planar graph P_H which admits no homomorphism into H. The union of all these planar graphs then has oriented chromatic number at least 18). Theorem 2 also greatly simplifies the existing proofs that $\chi_o(\mathcal{P}) \geq 16$ and 17.

We use the following notations. Let $\Gamma = (V, A)$ be an oriented graph, then $V(\Gamma) = V$, $A(\Gamma) = A$, $|\Gamma| = |V|$. For $u, v \in V$, we define $[u, v]$ to be 1, -1 or 0, when there is, respectively, an arc from u to v, an arc from v to u or no arc in either direction. We set $d^{\pm}(v) = |\Gamma^{\pm}(v)|$, (here, and throughout the paper, we use the $\pm$ notation to indicate that the statement is true for each choice of sign), and $d(v) = d^+(v) + d^-(v)$, the degree of v. For $U \subseteq V$, we let $[U]$ denote the subgraph of Γ induced by U, and $\Gamma - U = [V \setminus U]$. We write $\Gamma_1 \simeq \Gamma_2$ to indicate that the oriented graphs Γ_1 and Γ_2 are isomorphic, and $\Gamma_1 \supseteq \Gamma_2$ to indicate that Γ_2 is isomorphic to a subgraph of Γ_1. In the latter case we also say that Γ_1 is an *extension* of Γ_2. We will say that two vertices u and v *agree* (resp. *disagree*) on a vertex w if $[u, w][v, w] = 1$ (resp. -1), and that they (dis)agree on a vertex set S if they (dis)agree on every vertex in S. We let $d(u, v)$ (resp. $a(u, v)$) denote the set of vertices on which u and v disagree (resp. agree). Two vertices $u, v \in V(\Gamma)$ will be termed *complementary* or a *complementary pair* if they either agree or disagree on the set of all of their common neighbors.

2 Paley tournaments and Tromp graphs

Let q be a prime, $q \equiv 3 \bmod 4$. The *Paley tournament* P_q is defined by $V(P_q) = \mathbb{Z}_q$ $A(P_q) = \{(a, b) \mid b - a$ is a nonzero quadratic residue$\}$. Since -1 is not a square in $\mathbb{Z}_q$, P_q is an oriented graph. The automorphisms of P_q are the maps of the form $ax + b$, where $a, b \in \mathbb{Z}_q$, a is a nonzero quadratic residue. Clearly P_q is arc transitive.

We construct the *Tromp graph* T_{2q+2} of order $2q + 2$, by taking two copies of P_q, with vertex sets $\{0, 1, \ldots q - 1\}$ and $\{0', 1', \ldots q - 1'\}$, and two additional vertices ∞ and ∞'. We then add arcs such that $[\infty, x] = [\infty', x'] = 1$ for all $x \in \mathbb{Z}_q$, and $[x, y'] = [x', y] = -[x, y]$ and $[x, x'] = 0$ for distinct $x, y \in \mathbb{Z}_q \cup \{\infty\}$. These graphs are also arc transitive; their symmetries are discussed further in [2]. We will mostly be concerned with the graphs P_7, T_8 and T_{16} and their extensions.

Definition:- If G is a vertex transitive oriented graph and $v \in V(G)$, then G^+ (Resp. G^-) is the extension of G obtained by appending a new vertex $\tilde{v}$ to G, such that $[\tilde{v}, v] = 0$, and $[\tilde{v}, w] = [v, w]$ (resp. $-[v, w]$) for all $w \in V(G) \setminus \{v\}$. Since G is vertex transitive, G^+ and G^- are defined up to isomorphism independently of the choice of v. For convenience we will let $v = 0$, when $G = P_7$.

Observe that each of P_7^+ and P_7^- have one complementary pair, $\{0, \tilde{0}\}$, while T_8 has four, $\{\{0,0'\},\{1,1'\},\{2,2'\},\{\infty,\infty'\}\}$. The vertices of these pairs agree on all the other vertices for P_7^+, and disagree for P_7^- and T_8. In all these cases v and w are complementary if and only if $[v, w] = 0$. We will be concerned with extensions of order 8 of the graphs P_7^+, P_7^- and T_8; these are constructed simply by adding (possibly) an arc to P_7^+ and P_7^-, and up to four arcs to T_8. We gather some simple observations about these graphs. Except for the last one, the proofs are all routine (keeping in mind the arc-transitivity of P_7 and T_8), so we leave the details to the reader.

Lemma 5 *If C is a directed 3-cycle of $\Gamma = P_7$ or T_8 then there is a unique $v \in V(\Gamma)$, such that $V(C) = \Gamma^+(v)$ or $V(C) = \Gamma^-(v)$.*

Lemma 6 *If C_1 and C_2 are two disjoint directed cycles of $\Gamma = P_7$ then there is a unique $v \in V(\Gamma)$, such that $\{V(C_1), V(C_2)\} = \{\Gamma^+(v), \Gamma^-(v)\}$.*

Lemma 7 *If (u, v) is an arc in $\Gamma \simeq P_7$, then $|\Gamma^+(u) \cap \Gamma^-(v)| = 1$.*

Lemma 8 *If Γ is an order 8 extension of P_7^+ (resp. T_8), u and v are distinct vertices in Γ and $\alpha = \pm 1$, then $[\Gamma^\alpha(u) \cap \Gamma^{-\alpha}(v)]$ (resp. $[\Gamma^\alpha(u) \cap \Gamma^\alpha(v)]$) contains no directed cycle.*

Lemma 9 *Let G_1 and G_2 be oriented graphs of order 5 obtained by removing a directed 3-cycle from respectively a P_7^+ extension and a T_8 extension, then G_1 and G_2 are not isomorphic.*

Proof:- Suppose that the lemma is false, then there exists an oriented graph Γ with vertex sets A and B, of size 8, such that $[A] \supseteq T_8$, $[B] \supseteq P_7^+$ and $B \setminus A$ and $A \setminus B$ both induce directed 3-cycles. If $A \setminus B$ contains no pair which is complementary in $[A]$, then, by Lemma 5, $[A \cap B]$ contains two vertices x and x' which are complementary in A, together with a directed 3-cycle which lies in $\Gamma^+(x) \cap \Gamma^-(x')$, but, since $[B] \supseteq P_7^+$, this contradicts Lemma 8.

Thus $A \setminus B$ does contain a pair which is complementary in $[A]$, whence $A \cap B$ comprises two more such pairs, $\{x, x'\}$ and $\{y, y'\}$, together with a vertex z. Since $B \setminus A$ induces a triangular circuit, it does not contain the pair which is complementary in $[B]$, and so lies in an induced P_7. Hence, by Lemma 5, $[A \cap B]$ includes a vertex v and a cycle C in $\Gamma^+(v)$ or $\Gamma^-(v)$. Since C cannot contain both vertices of either pair $\{x, x'\}$ or $\{y, y'\}$, we may assume that $C = [\{x, y, z\}]$ and $v = x'$. Now

$$\begin{aligned} [x', y] &= [y, x] && \text{(because } x \text{ and } x' \text{ are complementary in } [A]) \\ &= [x, z] && \text{(because } \{x, y, z\} \text{ induces a directed cycle)} \\ &= -[x', z] \\ &= -[x', y] && \text{(because } C \subseteq \Gamma^+(x') \text{ or } C \subseteq \Gamma^-(x')), \end{aligned}$$

a contradiction. □

3 Proof of Theorem 1: first cases

We now begin the proof of Theorem 1. After some preliminary results, we prove that if Γ has properties A and B, then, $\Gamma \simeq T_{16}$ if $|\Gamma| \leq 16$ (Lemma 16), and $\Gamma \simeq T_{16}^+$ if $|\Gamma| = 17$ and has a vertex of degree less than 15 (Lemma 17). The proof is completed in Section 4 by showing that Γ must have such a vertex. To begin with it is convenient to slightly reformulate Property B.

An oriented graph Γ has Property B if, for each $u, v \in V(\Gamma)$, with $\alpha = [u, v] \neq 0$,

$$\exists w \in V(\Gamma) \quad \text{such that} \quad v \in \Gamma^{-\alpha}(w) \text{ and } |\Gamma^{\alpha}(u) \cap \Gamma^{-\alpha}(w)| \geq 5, \qquad (3)$$

that is u and w disagree "uniformly" on at least 5 vertices, which include v. For every adjacent u and v in an oriented graph with Property B, we will let $w(u, v)$ denote a vertex which satisfies (3). (Of course, there may be several such vertices, so we just choose arbitrarily.)

We first consider some implications of Property A alone. Sometimes the following simpler criterion suffices.

Corollary 10 *If Γ has Property A, and $[u, v] \neq 0$, then $|a(u, v)|, |d(u, v)| \geq 6$.*

Definition Let G be an oriented graph. G has property Q_1 if G contains a directed cycle. For $n > 1$, G has property Q_n if, for each $v \in V(G)$, $[G^+(v)]$ and $[G^-(v)]$ have property Q_{n-1}. The following is implicit in [2, Lemma 10].

Lemma 11 *If Γ has Property A, then it has property Q_3.*

The next result generalizes [2, Lemma 11]. We defer its proof to Section 5.

Theorem 12 *The only minimal graphs with property Q_2 of order 8 or less are P_7 and T_8.*

Corollary 13 *Every oriented graph of order 8 with property Q_2 is an extension of P_7^+, P_7^- or T_8.*

Proof:- Let Γ be of order 8 and have property Q_2. If Γ contains an imbedded P_7 and one other vertex v, then, by Lemma 6 and the definition of Q_2, there is a vertex w which either agrees or disagrees with v on all their common neighbors. In this case $\Gamma \supseteq P_7^+$ or $\Gamma \supseteq P_7^-$. Otherwise, by Theorem 12, $\Gamma \supseteq T_8$. □

Lemma 11, Theorem 12 and Corollary 13 now give

Lemma 14 *For every Γ which has Property A, and $v \in V(\Gamma)$, $d^{\pm}(v) \geq 7$, with $[\Gamma^{\pm}(v)] \simeq P_7$ if equality holds, and $[\Gamma^{\pm}(v)]$ being an extension of P_7^+, P_7^- or T_8 if $d^{\pm}(v) = 8$.*

Lemma 15 *If Γ has Properties A and B, and $u \in V(\Gamma)$, then*

$$\text{If } v \in \Gamma^{\alpha}(u), \text{ and } w(u,v) \text{ is adjacent to } u, \text{ then } d^{\alpha}(u) \geq 8. \tag{4}$$

$$\text{If } u \text{ is adjacent to every other vertex, then } d^{\pm}(u) \geq 8. \tag{5}$$

$$7 \leq d^{\pm}(u) \leq |\Gamma| - 9 \tag{6}$$

If, in addition, $|\Gamma| \leq 17$, then:

$$\text{If } u \text{ has a unique non-neighbor } u', \text{ then } u \text{ is the unique non-neighbor of } u', \text{ and } d(u,u') = V(\Gamma) \setminus \{u,u'\} \tag{7}$$

If u has two non-neighbors u', u'', then either $|d(u,u')|, |d(u,u'')| \geq 10$, or else $[u',u''] = 0$, and $d(u,u')$ or $d(u,u'')$ is $V(\Gamma) \setminus \{u,u',u''\}$ (8)

Proof:- If $v \in \Gamma^{\alpha}(u)$, and $w = w(u,v)$ is adjacent to u, then (1) and (3) give $|\Gamma^{\alpha}(u)| \geq |\Gamma^{\alpha}(u) \cap \Gamma^{\alpha}(w)| + |\Gamma^{\alpha}(u) \cap \Gamma^{-\alpha}(w)| \geq 3 + 5$, whence (4), and so (5). Lemma 14 gives the first inequality of (6), and also the second if $d(u) < |\Gamma| - 1$. If $d(u) = |\Gamma| - 1$, then this inequality follows from (5).

For the rest of the proof, we suppose that $|\Gamma| \leq 17$. If u has a unique non-neighbor u', then $d(u) \leq 15$, and so there is some $\alpha = \pm 1$ such that $|\Gamma^{\alpha}(u)| \leq 7$; thus by (4), we have $w(u,v) = u'$ for every $v \in \Gamma^{\alpha}(u)$, and so u and u' disagree on $\Gamma^{\alpha}(u)$. Now let $v \in \Gamma^{-\alpha}(u)$ and $w = w(u,v)$, so that $|\Gamma^{\alpha}(w) \cap \Gamma^{-\alpha}(u)| \geq 5$. If $w \neq u'$, then $|\Gamma^{\alpha}(w) \cap \Gamma^{\alpha}(u)| \geq 3$ and $|\Gamma^{\alpha}(w) \cap \{u,u'\}| \geq 1$ (since $\Gamma^{\alpha}(u) \subseteq \Gamma^{-\alpha}(u')$). But now $|\Gamma^{\alpha}(w)| \geq 5 + 3 + 1 = 9$, contrary to (6). Hence $w(u,v) = u'$ for all $v \in \Gamma^{-\alpha}(u)$, and (7) follows.

If u has a two non-neighbors u' and u'', then, by Lemma 14, we have $|\Gamma| = 17$ and $d(u) = 14$. It follows from (7), that $d(u') = d(u'') = 14$ also. Suppose that one of these vertices–we assume u''– disagrees with u on fewer than 10 vertices, then for some $\alpha \in \{-1, 1\}$, $|\Gamma^{\alpha}(u) \cap \Gamma^{-\alpha}(u')| < 5$. It follows using (4) that, for every $v \in \Gamma^{\alpha}(u)$, $w(u, v) = u'$; that is u and u' disagree on $\Gamma^{\alpha}(u)$. Now suppose for a contradiction that $[u', u''] \neq 0$. It follows from (7) that u' has a non-neighbor $v \in \Gamma^{-\alpha}(u)$, and so, using (4), that $w(u, v) = u''$. In particular $|\Gamma^{-\alpha}(u) \cap \Gamma^{\alpha}(u'')| \geq 5$. But now $|\Gamma^{\alpha}(u') \cap \Gamma^{-\alpha}(u'')| \leq 2$, contrary to (1). Thus $[u', u''] = 0$ and so $\Gamma^{\alpha}(u') = \Gamma^{-\alpha}(u)$, whence $d(u, u') = V(\Gamma) \setminus \{u, u', u''\}$. □

Note that (6) eliminates the case $|\Gamma| = 15$ immediately. We can now also easily deal with the case $|\Gamma| = 16$.

Lemma 16 *The only oriented graph* Γ *of order at most 16 with properties A and B is* T_{16}.

Proof:- By (6), $|\Gamma| = 16$, and each vertex of Γ has exactly one non-neighbor.
Let τ map each vertex of $V(\Gamma)$ to its non-neighbor. By (7), it follows that τ is an involution, and $[\tau(x), \tau(y)] = -[\tau(x), y] = [x, y]$, for $x, y \in V(\Gamma)$, so that τ is an automorphism of Γ. Let $\infty \in V(\Gamma)$; since each $[\Gamma^{\pm}(\infty)] \simeq P_7$ is a tournament, τ maps $\Gamma^{+}(\infty)$ to $\Gamma^{-}(\infty)$, and, in view of (7), it follows that $\Gamma \simeq T_{16}$. □

In order to deal with non-neighboring vertices more easily, it is convenient to use the *complement* $\overline{\Gamma}$ of an oriented graph Γ, defined as the (unoriented) graph on the same vertex set, where two vertices are joined by an edge in $\overline{\Gamma}$ if and only if they are not joined by an arc in Γ.

Lemma 17 *The only oriented graph* Γ *of order 17 with properties A and B and with at least one vertex of degree at most 14 is* T_{16}^{+}.

Proof:-. By (6), each vertex has degree at least 14 in Γ, and so at most two in $\overline{\Gamma}$, and our hypothesis is that this is attained for at least one vertex. By (7) the neighbor of a vertex of degree one in $\overline{\Gamma}$ also has degree one, thus the components of $\overline{\Gamma}$ are isolated vertices, K_2s and cycles, and we have at least one cycle C. Let C have vertices in order $v_1, v_2, \ldots v_n$. We will show first that $n = 3$. First we note that $n \leq 5$, for otherwise we would have two adjacent vertices u and v in Γ, with four non-neighbors altogether, none of which are common to both vertices. Thus $V(\Gamma)$ would contain u and v, together with their four non-neighbors and at least 12 common neighbours, whence $|\Gamma| \geq 18$, a contradiction.

Suppose now that $n = 4$. In this case v_1 and v_3 are adjacent vertices in Γ, which have in common their two non-neighbors v_2 and v_4. Suppose that $v_3 \in \Gamma^{\alpha}(v_1)$. By (3), there is a vertex $w = w(v_1, v_3)$ such that $v_3 \in \Gamma^{\alpha}(v_1) \cap \Gamma^{-\alpha}(w)$ and $|\Gamma^{\alpha}(v_1) \cap \Gamma^{-\alpha}(w)| \geq 5$. Since w cannot be v_2 or v_4, it must be adjacent to v_1. But, since $\Gamma^{\alpha}(v_1) \simeq P_7$, this contradicts (4).

Suppose now that $n = 5$. By (8) and Corollary 10, if i and j are consecutive (modulo 5), then $|d(v_i, v_j)| \geq 10$, and otherwise $|d(v_i, v_j)| \geq 6$. Thus

$$\sum_{1 \leq i < j \leq 5} |d(v_i, v_j)| \geq 50 + 30 = 80$$

On the other hand, each vertex v of $\Gamma - C$ is in at most 6 of the sets $d(v_i, v_j)$ (this value being attained when v has three out-neighbors and two in-neighbors in C, or *vice-versa*), and each vertex of C is in at most 1. Thus the above sum cannot exceed $6(12) + 5 = 77$, a contradiction.

Thus $|C| = 3$. We have $|d(v_1, v_2)| + |d(v_1, v_3)| + |d(v_2, v_3)| \leq 28$, because each vertex of $\Gamma - C$ contributes at most 2 to this sum, and each vertex of C, nothing. But, by (8), if this inequality is satisfied, then two of the values of $|d(v_i, v_j)|$ are 14, whence the third is zero; that is two of the vertices of C agree on all their common neighbors.

It follows that Γ can be obtained from an oriented graph G of order 16, by duplicating a vertex, and it is easy to check that G inherits the properties A and B from Γ. Thus, by Lemma 16, $G \simeq T_{16}$, and so $\Gamma \simeq T_{16}^{+}$.
□

4 Completion of proof of Theorem 1

In view of Lemmas 16 and 17, we can complete the proof of Theorem 1, by showing that there is no oriented graph of order 17 and minimum degree at least 15, which has properties A and B. Throughout this section we will suppose, for a contradiction, that Γ is such a graph. We choose a fixed vertex ∞ of Γ, and set $\Gamma_1 = [\Gamma^{+}(\infty)]$, $\Gamma_2 = [\Gamma^{-}(\infty)]$. By (7) we have immediately

Lemma 18 *Γ_1 and Γ_2 are both tournaments.*

We will call a vertex v of Γ *deficient* if $d(v) = 15$. By (7), each deficient vertex v has a unique non-neighbor v'; we will refer to $\{v, v'\}$ as a *deficient pair.* In view of the above lemma, each deficient pair which does not include ∞, has one vertex in each of Γ_1 and Γ_2.

By Lemma 14 and (6), each of Γ_1 and Γ_2 is an extension of T_8, P_7, P_7^{+} or P_7^{-}.

Lemma 19 *Let $\Gamma_1 \supseteq P_7^{+}$ (resp. P_7^{-} or T_8), with v and v' be complementary in Γ_1, and define, for $\epsilon = \pm 1$*

$$\Delta_{\epsilon}(v) = [\Gamma^{\epsilon}(v) \cap \Gamma^{-\epsilon}(v')] \quad (\textit{resp.}\ [\Gamma^{\epsilon}(v) \cap \Gamma^{\epsilon}(v')]),$$

then

$$\Delta_1(v) \text{ and } \Delta_{-1}(v) \text{ each contain a directed cycle in } \Gamma_2, \tag{9}$$

If $\Gamma_2 \supseteq \Gamma_3 \simeq P_7$, *and* $\Delta_1(v)$ *and* $\Delta_{-1}(v)$ *both lie in* Γ_3, *then there is a vertex* u *in* Γ_3, *such that* $[u, v] = 0$ *or* $[u, v'] = 0$, (10)

Γ *is either a tournament, or has at least two deficient pairs.* (11)

Proof:- If ∞ is deficient, then we let ∞' denote the (unique) non-neighbor of ∞. Let $S := V(\Gamma) - V(\Gamma_1)$, whence $S = V(\Gamma_2) \cup \{\infty, \infty'\}$ when ∞ is deficient, $S = V(\Gamma_2) \cup \{\infty\}$ otherwise. Since the arcs between ∞ and other vertices of S are all in the same direction, a directed cycle in $[S]$ does not contain ∞; nor (in view of (7)) does it contain ∞'. Thus all directed cycles in $[S]$ actually lie in Γ_2. It follows from Lemma 18 and (1) that $\Delta_1(v)$ and $\Delta_{-1}(v)$ each contain a directed cycle. These cycles have no vertex in Γ_1, so must lie in $[S]$, hence in Γ_2; this establishes (9).

Now let Γ_3 be as in (10). If ∞ is deficient, then $\Gamma_3 = \Gamma_2$, and we set $w = \infty'$; otherwise we let w be the single vertex in $\Gamma_2 \setminus \Gamma_3$. By Lemma 6, there is a vertex u in Γ_3 such that $\{\Delta_1(v), \Delta_{-1}(v)\} = \{[\Gamma_3^+(u)], [\Gamma_3^-(u)]\}$. We claim that either $[u, v] = 0$ or $[u, v'] = 0$. Suppose for a contradiction that u is adjacent to both v and v'.

Suppose first that $\Gamma_1 \supseteq T_8$ or P_7^-, then u either agrees with both v and v' on $\Gamma_3 \setminus \{u\}$ or disagrees with both. In the former case

$$12 \leq |d(u, v)| + |d(u, v')| \leq 6 + 2 + 2 + 2.$$

The first equality follows from Corollary 10, and the summands on the right are contributed by the vertex sets $V(\Gamma_1) \setminus \{v, v'\}$, $\{v, v'\}$, $\{w\}$ and $\{\infty\}$, respectively. Hence equality holds, and so v and v' both disagree with u, hence agree with each other, on w; that is, for some $\eta \in \{-1, 1\}$, $w \in \Gamma^\eta(v) \cap \Gamma^\eta(v')$. Now each of $[\Gamma^{-\eta}(v) \cap \Gamma^\eta(u)]$ and $[\Gamma^{-\eta}(v') \cap \Gamma^\eta(u)]$ contains a directed cycle, and these must lie in $\Gamma_1 \cup \{\infty\}$, hence (since all arcs between ∞ and Γ_1 are in the same direction) in Γ_1. Since these cycles are disjoint, $|\Gamma^\eta(u) \cap V(\Gamma_1)| \geq 6$, whence $|\Gamma^{-\eta}(u) \cap V(\Gamma_1)| \leq 2$, contrary to (1).

If u disagrees with v and v' on $C_1 \cup C_2$, then, arguing as before we have

$$12 \leq |a(u, v)| + |a(u, v')| \leq 6 + 2 + 2 + 0$$

(this time ∞ contributes nothing to the sum), for an immediate contradiction.

Now suppose that $\Gamma_1 \supseteq P_7^+$. Now u agrees with one of the vertices v, v' on $\Gamma_3 \setminus \{u\}$ and disagrees with the other. We suppose that u agrees with v. By a similar argument to before

$$12 \leq |d(u,v)| + |a(u,v')| \leq 6+2+2+1,$$

(this time ∞ contributes 1 to the sum), again giving a contradiction. We have thus proved (10).

Now (11) easily follows. If Γ is not a tournament then we may assume that ∞ is deficient, whence $\Gamma_2 \simeq P_7$. The existence of another deficient pair now follows from (10), with $\Gamma_3 = \Gamma_2$. □.

Lemma 20 *Neither* Γ_1 *nor* Γ_2 *is an extension of* P_7^-.

Proof:- By symmetry (properties A and B are preserved under reversal of arcs), we may suppose for a contradiction that $\Gamma_1 \supseteq P_7^-$. Let $\{v, v'\}$ be the complementary pair in Γ_1. Let $A = \Delta_1(v) \cup \Delta_{-1}(v)$, as defined in (9), and $S = V(\Gamma) \setminus (\Gamma_1 \cup \{\infty\})$. For each $w \in A$, there is an $\epsilon \in \{-1, 1\}$ such that $v, v' \notin \Gamma^\epsilon(w)$, and so, by (1) and Lemma 6,

$$\forall w \in A, \text{ there are } \epsilon, \eta \in \{-1,1\} \text{ such that } \Gamma^\epsilon(w) \supseteq \Gamma_1^\eta(v) \setminus \{v'\} \qquad (12)$$

By the pigeonhole principle, there is thus an $\eta = \pm 1$ such that the vertices of the directed cycle $C = \Gamma_1^\eta(v) \setminus \{v'\}$ agree (pairwise) on a set $B \subseteq A$, of at least three vertices. Thus the set $D := S \setminus B$ has at most 5 vertices. Let the arcs of C be (α_0, α_1), (α_1, α_2) and (α_2, α_0), then each $w \in D$ is $\Gamma^+(\alpha_i) \cap \Gamma^-(\alpha_{i+1})$ for at most one value of i (reading subscripts modulo 3). Hence (again by the pigeonhole principle), there is an arc (α_i, α_{i+1}) of C for which $\Gamma^+(\alpha_i) \cap \Gamma^-(\alpha_{i+1})$ has at most one vertex in D, hence in S. But since α_i and α_{i+1} agree on v and on v', Lemma 7, shows that $|\Gamma_1^+(\alpha_i) \cap \Gamma_1^-(\alpha_{i+1})| = 1$; hence $|\Gamma^+(\alpha_i) \cap \Gamma^-(\alpha_{i+1})| \leq 2$, contrary to (1). □

Completion of Proof of Theorem 1:- To begin with, we consider the case where one or both of Γ_1, Γ_2 is a T_8 extension. By symmetry, we assume that $\Gamma_1 \supseteq T_8$. We first show that

If $u \in V(\Gamma_2)$ is deficient, and $\{v, v'\}$ are complementary in Γ_1, then v and v' do not agree on u. (13)

Let $w \in V(\Gamma_1)$ be the non-neighbor of u. Note that (13) is immediate if $w \in \{v, v'\}$. Otherwise, by (7), we have $[v,u] = -[v,w] = [v',w] = -[v',u]$. By (9), we have

Every complementary pair in Γ_1 agrees on at least 6 vertices of Γ_2. (14)

From (11), (13) and (14), it follows that Γ either has exactly two deficient pairs, or none. In particular,

$$\text{There is a complementary pair of non-deficient vertices in } \Gamma_1. \tag{15}$$

We now begin to eliminate cases. It follows from (10) and (15) that $\Gamma_2 \not\simeq P_7$. Next we suppose that $\Gamma_2 \supseteq P_7^+$, and let 0 and $\tilde{0}$ be complementary in Γ_2, with $[\tilde{0}, 0] = 1$. We show first that

$$\text{Every complementary pair in } \Gamma_1 \text{ agrees on } \tilde{0}. \tag{16}$$

We first we note, using (10), (14) and the fact that $\Gamma_2 - \{\tilde{0}\} \simeq P_7$, that if v and v' are complementary and fail to agree on $\tilde{0}$, then one of the pair is deficient. Thus if (16) fails, then Γ_2 has at least one deficient vertex, and hence it has two–call these u_1 and u_2. If $\tilde{0}$ is one of these, then, by (13), *no* complementary pair agrees on $\tilde{0}$, in which case every pair contains a deficient vertex, contrary to (15). Thus $\tilde{0} \notin \{u_1, u_2\}$, but now (16) follows from (13) and (14).

Using (16), $\Gamma^-(\tilde{0})$ comprises a set of complementary pairs in Γ_1, and a 3-cycle in Γ_2 (0 and ∞ are both in $\Gamma^+(\tilde{0})$). Thus $|\Gamma^-(\tilde{0})|$ is odd, whence, by (5) and (6), $\tilde{0}$ is deficient, contrary to (13) and (16). Together with Lemmas 14 and 20, this shows that, if $\Gamma_1 \supseteq T_8$, then $\Gamma_2 \not\supseteq P_7$. By symmetry, it follows that if one of the graphs Γ_1, Γ_2 is an extension of T_8, then both are.

To complete the proof, we choose ∞ to be non-deficient. This is justified because $|\Gamma| = 17$ and there are an even number of deficient vertices. The only cases that we have not eliminated are $\Gamma_1, \Gamma_2 \supseteq P_7^+$ and $\Gamma_1, \Gamma_2 \supseteq T_8$. Let v and v' be complementary in Γ_1 with $[v', v] = 1$. When $\Gamma_1 \supseteq T_8$, we also suppose, using (15), that v is non-deficient. Setting $C = \Gamma_1^-(v) \setminus \{v'\}$, $\Gamma^-(v)$ contains v', ∞, C and the vertices of a directed 3-cycle from Γ_2. By (6), $\Gamma^-(v)$ contains only these eight vertices. When $\Gamma_1 \supseteq T_8$ (resp. P_7^+), $C \subseteq \Gamma^+(\infty) \cap \Gamma^+(v')$ (resp. $\Gamma^+(\infty) \cap \Gamma^-(v')$), whence, by Lemma 8, $[\Gamma^-(v)] \supseteq P_7^+$ (resp. T_8). If $|\Gamma^+(v)| = 8$, then $|\Gamma^+(v) \cap \Gamma_2| = 5$, and, by Lemma 9, $[\Gamma^+(v)] \supseteq T_8$ (resp. P_7^+). If $|\Gamma^+(v)| = 7$, then v is deficient, whence $\Gamma_1 \supseteq P_7^+$ and $[\Gamma^-(v)] \supseteq T_8$. Thus in either case, one of the graphs $[\Gamma^+(v)]$, $[\Gamma^-(v)]$ is an extension of T_8, and the other is an extension of P_7, a case we have already eliminated. □

5 Proof of Theorem 12

It suffices to show that, if T is a tournament of order at most 8 with property Q_2, then T contains an imbedded P_7 or T_8. We suppose first that $|T| = 8$; we may assume that $V(T) = \mathbb{Z}_7 \cup \{\tilde{0}\}$. For each $v \in V(T)$ $3 \le d^{\pm}(v) \le 4$. Let $A_{\pm} = \{v \in V(T) \mid d^{\pm}(v) = 3\}$, and for $v \in A_{\pm}$, define $\rho(v) = |T^{\pm}(v) \cap A_{\mp}|$. By counting indegrees and outdegrees, $|A_+| = |A_-| = 4$ and the mean value of $\rho(v)$ is 3/2. We assume that $\rho(v)$ attains its maximum value at $v = 0$; thus $\rho(0) \geq 2$. We assume that $0 \in A_+$ (this is justified since the P_7 and T_8 are both self anti-isomorphic and property Q_2 is also preserved under anti-isomorphism). We may further assume that $T^+(0) = \{1, 2, 4\}$, that $\{1, 2\} \subseteq A_-$ and that $[1, 2] = 1$, whence $[2, 4] = [4, 1] = 1$, and $T^-(0) = \{3, 5, 6, \tilde{0}\}$ (the motivation for these and subsequent choices is to make T resemble P_7). Next, we may assume that $T^-(1) = \{0, 4, 6\}$, whence $T^+(1) = \{2, 3, 5, \tilde{0}\}$, and $[4, 6] = [2, 6] = 1$ (because if $[6, 2] = 1$, then $T^-(2) = \{0, 1, 6\}$ spans a transitive triangle). Next we may assume that $T^-(2) = \{0, 1, 5\}$, whence $T^+(2) = \{3, 4, 6, \tilde{0}\}$.

Suppose now that $\rho(0) = 3$, then $4 \in A_-$, and so $[4, 5] = 1$ (else $[T^-(4)]$ is transitive). By the remaining symmetry (between vertices 3 and $\tilde{0}$), we may assume that $T^-(4) = \{0, 2, 3\}$, whence $T^+(4) = \{1, 5, 6, \tilde{0}\}$. Now $\{0\} \subseteq T^+(\tilde{0}) \subseteq \{0, 3, 5, 6\}$, whence $\tilde{0} \in A_-$ (else $T^+(\tilde{0}) = \{0, a, b\}$, with $\{a, b\} \subseteq \{3, 5, 6\}$, which induces a transitive triangle in all cases); thus $T^+(\tilde{0}) = \{0, 3, 5, 6\}$, $T^-(\tilde{0}) = \{1, 2, 4\}$. We now have $A_- = \{1, 2, 4, \tilde{0}\}$, and so $A_+ = \{0, 3, 5, 6\}$. Now T is determined by the value of $[3, 5]$. If $[3, 5] = 1$, then $[\mathbb{Z}_7] = P_7$; otherwise $T \supseteq T_8$ (with complementary pairs $\{0, \tilde{0}\}$,$\{1, 3\}$,$\{2, 6\}$ and $\{4, 5\}$).

Finally suppose that $\rho(0) = 2$. By our maximality assumption, we then have $\rho(v) \leq 2$ for all v. We now have $4 \in A_+$, and so, since $\rho(1) \leq 2$, we then have $6 \in A_-$. It follows that $T^-(6) = \{2, 4, 5\}$, since $\{2, 4, 3\}$ and $\{2, 4, \tilde{0}\}$ both span transitive triangles. Hence $T^+(6) = \{0, 1, 3, \tilde{0}\}$. A similar argument then gives $T^+(4) = \{1, 5, 6\}$, whence $T^-(4) = \{0, 2, 3, \tilde{0}\}$. Since $\rho(4) \leq 2$, $5 \in A_+$, whence $T^+(5) = \{0, 2, 6\}$, $T^-(5) = \{1, 3, 4, \tilde{0}\}$. Now again $[\mathbb{Z}_7] = P_7$.

If $|T| = 7$ then we can select any $v \in V(T)$, and construct an extension $\tilde{T}$ of T by appending a new vertex w in such a way that v and w agree on $V(T) \setminus \{v\}$, and v and w are joined by an arc in either direction. Since $\tilde{T}$ also has property Q_2, the result just proved shows that $\tilde{T}$ contains an imbedded graph G isomorphic to P_7 or T_8. In neither case can G contain two vertices which agree on all their common neighbors, so we must have $G = \tilde{T} - \{w\} = T$ or $T = \tilde{T} - \{v\} \simeq T$. In both cases $T \simeq P_7$. □

6 $\mathcal{P}$-bounds

In this section we recall from [2] some necessary conditions for an oriented graph to be a $\mathcal{P}$-bound.

Definition:- A class $\mathcal{C}$ of oriented graphs is *k-complete* if the graph obtained by pasting together two graphs in $\mathcal{C}$ along isomorphic l-tournaments ($l \leq k$) is also in $\mathcal{C}$. In particular, observe that $\mathcal{P}$ is 2-complete.

We say that an oriented graph Γ is *minimal* with some property P if Γ has this property, but no proper subgraph of Γ has.

Lemma 21 ([6], Lemma 3) *Let C be a 2-complete class of oriented graphs, Γ a minimal C-bound, G a graph in C, (u, v) an arc in G and (u', v') an arc in Γ, then there is a homomorphism $\phi : G \to \Gamma$, such that $\phi(u) = u'$ and $\phi(v) = v'$.*

Proof of Theorem 2:- Let Γ be a $\mathcal{P}$-bound. From [2, Lemma 8], Γ has property A. To prove property B, let P be the oriented graph constructed from a directed 5-cycle C, and two other vertices a and b, with arcs from a to each vertex in C, and from each vertex in C to b. Then P is planar, and every homomorphism of P is injective. Now, if $u, v \in V(\Gamma)$ and $[u, v] = 1$, then Lemma 21 gives a homomorphism $\phi : P \to \Gamma$, which maps a to u, and one of the vertices of C to v, whence (3) holds if we set $w = \phi(b)$. Similarly, if $[u, v] = -1$, then there is a homomorphism mapping b to u, and some vertex of C to w, and we can take $w = \phi(a)$. □

Proof of Lemma 3:- Modify the graph P in the proof of Theorem 2 above, by reversing one of the arcs beween b and C, to conclude, by a similar argument, that for each vertex u in a minimal $\mathcal{P}$-bound, there exists a vertex v which disagrees with u and at least four vertices, and agrees on at least one. But this is false for T_{16}.

Since there is a homomorphism from T_{16}^+ to T_{16} (identify the duplicated vertices), it follows that T_{16}^+ does not bound $\mathcal{P}$ either. □

7 Concluding Remarks

We can obtain results similar to Theorem 1 using property A alone. The following is implicit in [2].

Theorem 22 *The only connected oriented graphs with property A and maximum degree 15 or less are T_{16} and T_{16}^+,*

and in an earlier version of this paper we proved

Theorem 23 *The only oriented graphs with property A of order 17 or less are T_{16}, T_{16}^+ and the graph $\overline{T_{16}}$ constructed by adding a new vertex ω, with $\overline{T_{16}}^+(\omega) = \{0, 1, 2, 4, 0', 1', 2', 4'\}$, and $\overline{T_{16}}^-(\omega) = \{\infty, 3, 5, 6, \infty', 3', 5', 6'\}$.*

The details of the proof, which we omit, are tedious. Theorem 1 follows from Theorem 23, by showing (for example through (7)) that $\overline{P_{16}}$ does not have property B; but the arguments are much simpler if this condition is incorporated from the start.

References

[1] P. Hell and J. Nešetřil, "Graphs and Homomorphisms", Oxford Lecture Series in Mathematics and its Applications, 28, Oxford University Press (2004)

[2] T. H. Marshall "Homomorphism bounds for oriented planar graphs", *J. Graph Theory* **55** (2007), 175-190.

[3] A. Pinlou, "An oriented coloring of planar graphs with girth at least five", *Discrete Math.*, **309** (2009), 2108-2118

[4] A. Raspaud and E. Sopena, "Good and semi-strong colorings of oriented planar graphs", *Inform. Proc. Letters* **51** (1994), 171-174.

[5] E. Sopena, "The chromatic number of oriented graphs", *J. Graph Theory* **25** (1997), 191-205.

[6] E. Sopena, "There exist oriented planar graphs with oriented chromatic number at least sixteen", *Inform. Proc. Letters* **81** (2002), 309-312.