EQUITABLE BLOCK COLOURINGS

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ABSTRACT. Let $\Sigma = (X, \mathcal{B})$ a 4-cycle system of order $v = 1 + 8k$. A c-colouring of type s is a map $\phi: \mathcal{B} \to \mathcal{C}$, with $\mathcal{C}$ set of colours, such that exactly c colours are used and for every vertex x all the blocks containing x are coloured exactly with s colours. Let $4k = qs + r$, with $q, r \geq 0$. ϕ is *equitable* if for every vertex x the set of the $4k$ blocks containing x is parted in r colour classes of cardinality $q + 1$ and $s - r$ colour classes of cardinality q. In this paper we study colourings for which $s|k$, giving a description of equitable block colourings for $c \in \{s, s+1, \dots, \lfloor \frac{2s^2+s}{3} \rfloor\}$.

1. INTRODUCTION

Block colourings of 4-cycle systems have been introduced and studied in [3, 4]. For any vertex x these colourings require particular conditions on the colours of the blocks containing x.

Let K_v be the complete simple graph on v vertices. The graph having vertices $a_1, a_2, \dots, a_k$, with $k \geq 3$, and having edges $\{a_k, a_1\}$ and $\{a_i, a_{i+1}\}$ for $i = 1, \dots, k-1$ is a k-cycle and it will be denoted by $(a_1, a_2, \dots, a_k)$. A 4-cycle system of order v, briefly $4CS(v)$, is a pair $\Sigma = (X, \mathcal{B})$, where X is the set of vertices and $\mathcal{B}$ is a set of 4-cycles, called *blocks*, that partitions the edges of K_v. It is known that a $4CS(v)$ exists if and only if $v = 1 + 8k$, for some $k \geq 0$.

A colouring of a $4CS(v)$ $\Sigma = (X, \mathcal{B})$ is a mapping $\phi: \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours. A c-colouring is a colouring in which exactly c colours are used. The set of blocks coloured with a colour of $\mathcal{C}$ is a *colour class*. A c-colouring of type s is a colourings in which, for every vertex x, all the blocks containing x are coloured exactly with s colours.

Let $\Sigma = (X, \mathcal{B})$ a $4CS(v)$, with $v = 1+8k$, let $\phi: \mathcal{B} \to \mathcal{C}$ be a c-colouring of type s and let $4k = qs + r$, with $q, r \geq 0$. Note that each vertex of a $4CS(v)$, with $v = 1 + 8k$, is contained in exactly $\frac{v-1}{2} = 4k$ blocks. ϕ is *equitable* if for every vertex x the set of the $4k$ blocks containing x is parted in r colour classes of cardinality $q+1$ and $s-r$ colour classes of cardinality q. A bicolouring, tricolouring or quadricolouring is an equitable colouring with $s = 2$, $s = 3$ or $s = 4$.

The colour spectrum of a $4CS(v)$ $\Sigma = (X, \mathcal{B})$ is the set:

$$\Omega_s(\Sigma) = \{c \mid \text{ there exists a an } c\text{-block-colouring of type } s \text{ of } \Sigma\}.$$

It is also considered the set $\Omega_s(v) = \bigcup \Omega_s(\Sigma)$, where Σ varies in the set of all the $4CS(v)$.

Let us recall that the *lower s-chromatic index* is $\chi'_s(\Sigma) = \min \Omega_s(\Sigma)$ and the *upper s-chromatic index* is $\overline{\chi}'_s(\Sigma) = \max \Omega_s(\Sigma)$. If $\Omega_s(\Sigma) = \varnothing$, then we say that Σ is uncolourable.

In the same way we consider $\chi'_s(v) = \min \Omega_s(v)$ and $\overline{\chi}'_s(v) = \max \Omega_s(v)$.

Block colourings for $s = 2$, $s = 3$ and $s = 4$ have been studied in [1, 3, 4]. The problem arose as a consequence of colourings of Steiner systems studied in [2, 5, 6, 8].

In this paper we study colourings for which $s|k$, giving a description of equitable block colourings for $c \in \{s, s+1, \dots, \lfloor \frac{2s^2+s}{3} \rfloor\}$.

2. Main result

In this section we prove the main result of the paper, giving in each case the construction of the desired colouring. In these constructions we will use the following symbolism. Let $A = \{a_1, a_2, \dots, a_{2p}\}$ and $B = \{b_1, b_2, \dots, b_{2q}\}$ be two sets such that $A \cap B = \varnothing$. We denote by $[A, B]$ the following family of 4-cycles:

$$[A, B] = \{(a_i, b_j, a_{i+p}, b_{j+p}) \mid 1 \le i \le p,\ 1 \le j \le q\}.$$

Note that $|[A, B]| = pq$.

Theorem 2.1. *If $s \mid k$, then $s, s+1, \dots, \lfloor \frac{2s^2+s}{3} \rfloor \in \Omega_s(v)$.*

Proof. Let $k = hs$, with $h \in \mathbb{N}$. Let $c \in \{s, s+1, \dots, \lfloor \frac{2s^2+s}{3} \rfloor\}$.

Case c = s. $\Sigma = (\mathbb{Z}_v, \mathcal{B})$ with starter blocks $\{(0, i, 4k+1, k+i) \mid 1 \le i \le k\}$ is a 4CS(v). If we assign the colour j to the blocks obtained for $i = (j-1)h + 1, \dots, jh$ and all their translated forms, we get an s-block-colouring of type s of Σ.

Case c = s + 1. Consider the sets $A_i = \{a_1^{(i)}, \dots, a_{8h}^{(i)}\}$ for $i = 1, \dots, s$ with $A_i \cap A_j = \varnothing$ for $i \ne j$. Take $\infty \notin A_1 \cup \dots \cup A_s$ and consider the following 4-cycle systems of order $1 + 8h$:

$$\Sigma_i = (A_i \cup \{\infty\}, \mathcal{B}_i)$$

for $i = 1, \dots, s$. We define the following 4-cycle system $\Sigma = (X, \mathcal{B})$ of order $v = 1 + s \cdot 8h = 1 + 8k$, with:

$$X = \bigcup_{i=1}^{s} A_i \cup \{\infty\}$$

and

$$\mathcal{B} = \bigcup_{i=1}^{s} \mathcal{B}_i \cup \bigcup_{p<q} [A_p, A_q].$$

We define a block-colouring $f: \mathcal{B} \to \{1, \ldots, s+1\}$ as follows:

$$f(\Box) = i \quad \forall \Box \in \mathcal{B}_i$$
$$f([A_p, A_q]) = i \quad \forall p, q \text{ such that } p + q \equiv i \mod s+1.$$

In this way we get a $s+1$-block-colouring of type s of Σ.

Case $\mathbf{s+2 \le c \le \frac{s^2+s}{2}}$. Take $p_1, \ldots, p_{c-s-1}, q_1, \ldots, q_{c-s-1} \in \{1, \ldots, s\}$ such that $(p_l, q_l) \neq (p_m, q_m)$ for $l \neq m$. We define a block-colouring

$$g: \mathcal{B} \to \{1, \ldots, c\}$$

as follows:

$$g(\Box) = f(\Box) \quad \forall \Box \in \mathcal{B} \setminus \{[A_{p_1}, A_{q_1}], \ldots, [A_{p_{c-s-1}}, A_{q_{c-s-1}}]\}$$
$$g([A_{p_i}, A_{q_i}]) = s + 1 + i \quad \forall i = 1, \ldots, c - s - 1.$$

In this way we get a c-block-colouring of type s of Σ.

Case $\mathbf{\frac{s^2+s}{2} + 1 \le c \le \lfloor \frac{2s^2+s}{3} \rfloor}$. Consider the sets $A_1, \ldots, A_{2s}$ defined in the following way:

- $A_i = \{a_1^{(i)}, \ldots, a_{4h}^{(i)}\}$ for $i = 1, 3, \ldots, 2s-1$
- $A_{i+1} = \{a_{4h+1}^{(i)}, \ldots, a_{8h}^{(i)}\}$ for $i = 1, 3, \ldots, 2s-1$
- $A_i \cap A_j = \emptyset$ for $i \neq j$.

Take $\infty \notin A_1 \cup \cdots \cup A_{2s}$ and consider the following 4-cycle systems of order $1 + 8h$:

$$\Sigma_i = (A_i \cup A_{i+1} \cup \{\infty\}, \mathcal{B}_i)$$

for $i = 1, 3, \ldots, 2s-1$. Let us consider also the set:

$$F = \{(p, q) \mid p, q = 1, \ldots, 2s,\ p < q\} \setminus \{(1,2), (3,4), \ldots, (2s-1, 2s)\}.$$

We define the following 4-cycle system $\Sigma = (X, \mathcal{B})$ of order $v = 1 + 2s \cdot 4h = 1 + 8k$, with:

$$X = \bigcup_{j=1}^{2s} A_j \cup \{\infty\}$$

and

$$\mathcal{B} = \bigcup_{j=0}^{s-1} \mathcal{B}_{2j+1} \cup \bigcup_{(p,q) \in F} [A_p, A_q].$$

Let $c = \frac{s^2+s}{2} + t$, with $1 \le t \le \frac{s^2-s}{6}$. Then:

$$(1) \qquad 2s^2 - 2s = 4t \cdot 3 + \left(\frac{s^2-s}{2} - 3t\right) \cdot 4.$$

Let us consider the complete graph K_{2s} and the 1-factor

$$I = \{(1,2), (3,4), \ldots, (2s-1, 2s)\}.$$

By (1) and by [7, Theorem 2.4] it is possible to divide $K_{2s} - I$ in 3-cycles $C_1, \ldots, C_{4t}$ and in 4-cycles $C_{4t+1}, \ldots, C_{\frac{s^2-s}{2}+t}$.

If $C_m = (i, j, k)$ is one of the 3-cycles, let us consider

$$\mathcal{C}_m = \{[A_i, A_j], [A_j, A_k], [A_i, A_k]\};$$

if $C_m = (i, j, k, l)$ is one of the 4-cycles, let us consider

$$\mathcal{C}_m = \{[A_i, A_j], [A_j, A_k], [A_k, A_l], [A_l, A_i]\}.$$

We define a block-colouring $f: \mathcal{B} \to \{1, \ldots, c\}$ as follows:

$$f(\square) = \frac{i+1}{2} \quad \forall \square \in \mathcal{B}_i$$

$$f([A_p, A_q]) = s + i \quad \forall [A_p, A_q] \in \mathcal{C}_i.$$

In this way we get a c-block-colouring of type s of Σ. □

Corollary 2.2. *If $s|k$, then $\chi_s'(v) = s$.*

Theorem 2.3. *Let us suppose that $s \mid k$. Then $\lfloor \frac{2s^2+s}{3} \rfloor \leq \overline{\chi}_s'(v) \leq \frac{s^2 v}{v+s-1}$.*

Proof. Let $\Sigma = (V, \mathcal{B})$ be a 4CS$(1+8k)$ and let $\varphi: \mathcal{B} \to C$ be a c-colouring of type s of Σ and let $k = hs$. Let $x \in C$ and take element $v \in V$ incident with blocks of colour x. v is contained in $4h$ blocks colour x and so in V there are at least $1 + 8h$ elements incident with blocks of colour x. Then $c(1+8h) \leq sv$. □

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