ON CLOSED GRAPHS I

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ABSTRACT. A graph is closed when its vertices have a labeling by $[n]$ with a certain property first discovered in the study of binomial edge ideals. In this article, we prove that a connected graph has a closed labeling if and only if it is chordal, claw-free, and has a property we call *narrow*, which holds when every vertex is distance at most one from all longest shortest paths of the graph.

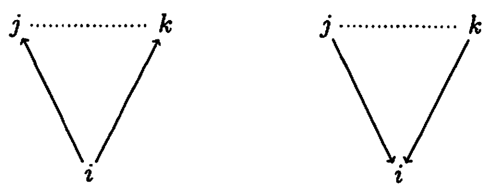
1. INTRODUCTION

In this paper, G will be a simple graph with vertex set $V(G)$ and edge set $E(G)$.

Definition 1.1. A *labeling* of G is a bijection $V(G) \simeq [n] = \{1, \ldots, n\}$, and given a labeling, we typically assume $V(G) = [n]$. A labeling is *closed* if whenever we have distinct edges $\{j, i\}, \{i, k\} \in E(G)$ with either $j > i < k$ or $j < i > k$, then $\{j, k\} \in E(G)$. Finally, a graph is *closed* if it has a closed labeling.

A labeling of G gives a direction to each edge $\{i, j\} \in E(G)$ where the arrow points from i to j when $i < j$, i.e., the arrow points to the bigger label. The following picture illustrates what it means for a labeling to be closed:

(1.1)



Whenever the arrows point away from i (as on the left) or towards i (as on the right), closed means that j and k are connected by an edge.

1991 *Mathematics Subject Classification.* 05C75 (primary), 05C25, 05C78 13P10 (secondary).
Key words and phrases. closed graph, chordal graph, binomial edge ideal.

Closed graphs were first encountered in the study of binomial edge ideals. The *binomial edge ideal* of a labeled graph G is the ideal J_G in the polynomial ring $\mathsf{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ (k a field) generated by the binomials

$$f_{ij} = x_i y_j - x_j y_i$$

for all i, j such that $\{i, j\} \in E(G)$ and $i < j$. A key result, discovered independently in [7] and [8], is that the above binomials form a Gröbner basis of J_G for lex order with $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ if and only if the labeling is closed. The name "closed" was introduced in [7].

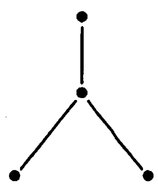
Binomial edge ideals are explored in [3] and [10], and a generalization is studied in [9]. The paper [2] characterizes closed graphs using the clique complex of G, and closed graphs also appear in [4, 5, 6].

The goal of this paper is to characterize when a graph G has a closed labeling in terms of properties that can be seen directly from the graph. Our starting point is the following result proved in [7].

Proposition 1.2. *Every closed graph is chordal and claw-free.*

"Claw-free" means that G has no induced subgraph of the form

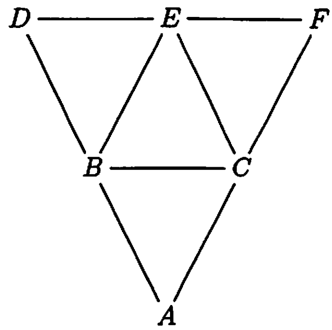
(1.2)



Besides being chordal and claw-free, closed graphs also have a property called *narrow*. The *distance* $d(v, w)$ between vertices v, w of a connected graph G is the length of the shortest path connecting them, and the *diameter* of G is $\text{diam}(G) = \max\{d(v, w) \mid v, w \in E(G)\}$. Given vertices v, w of G satisfying $d(v, w) = \text{diam}(G)$, a shortest path connecting v and w is called a *longest shortest path* of G.

Definition 1.3. A connected graph G is *narrow* if for every $v \in V(G)$ and every longest shortest path P of G, either $v \in V(P)$ or $\{v, w\} \in E(G)$ for some $w \in V(P)$.

Thus a connected graph is narrow if every vertex is distance at most one from every longest shortest path. Here is a graph that is chordal and claw-free but not narrow:

(1.3) 

Narrowness fails because D is distance two from the longest shortest path ACF.

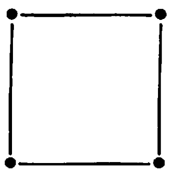
We can now state the main result of this paper.

Theorem 1.4. *A connected graph is closed if and only if it is chordal, claw-free, and narrow.*

This theorem is cited in [4, 5, 6]. Since a graph is closed if and only if its connected components are closed [2], we get the following corollary of Theorem 1.4.

Corollary 1.5. *A graph is closed if and only if it is chordal, claw-free, and its connected components are narrow.*

The independence of the three conditions (chordal, claw-free, narrow) is easy to see. The graph (1.2) is chordal and narrow but not claw-free, and the graph (1.3) is chordal and claw-free but not narrow. Finally, the 4-cycle



is claw-free and narrow but not chordal.

The paper is organized as follows. In Section 2 we recall some known properties of closed graphs and prove some new ones, and in Section 3 we introduce an algorithm for labeling connected graphs. Section 4 uses the algorithm to prove Theorem 1.4.

In a subsequent paper [1] we will explore further properties of closed graphs.

2.1. Directed Paths. A path in a graph G is $P = v_0 v_1 \cdots v_{\ell-1} v_\ell$ where $\{v_j, v_{j+1}\} \in E(G)$ for $j = 0, \ldots, \ell-1$. A single vertex is regarded as a path of length zero. When G is labeled, we assume as usual that $V(G) = [n]$. Then a path $P = i_0 i_1 \cdots i_{\ell-1} i_\ell$ is *directed* if either $i_j < i_{j+1}$ for all j or $i_j > i_{j+1}$ for all j. Here is a result from [7].

Proposition 2.1. *A labeling on a graph G is closed if and only if for all vertices $i, j \in V(G) = [n]$, all shortest paths from i to j are directed.*

2.2. Neighborhoods and Intervals. Given a vertex $v \in V(G)$, the *neighborhood* of v in G is

$$N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}.$$

When G is labeled and $i \in V(G) = [n]$, we have a disjoint union

$$N_G(i) = N_G^{>}(i) \cup N_G^{<}(i),$$

where

$$N_G^{>}(i) = \{j \in N_G(i) \mid j > i\}, \ N_G^{<}(i) = \{j \in N_G(i) \mid j < i\}.$$

This is the notation used in [2], where it is shown that a labeling is closed if and only if $N_G^{>}(i)$ and $N_G^{<}(i)$ are complete for all $i \in V(G) = [n]$.

Vertices $i, j \in [n]$ with $i \leq j$ give the *interval* $[i, j] = \{k \in [n] \mid i \leq k \leq j\}$. Here is a characterization of when a labeling of a connected graph is closed.

Proposition 2.2. *A labeling on a connected graph G is closed if and only if for all $i \in V(G) = [n]$, $N_G^{>}(i)$ is complete and equal to $[i+1, i+r]$, $r = |N_G^{>}(i)|$.*

Proof. Assume that the labeling is closed. Then Definition 1.1 easily implies that $N_G^{>}(i)$ is complete. It remains to show that $N_G^{>}(i)$ is an interval of the desired form.

Pick $j \in N_G^{>}(i)$ and $k \in [n]$ with $i < k < j$. A shortest path $P = i_0 i_1 i_2 \cdots i_m$ from $i = i_0$ to $k = i_m$ is directed by Proposition 2.1. Since $i < k$, we have $i = i_0 < i_1 < i_2 < \cdots < i_m = k$. Thus $i_1 \in N_G^{>}(i)$ and hence $\{i_1, j\} \in E(G)$ since $N_G^{>}(i)$ is complete. Since $i_1 < j$, we have $j \in N_G^{>}(i_1)$.

We now prove by induction that $j \in N_G^{>}(i_u)$ for all $u = 1, \ldots, m$. The base case is proved in the previous paragraph. Now assume $j \in N_G^{>}(i_u)$. Then $\{j, i_{u+1}\} \in E(G)$ since $\{i_u, i_{u+1}\} \in E(G)$ and the labeling is closed. This completes the induction. Since $k = i_m$, it follows that $j \in N_G^{>}(k)$. Then we have $\{i, j\}, \{k, j\} \in E(G)$ with $i < j > k$. Thus $\{i, k\} \in E(G)$ since the labeling is closed, so $k \in N_G^{>}(i)$ since $i < k$. Hence $N_G^{>}(i)$ is an interval of the desired form.

Conversely, suppose that $N_G^{\geq}(i)$ is complete and $N_G^{\geq}(i) = [i+1, \ldots, i+r]$, $r = |N_G^{\geq}(i)|$, for all $i \in V(G)$. Take $\{j,i\},\{i,k\} \in E(G)$ with $j > i < k$ or $j < i > k$. The former implies $\{j,k\} \in E(G)$ since $N_G^{\geq}(i)$ is complete. For the latter, assume $j < k$. Then $j < k < i$ with $i \in N_G^{\geq}(j)$. Since $N_G^{\geq}(j)$ is an interval containing $j+1$ and i, $N_G^{\geq}(j)$ also contains k. Hence $\{j,k\} \in E(G)$. □

2.3. **Layers.** The following subsets of $V(G)$ will play a key role in what follows.

Definition 2.3. Let G be a connected graph labeled so that $V(G) = [n]$. Then the N^{th} *layer of* G is the set

$$L_N = \{i \in [n] \mid d(i,1) = N\}.$$

Thus L_N consists of all vertices that are distance N from the vertex 1. Note that $L_0 = \{1\}$ and $L_1 = N_G(1) = N_G^{\geq}(1)$. Furthermore, since G is connected, we have a disjoint union

$$V(G) = L_0 \cup L_1 \cup \cdots \cup L_h,$$

where $h = \max\{d(i,1) \mid i \in [n]\}$. We omit the easy proof of the following lemma.

Lemma 2.4. *Let G be a connected graph labeled so that $V(G) = [n]$. Then:*

(1) *If $i \in L_N$ and $\{i,j\} \in E(G)$, then $j \in L_{N-1}$, L_N, or L_{N+1}.*
(2) *If P is a path in G connecting $i \in L_N$ to $j \in L_M$ with $N \leq M$, then for every integer $N \leq m \leq M$, there exists $k \in V(P)$ with $k \in L_m$.*

Proposition 2.5. *Let G be a connected graph with a closed labeling satisfying $V(G) = [n]$. Then:*

(1) *Each layer L_N is complete.*
(2) *If $d = \max\{L_N\}$, then $L_{N+1} = N_G^{\geq}(d)$.*

Proof. We first show that

$$r \in L_M,\ s \in L_{M+1},\ \{r,s\} \in E(G) \Longrightarrow r < s. \tag{2.1}$$

To see why, take a shortest path from 1 to $r \in L_M$. This path has length M, so appending the edge $\{r,s\}$ gives a path of length $M+1$ to s. Since $s \in L_{M+1}$, this is a shortest path and hence is directed by Proposition 2.1. Thus $r < s$.

For (1), we use induction on $N \geq 0$. The base case is trivial since $L_0 = \{1\}$. Now assume L_N is complete and take $i,j \in L_{N+1}$ with $i \neq j$. A shortest path P_1 from 1 to $i \in L_{N+1}$ has a vertex $k \in L_N$ adjacent to i, and a shortest path P_2 from 1 to $j \in L_{N+1}$ has a vertex $l \in L_N$ adjacent to j. Then $k < i$ and $l < j$ by (2.1).

If $k = l$, then $i > k < j$, which implies $\{i, j\} \in E(G)$ since the labeling is closed. If $k \neq l$, then $\{l, k\} \in E(G)$ since L_N is complete. Assume $l > k$. Then $l > k < i$ and closed imply $\{l, i\} \in E(G)$. Since $l \in L_N$ and $i \in L_{N+1}$, we have $l < i$ by (2.1). Then $i > l < j$ and closed imply $\{i, j\} \in E(G)$. Hence L_{N+1} is complete.

We now turn to (2). To prove $L_{N+1} \subseteq N_G^{>}(d)$, $d = \max\{L_N\}$, take $i \in L_{N+1}$. A shortest path from 1 to i will have a vertex $k \in L_N$ such that $\{k, i\} \in E(G)$. Then $k < i$ by (2.1), hence $i \in N_G^{>}(k)$. Also, $k \leq d$ since $d = \max(L_N)$. If $k = d$, then $i \in N_G^{>}(d)$. If $k < d$, then $\{k, d\} \in E(G)$ since L_N is complete. Then $i > k < d$ and closed imply $\{d, i\} \in E(G)$, and then $d < i$ by (2.1). Thus $i \in N_G^{>}(d)$.

To prove the opposite inclusion, take $i \in N_G^{>}(d)$. Since $\{d, i\} \in E(G)$ and $d \in L_N$, we have $i \in L_M$ for $M = N-1, N, N+1$ by Lemma 2.4. If $i \in L_{N-1}$, then (2.1) would imply $i < d$, contradicting $i \in N_G^{>}(d)$. If $i \in L_N$, then $i \leq \max\{L_N\} = d$, again contradicting $i \in N_G^{>}(d)$. Hence $i \in L_{N+1}$. □

2.4. **Longest Shortest Paths.** When the labeling of a connected graph is closed, the diameter of the graph determines the number of layers as follows.

Proposition 2.6. *Let G be a connected graph with a closed labeling. Then:*

(1) $\operatorname{diam}(G)$ *is the largest integer h such that $L_h \neq \emptyset$.*
(2) *If P is a longest shortest path of G, then one endpoint of P is in L_0 or L_1 and the other is in L_h, where $h = \operatorname{diam}(G)$.*

Proof. For (1), let h be the largest integer with $L_h \neq \emptyset$. Since points in L_h have distance h from 1, we have $h \leq \operatorname{diam}(G)$.

For the opposite inequality, it suffices to show that $d(i, j) \leq h$ for all $i, j \in V(G)$ with $i \neq j$. We can assume G has more than one vertex, so that $h \geq 1$. Suppose $i \in L_N$ and $j \in L_M$ with $N \leq M$. If $N = 0$, then $i = 1$ and $d(i, j) = d(1, j) = M \leq h$ since $j \in L_M$. Also, if $M = N$, then $i, j \in L_N$, so that $d(i, j) = 1 \leq h$ since L_N is complete by Proposition 2.5. Finally, if $0 < N < M$, let $d_u = \max(L_u)$ for each integer u. By Proposition 2.5, we know that $j \in N_G^{>}(d_{M-1})$. Hence, if $i \neq d_N$, then $P = i d_N d_{N+1} \cdots d_{M-2} d_{M-1} j$ is a path of length $M - N + 1$. If $i = d_N$, then $P = i d_{N+1} \cdots d_{M-1} j$ is a path of length $M - N$. Thus we have a path from i to j of length at most $M - N + 1$, so that $d(i, j) \leq M - N + 1 \leq M \leq h$.

For (2), let i and j be the endpoints of the longest shortest path P with $i \in L_N$, $j \in L_M$ and $N \leq M$. If $0 < N < M$, then the previous paragraph implies

$$\operatorname{diam}(G) = d(i, j) \leq M - N + 1 \leq M \leq h = \operatorname{diam}(G),$$

which forces $N = 1$ (so $i \in L_1$) and $M = h$ (so $j \in L_h$). The remaining cases $N = 0$ and $N = M$ are straightforward and are left to the reader. □

Recall from Definition 1.3 that a connected graph G is narrow when every vertex is distance at most one from every longest shortest path. Narrowness is a key property of connected closed graphs.

Theorem 2.7. *Every connected closed graph is narrow.*

Proof. Let G be a connected graph with a closed labeling. Pick a vertex $i \in V(G)$ and a longest shortest path P. Since G is connected, $i \in L_N$ for some integer N. By Proposition 2.6, the endpoints of P lie in L_0 or L_1 and L_h, $h = \text{diam}(G)$. Then Lemma 2.4 implies that P has a vertex i_M in L_M for every $1 \leq M \leq h$.

If $N \geq 1$, then either $i = i_N \in V(P)$ or $i \neq i_N$, in which case $\{i, i_N\} \in E(G)$ since L_N is complete by Proposition 2.5. On the other hand, if $N = 0$, then $i \in L_0$, hence $i = 1$. Then $\{i, i_1\} = \{1, i_1\} \in E(G)$ since $i_1 \in L_1 = N_G(1)$. In either case, i is distance at most one from P. □

3. A Labeling Algorithm

Algorithm 1, stated on the next page, labels the vertices of a connected graph. This algorithm will play a key role in the proof of Theorem 1.4.

The algorithm works as follows. Among the endpoints of all longest shortest paths, we select one of minimal degree and label it as 1. We then go through the vertices in $N_G(1)$ and label them $2, 3, \ldots$, first labeling vertices with the fewest number of edges connected to unlabeled vertices. This process is repeated for the unlabeled vertices connected to vertex 2, and vertex 3, and so on until every vertex is labeled. Furthermore, every vertex will be labeled because we first label everything in $N_G^{>}(1)$, then label everything in $N_G^{>}(2)$ not already labeled, and so on. Since the input graph is connected, this process must eventually reach all of the vertices. Hence we get a labeling of G.

The following lemma explains the function l that appears in Algorithm 1.

Lemma 3.1. *Let G be a connected graph with the labeling from Algorithm 1. Then:*

(1) *$l(1) = 0$, and for every $i \in [n]$ with $i > 1$, $l(i) = \min(N_G(i))$.*
(2) *If $l(t) < l(s)$, then $t < s$.*

Proof. Algorithm 1 defines $l(1) = 0$. Now assume $i > 1$ and let v be the vertex assigned the label i. By lines 13 and 14 of the algorithm, we need to show that when the label i is assigned to v, the variable j equals $\min(N_G(i))$. This follows because for any smaller value $j' < j$, line 11 implies that everything in the neighborhood of j' is labeled before j' is incremented. However, lines 10–12 show that v is adjacent to j and unlabeled at the start of the loop on line 11. Hence v cannot link to any smaller value of j, and since v has label i, $j = \min(N_G(i))$ follows.

Algorithm 1: Labeling Algorithm (comments enclosed by /* and */)

Input: A connected graph G with n vertices

Output: A labeling $V(G) = [n]$ and a function $l : [n] \to \{0, \ldots, n-1\}$

1 $i := 1$;
2 $j := 0$;
3 $v_0 :=$ endpoint of a longest shortest path with minimal degree;
4 label v_0 as i;
5 $l(i) := j$;
6 $i := i + 1$;
7 $J := \{v_0\}$ `/* initial value of set of labeled vertices */`;
8 $j := j + 1$;
9 **while** $j \leq |V(G)|$ **do**
10     $S := N_G(j) \setminus J$ `/* unlabeled vertices adjacent to j */`;
11   **while** $S \neq \emptyset$ **do**
12         $v :=$ pick $v \in S$ such that $|N_G(v) \setminus J| = \min_{u \in S}\{|N_G(u) \setminus J|\}$;
13   label v as i;
14         $l(i) := j$;
15         $i := i + 1$;
16         $J := J \cup \{v\}$ `/* add v to labeled vertices */`;
17         $S := S \setminus \{v\}$ `/* remove v from unlabeled adjacent vertices */`;
18   **end**
19     $j := j + 1$;
20 **end**

(2) Suppose that $s, t \in [n]$ satisfy $l(t) < l(s)$. Since $l(t)$ (resp. $l(s)$) is the value of j when the label t (resp. s) was assigned in Algorithm 1, $l(t) < l(s)$ implies that the label s was assigned later than t in the algorithm. Since the labels are assigned in numerical order, we must have $t < s$. □

The labeling produced by Algorithm 1 allows us to define the layers L_N. These interact with the function l as follows:

Lemma 3.2. *Let G be a connected graph with the labeling from Algorithm 1. Then:*

(1) *If $t \in L_N$, then $l(t) \in L_{N-1}$ if $N > 0$.*
(2) *If $t \in L_N$ and $s \in L_M$ with $N < M$, then $t < s$.*

Proof. We prove (1) and (2) simultaneously by induction on $N \geq 1$ (the case $N = 0$ of (2) is trivially true). The first time Algorithm 1 gets to Line 10, we have $S = N_G(1) \setminus J = N_G(1) = L_1$. Every vertex in $S = L_1$, is labeled during the loop starting on Line 9, so $l(t) = 1$ for all $t \in L_1$. Hence (1) holds when $N = 1$. Also, if $s \in L_M$ with $1 < M$, then the vertex s is not labeled at this stage. Since labels are assigned in numerical order, we must have $t < s$ for all $t \in L_1$. Hence (2) holds when $N = 1$.

Now assume that (1) and (2) hold for M and every $N \leq N_0$. Given $t \in L_{N_0+1}$, a shortest path from 1 to t gives $v \in L_{N_0}$ with $v \in N_G(t)$. Since $l(t) = \min(N_G(t))$ by Lemma 3.1(1), we have $l(t) \leq v$. We have $l(t) \in L_u$ for some u. If $u > N_0$, then the inductive hypothesis for (2) would imply $l(t) > v$, which contradicts $l(t) \leq v$. Hence $l(t) \in L_u$ for some $u \leq N_0$. But $t \in L_{N_0+1}$ and $\{t, l(t)\} \in E(G)$ imply $l(t) \in L_u$ for $u \geq N_0$ by Lemma 2.4(1). Hence $l(t) \in L_{N_0}$, proving (1) for $N_0 + 1$.

Turning to (2), pick $t \in L_{N_0+1}$ and $s \in L_M$ with $N_0 + 1 < M$. We just showed that $l(t) \in L_{N_0}$, and Lemma 2.4(1) implies that $l(s) \in L_u$, $u \geq M - 1$, since $s \in L_M$. Then $N_0 < M - 1 \leq u$, so our inductive hypothesis, applied to $l(t) \in L_{N_0}$ and $l(s) \in L_u$, implies $l(t) < l(s)$. Then $t < s$ by Lemma 3.1(2), proving (2) for $N_0 + 1$. □

4. Proof of the Main Theorem

We now turn to the main result of the paper. Theorem 1.4 from the Introduction states that a connected graph is closed if and only if it is chordal, claw-free and narrow. One direction is now proved, since closed graphs are chordal and claw-free by Proposition 1.2, and connected closed graphs are narrow by Theorem 2.7.

The proof of converse is harder. The key idea that the labeling constructed by Algorithm 1 is closed when the input graph is chordal, claw-free and narrow. Thus the proof of Theorem 1.4 will be complete once we prove the following result.

Theorem 4.1. *Let G be a connected, chordal, claw-free, narrow graph. Then the labeling produced by Algorithm 1 is closed.*

Proof. By Proposition 2.2, it suffices to show that the labeling produced by Algorithm 1 has the property that for all $m \in V(G) = [n]$,

(4.1) $N_G^>(m)$ is complete and $N_G^>(m) = [i+m, i+r_m]$ for $r_m = |N_G^>(m)|$.

We will prove this by induction on m. In (4.2) below, we show that (4.1) holds for $m = 1$, and in (4.3) below, we show that if (4.1) holds for all $1 \leq u < m$, then it also holds for m. Thus, we will be done after proving (4.2) and (4.3). □

4.1. The Base Case.

After Algorithm 1 runs on a chordal, claw-free and narrow graph G, the base case of the induction in the proof of Theorem 4.1 is the following assertion:

(4.2) $\quad N_G^>(1) = [2, 1 + r]$, $r = |N_G^>(1)|$, and $N_G^>(1)$ is complete.

We will first show that $N_G^{\geq}(1) = [2, 1+r]$, $r = |N_G^{\geq}(1)|$. The first time through the the loop beginning on Line 9 in Algorithm 1, $j = 1$ and $i = 2$ and $S = N_G(1)$. For each vertex in S, the loop beginning on Line 11 labels that vertex i, removes it from S, and increments i. This continues until $S = \emptyset$, at which point every vertex in S has been labeled $2, 3, \ldots, 1+r$, where r is the initial size of S. Hence $N_G^{\geq}(1) = N_G(1) = [2, 1+r]$.

To prove that $N_G^{\geq}(1)$ is complete, there are several cases to consider. Pick distinct vertices $s, t \in N_G^{\geq}(1)$ and assume that $\{s, t\} \notin E(G)$. Note that $s, t \in L_1$ are distance 2 apart and therefore $h = \text{diam}(G) \geq 2$. Our choice of vertex 1 guarantees that there is a longest shortest path P with 1 as an endpoint. Let $z \in V(G)$ be the other, so that $P = v_0 v_1 \cdots v_h$, $1 = v_0$ and $v_h = z$. Since $v_1 \in V(P)$ is the only vertex of P in L_1, s and t cannot both lie on P.

Therefore, either $s \in V(P)$, $t \in V(P)$, or $s, t \notin V(P)$. We will show that each possibility leads to a contradiction, proving that $\{s, t\} \in E(G)$.

Case 1. Both $s, t \notin V(P)$. If s has distance $h-1$ from z, then appending the edge $\{1, s\}$ to a shortest path from s to z gives a longest shortest path P' from 1 to z that contains s. Replacing P with P', we get $s \in V(P)$, which is Case 2 to be considered below. Similarly, if t has distance $h-1$ from z, then replacing P allows us to assume $t \in V(P)$, which is also covered by Case 2 below.

Thus we may assume that neither s nor t has distance $h-1$ from z. Since $d(s, z) < h - 1$ would imply $d(1, z) < h$, we conclude that s has distance h from z, and the same holds for t. It follows that $\{s, v_2\}, \{t, v_2\} \notin E(G)$, since otherwise there is a path shorter than length h from s or t to z.

Since the subgraph induced on vertices $1, s, t, v_1$ cannot be a claw, either $\{v_1, s\} \in E(G)$ or $\{v_1, t\} \in E(G)$ or both. We consider each possibility separately.

Case 1A. Both $\{v_1, s\}$, $\{v_1, t\} \in E(G)$, as shown in Figure 1(a) on the next page. Then the subgraph induced on v_2, v_1, s, t is a claw, contradicting our assumption of claw-free.

Case 1B. Exactly one of $\{s, v_1\}, \{t, v_1\}$ is in $E(G)$. Without loss of generality, we may assume $\{s, v_1\} \in E(G)$ and $\{t, v_1\} \notin E(G)$, as shown in Figure 1(b). Recall that $\{t, v_2\} \notin E(G)$ and t is distance h to z.

Since t and 1 are both endpoints of longest shortest paths, Line 3 of Algorithm 1 implies that $\deg(1) \leq \deg(t)$. Since v_1 is adjacent to 1 but not t, there must be at least one t_2 adjacent to t but not 1, i.e., $t_2 \in N_G(t)$ with $t_2 \notin N_G(1)$.

For this t_2, it follows that $t_2 \in L_2$. We also have $t_2 \neq v_2$ since $\{t, v_2\} \notin E(G)$. Furthermore, $\{t_2, s\} \notin E(G)$, since otherwise we would have the 4-cycle $t_2 s 1 t t_2$ with no chords as $\{t_2, 1\}$, $\{t, s\} \notin E(G)$. Similarly, $\{t_2, v_1\} \notin E(G)$ or else we would have the 4-cycle $t_2 v_1 1 t t_2$ with no chords since $\{t_2, 1\}$, $\{t, v_1\} \notin E(G)$. Note also that $\{t_2, v_2\} \notin E(G)$, since otherwise we would

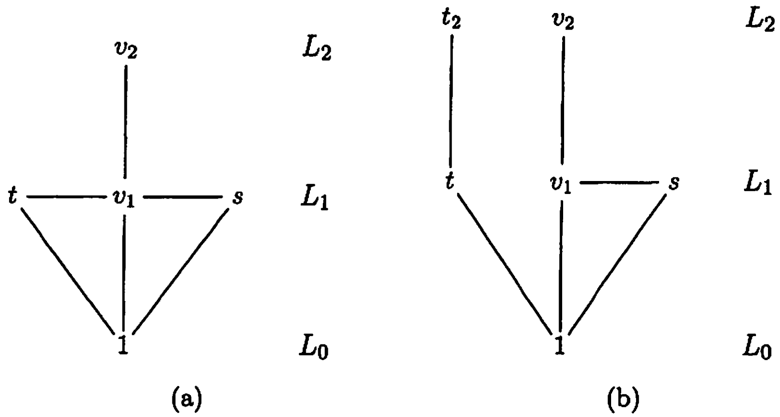


FIGURE 1. The portion of the graph relevant to (a) Case 1A and (b) Case 1B.

have the 5-cycle $t_2v_2v_11tt_2$ with no chords as $\{1, v_2\}$, $\{1, t_2\}$, $\{t, v_2\}$, $\{t, v_1\}$, $\{t_2, v_1\} \notin E(G)$, contradicting chordal. Hence t_2 gives Figure 1(b) as an induced subgraph.

Since G is narrow, either $t_2 \in V(P)$ or t_2 is adjacent to a vertex of P. However, $t_2 \in V(P)$ would imply $t_2 = v_2$ since both lie in L_2, contradicting $t_2 \neq v_2$. Thus $\{t_2, v_u\} \in E(G)$ for some $u > 1$. Since $t_2 \in L_2$ and $v_u \in L_u$, we have $u \leq 3$ by Lemma 2.4(1). We just proved $\{t_2, v_2\} \notin E(G)$, so we must have $\{t_2, v_3\} \in E(G)$. This gives the 6-cycle $t_2v_3v_2v_11tt_2$. Since Figure 1(b) is an induced subgraph, the only possible chords are $\{1, v_3\}$, $\{t, v_3\}$, $\{v_1, v_3\}$, but by Lemma 2.4(1) none of these are in $E(G)$ since $v_3 \in L_3$ and $1, t, v_1 \in L_0 \cup L_1$. Hence the 6-cycle has no chords, contradicting chordal.

Case 2. $s \in V(P)$ or $t \in V(P)$. We may assume $s = v_1$. Arguing as in Case 1B, there is $t_2 \in N_G(t)$ with $t_2 \notin N_G(1)$ and $t_2 \in L_2$. We also have $\{t, v_2\} \notin E(G)$, since otherwise the 4-cycle $1sv_2t1$ has no chords as $\{t, s\}$, $\{1, v_2\} \notin E(G)$.

Since G is narrow, t_2 must either be in P or be adjacent to a vertex in P. However, $t_2 \in V(P)$ would imply $t_2 = v_2$ since $t_2, v_2 \in L_2$, and the latter would give $\{t, v_2\} = \{t, t_2\} \in E(G)$, which we just showed to be impossible. Hence $t_2 \notin V(P)$, so that $\{t_2, v_u\} \in E(G)$ for some u. Note that $u < 4$ by Lemma 2.4(1). We claim that $u = 3$.

To see why, first note that $\{t_2, s = v_1\} \notin E(G)$, since otherwise we would have the 4-cycle $1tt_2v_11$ with no chords as $\{t, s\}$, $\{t_2, 1\} \notin E(G)$. We also know that $\{t_2, v_2\} \notin E(G)$, as otherwise we would have the 5-cycle

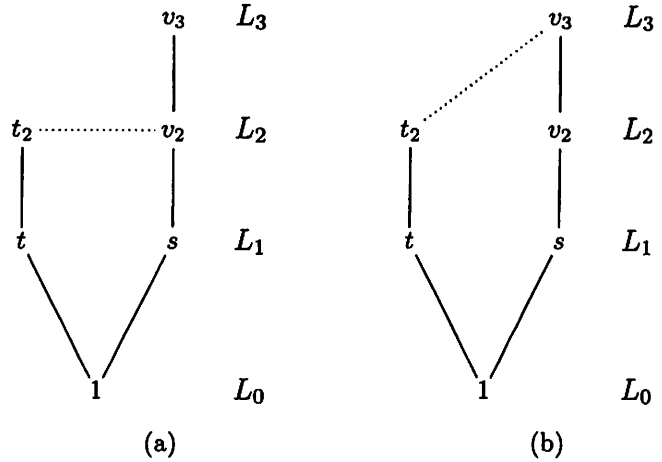


FIGURE 2. Both (a) and (b) cannot have the dotted edge or the graph has a 5-cycle or 6-cycle with no chord.

$t_2v_2s1tt_2$ with no chords since $\{t_2, s\}$, $\{t_2, 1\}$, $\{s, t\}$, $\{t, v_2\}$, $\{v_2, 1\} \notin E(G)$. See Figure 2(a).

Thus we must have $\{t_2, v_3\} \in E(G)$. However, this gives a 6-cycle $t_2v_3v_2s1tt_2$ with the same impossible chords as before along with $\{t, v_3\}$, $\{1, v_3\}$, $\{s, v_3\}$, $\{v_2, t_2\} \notin E(G)$, as in Figure 2(b). This contradicts chordal, and (4.2) follows.

4.2. **The Inductive Step.** After Algorithm 1 runs on a chordal, claw-free and narrow graph G, we now prove that the resulting labeling satsifies the inductive step in the proof of Theorem 4.1:

(4.3)

$$\text{If } N_G^{\gtrless}(u) = [u+1, u+r_u],\ r_u = |N_G^{\gtrless}(u)|,\ N_G^{\gtrless}(u) \text{ is complete, } 1 \le u < m,$$
$$\text{then } N_G^{\gtrless}(m) = [m+1, m+r_m],\ r_m = |N_G^{\gtrless}(m)|, \text{ and } N_G^{\gtrless}(m) \text{ is complete.}$$

For the first assertion of (4.3), we know that $N_G^{\gtrless}(m-1) = [m, m-1+r_{m-1}]$ is complete, which implies that $m+1, \ldots, m-1+r_{m-1} \in N_G^{\gtrless}(m)$. By analyzing the loop beginning on Line 11 at this stage of Algorithm 1, one finds that every vertex in S will be labeled with consecutive integers, starting at $i = m + r_{m-1}$ and continuing until the final vertex in $N_G(m)$ is labeled $i = m + r_{m-1} + r - 1$, where r is the original size of S. It follows that $N_G(m)$ is an interval of the desired form.

To show that $N_G^{\gtrless}(m)$ is complete, pick $s \neq t$ in $N_G^{\gtrless}(m)$. Let $P = v_0v_1 \cdots v_{q-1}v_q$ be a shortest path from $1 = v_0$ to $v_q = m$, with $v_u \in L_u$ for all u. Lemmas 2.4(1) and 3.2(2) imply that $s, t \in L_q \cup L_{q+1}$. Hence, s

and t are either both distance $q+1$ from 1, both distance q from 1, or one of s and t is distance q from 1 and the other is distance $q+1$ from 1. We consider each case separately.

Case 1. $s, t \in L_{q+1}$. Then $\{s, v_{q-1}\}, \{t, v_{q-1}\} \notin E(G)$ by Lemma 2.4(1). Since the subgraph induced on s, t, m, v_{q-1} cannot be a claw, we must have $\{s, t\} \in E(G)$.

Case 2. $s, t \in L_q$. We can assume $s < t$ and choose a shortest path $P_1 = w_0 w_1 \cdots w_q$ from $1 = w_0$ to $w_q = t$ with $w_u \in L_u$. Then $w_{q-1} < m$ by Lemma 3.2(2), giving $w_{q-1} < m < s < t$. Since $t \in N_G^{\geq}(w_{q-1})$ and $N_G^{\geq}(w_{q-1})$ is an interval by hypothesis, we have $s \in N_G^{\geq}(w_{q-1})$. But then $\{s, t\} \in E(G)$ since we are also assuming that $N_G^{\geq}(w_{q-1})$ is complete.

Case 3. We can assume $s \in L_q$ and $t \in L_{q+1}$, so $s < t$ by Lemma 3.2(2). We also have $l(m) \leq l(s)$ by Lemma 3.1(2) since $m < s$. We will consider separately the two possibilities that $l(m) < l(s)$ and $l(m) = l(s)$.

Case 3A. Suppose that $l(m) < l(s)$. Then $\{l(m), s\} \notin E(G)$ since $l(s) = \min(N_G(s))$ by Lemma 3.1(1). We also have $\{l(m), t\} \notin E(G)$, for otherwise we would have $l(t) \leq l(m)$ since $l(t) = \min(N_G(t))$. Then $l(t) \leq l(m) < l(s)$, which implies $t < s$ by Lemma 3.1(2), contradicting $s < t$. Since the subgraph induced on $l(m), m, s, t$ cannot be a claw, we must have $\{s, t\} \in E(G)$.

Case 3B. Suppose that $l(m) = l(s)$. We will assume $\{s, t\} \notin E(G)$ and derive a contradiction. The equality $l(m) = l(s)$ means that m and s were both labeled when $j = l(m) = l(s)$ in the loop starting on Line 9 of Algorithm 1. Consider the moment in the algorithm when the label m is assigned. Since $m < s$ and $j = l(m) = l(s)$, this happens during an iteration of the loop on Line 11 for which $m, s \in S$. Line 12 guarantees that the vertices assigned the labels m and s satisfy $|N_G(m) \setminus J| \leq |N_G(s) \setminus J|$. Since s is not yet labeled at this point and $s < t$, t is also not yet labeled and therefore $t \notin J$. It follows that $t \in N_G(m) \setminus J$ and $t \notin N_G(s) \setminus J$. But, in order for $|N_G(m) \setminus J| \leq |N_G(s) \setminus J|$ to hold, there must be $s_2 \in N_G(s)$ with $s_2 > m$ and $s_2 \notin J$ and $s_2 \notin N_G(m)$.

Let us study s_2. If $\{s_2, l(m)\} \in E(G)$, then $s_2 \in N_G^{\geq}(l(m))$. But we also have $m \in N_G^{\geq}(l(m))$. Since $l(m) < m$, $N_G^{\geq}(l(m))$ is complete by the hypothesis of (4.3), so we would have $\{m, s_2\} \in E(G)$. This contradicts our choice of s_2. Hence $\{s_2, l(m)\} \notin E(G)$. We also have $\{s_2, t\} \notin E(G)$, since otherwise the 4-cycle $s_2 t m s s_2$ would have no chords as $\{s_2, m\}, \{s, t\} \notin E(G)$. Also, since $m \in L_q$, Lemma 3.2(1) implies that $j = l(m) = l(s) \in L_{q-1}$.

We claim that $s_2 \in L_{q+1}$. Lemma 2.4(1), $s \in L_q$, and $s_2 > m \in L_q$ imply that $s_2 \in L_q$ or L_{q+1}. If $s_2 \in L_q$, then $l(s_2) \in L_{q-1}$ by Lemma 3.2(2). From here, $m \in L_q$ implies $m > l(s_2)$ by Lemma 3.2(2). Hence we have $l(s_2) < m < s_2$. The hypothesis of (4.3) implies that $N_G^{\geq}(l(s_2))$ is complete and

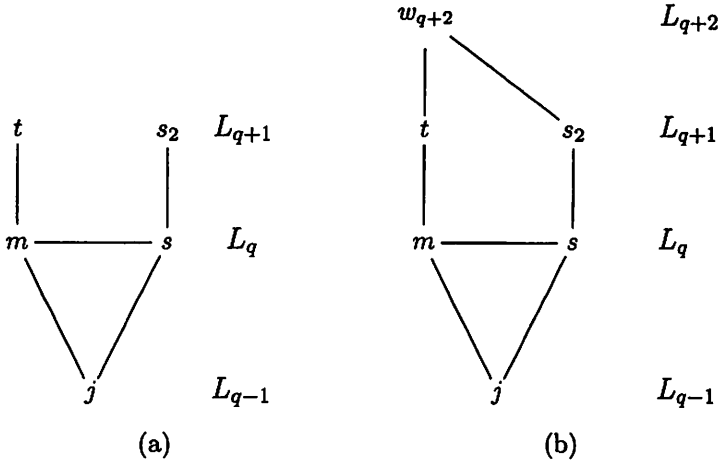


FIGURE 3. (a) Induced subgraph on $j = l(m) = l(s), m, s, t, s_2$ and (b) 5-cycle with no chords.

is an interval. Since $s_2 \in N_G^{\geq}(l(s_2))$, it follows that $m \in N_G^{\geq}(l(s_2))$, which contradicts our choice of s_2. Hence $s_2 \in L_{q+1}$ and we have Figure 3(a).

Let z be a vertex of distance $h = \text{diam}(G)$ from 1 and pick a longest shortest path $P_2 = w_0 w_1 \cdots w_h$ from $1 = w_0$ to $w_h = z$, so $w_u \in L_u$. Since G is narrow, t and s_2 must each either be in P_2 or be adjacent to a vertex in P_2. We will consider each of these cases.

First, suppose that $t \in V(P_2)$. Then $t \in L_{q+1}$ implies that $t = w_{q+1}$. Since $l(m) \in L_{q-1}$, there is a path of length $q-1$ connecting 1 to $l(m)$. Using $t = w_{q+1}$, it follows that $P_3 = 1 \cdots l(m) m t w_{q+2} \cdots z$ is a path of length $h = \text{diam}(G)$. Since G is narrow, s_2 must be adjacent to some vertex P_3. Then $\{s_2, t\}$, $\{s_2, m\}, \notin E(G)$ and Lemma 2.4(1) imply that $\{s_2, w_{q+2}\} \in E(G)$. This gives the 5-cycle $m s s_2 w_{q+2} t m$ with no chords since $\{s_2, t\}$, $\{m, s_2\}$, $\{s, t\} \notin E(G)$ and $\{w_{q+2}, m\}$, $\{w_{q+2}, s\} \notin E(G)$ since $w_{q+2} \in L_{q+2}$ but $s, m \in L_q$. See Figure 3(b). Hence we have a contradiction since G is chordal.

Second, suppose that $s_2 \in V(P_2)$. Then $s_2 = w_{q+1}$. Arguing as in the *First*, we arrive at Figure 3(b) with the same 5-cycle with no chords, again a contradiction.

Third, suppose that $s_2, t \notin V(P_2)$. First note that P_2 was an arbitrary longest shortest path starting at 1. Thus the above *First* and *Second* give a contradiction whenever s_2 or t are on *any* longest shortest path starting at 1. Hence we may assume that s_2 and t are not on any shortest path of length h starting at 1.

Since G is narrow, $s_2 \in L_{q+1}$ is adjacent to a vertex of P_2, which must be w_q, w_{q+1}, or w_{q+2} by Lemma 2.4(1). However, if $\{s_2, w_{q+2}\} \in E(G)$, then we would get a path of length h from 1 to z by taking any shortest path from 1 to s_2, followed by $\{s_2, w_{q+2}\}$, and then continuing along P_2 from w_{q+2} to z. This longest shortest path starts at 1 and contains s_2, contradicting the previous paragraph. Hence $\{s_2, w_{q+2}\} \notin E(G)$ and s_2 must be adjacent to w_q or w_{q+1}, and the same is true for t by a similar argument.

In fact, we must have $\{s_2, w_q\} \in E(G)$, since otherwise $\{s_2, w_{q+1}\} \in E(G)$ and the subgraph induced on $w_q, w_{q+1}, w_{q+2}, s_2$ would be a claw. A similar argument shows that $\{t, w_q\} \in E(G)$. Since $w_{q-1} \in L_{q-1}$ and $s_2, t \in L_{q+1}$, this implies that the subgraph induced on t, s_2, w_q, w_{q-1} is a claw, again contradicting claw-free. This final contradiction completes the proof of (4.3), and Theorem 4.1 is proved.

Remark 4.2. In (4.2) and (4.3), the chordal hypothesis is applied only to cycles of length 4, 5, or 6. Hence, in Theorem 1.4 and Corollary 1.5, we can replace chordal with the weaker hypothesis that all cycles of length 4, 5, or 6 have a chord.

Acknowledgements

Theorem 1.4 is based on the senior honors thesis of the second author, written under the direction of the first author. We are grateful to Amherst College for the Post-Baccalaureate Summer Research Fellowship that supported the writing of this paper. Thanks also to Michael Ching for Remark 4.2.

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