The bondage number of $(n-3)$-regular graphs of order n*

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Abstract: Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to a vertex in D. The domination number of G is the smallest cardinality of a dominating set of G. The bondage number of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with larger domination number of G. In this paper, we determine that the exact value of the bondage number of $(n-3)$-regular graph G of order n is $n-3$.

Keywords: domination number, bondage number, regular graph.
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1 Introduction

For graph-theoretical terminology and notation not defined here, we follow [14]. Specifically, a graph $G = (V, E)$ is considered as an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For a vertex x in G, let $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $E_G(x) = \{xy : y \in N_G(x)\}$. The cardinality $|E_G(x)|$ is the degree of x. For two disjoint nonempty and proper subsets S and T in $V(G)$, we use $E_G(S, T)$ to denote the set of edges between S and T in G, and $G[S]$ to denote a subgraph of G induced by S.

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A vertex y is said to be dominated by a vertex x if $y \in N_G(x)$ in graph G. A subset $D \subset V$ is a *dominating set* of G if $N_G(x) \cap D \neq \emptyset$ for every vertex x in $G - D$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets of G.

In 1990, Fink *et al.* [3] introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network under link failure. The *bondage number* of a nonempty graph G, denoted by $b(G)$, is the minimum number of edges whose removal from G results in a graph with larger domination number than $\gamma(G)$, that is,

$$b(G) = \min\{|B| : B \subseteq E(G), \gamma(G - B) > \gamma(G)\}.$$

A nonempty subset $B \subseteq E(G)$ is said a *bondage set* of G if $\gamma(G-B) > \gamma(G)$. A bondage set B is said to be *minimum* if $|B| = b(G)$. In fact, if B is a minimum bondage set, then $\gamma(G - B) = \gamma(G) + 1$, because the removal of one single edge can not increase the domination number by more than one.

It is quite difficult to compute the exact value of the bondage number for general graphs since it strongly depends on the domination number of the graphs. Very recently, Hu and Xu [6] showed that the problem determining bondage number for general graphs is NP-hard. However, the bondage number has received considerable attention in [1–5, 7–13, 15]. Much work focused on the bounds of the bondage number as well as the restraints on particular classes of graphs. In particular, Fink *et al.* [3] showed $b(K_n) = \lceil \frac{n}{2} \rceil$ for an $(n-1)$-regular graph K_n of order $n \geqslant 2$, $b(G) = n - 1$ for an $(n-2)$-regular graph G of order $n \geqslant 2$, where G is a t-partite graph $K_{n_1,n_2,\ldots,n_t}$ with $n_1 = n_2 = \cdots = n_t = 2$ and $t = \frac{n}{2}$ for an even integer $n \geq 4$.

In this paper, we show that $b(G) = n-3$ for every $(n-3)$-regular graph G of order $n \geq 4$.

2 Main results

Lemma 2.1 *$\gamma(G) = 2$ for any $(n-3)$-regular graph G of order $n \geq 4$.*

Proof. Let G be an $(n-3)$-regular graph of order $n \geq 4$. It is clear that $\gamma(G) \geq 2$ since there exists no such a vertex that dominates all vertices in G. We only need to construct a dominating set of G with two vertices. Let x be any vertex, and let y and z be the only two vertices not adjacent to x in G.

If $yz \in E(G)$, let $D = \{x, y\}$. If $yz \notin E(G)$, then there is a vertex w adjacent to both y and z in G since $n \geq 4$. Let $D = \{x, w\}$. Then D is a dominating set of G. Thus $\gamma(G) \leq 2$ and hence $\gamma(G) = 2$. ∎

Lemma 2.2 *$b(G) \leq n - 3$ for any $(n-3)$-regular graph G of order $n \geq 4$.*

Proof. For a vertex x in G, let $G' = G - x$. Then $\gamma(G') \geq 2$ since any vertex y in G' can not dominate all vertices in G'. Thus, $\gamma(G - E_G(x)) \geq 3 > 2 = \gamma(G)$ by Lemma 2.1, which implies that $b(G) \leq n - 3$. ∎

Lemma 2.3 *Let G be an $(n-3)$-regular graph of order $n \geq 7$ and B be a minimum bondage set of G. If $|B| \leq n - 4$, then there are at most two vertices x and y in G such that $E_G(x) \cap B = E_G(y) \cap B = \emptyset$.*

Proof. Let $G' = G - B$. Then $\gamma(G') = 3$ since B is minimum and $\gamma(G) = 2$ by Lemma 2.1. Suppose to the contrary that there are three vertices x_1, x_2 and x_3 such that

$$E_G(x_i) \cap B = \emptyset \text{ for each } i = 1, 2, 3. \tag{2.1}$$

Let $X = \{x_1, x_2, x_3\}$. We claim that X is a dominating set of G'. In fact, if there is some vertex u in G' that is not dominated by X, that is, u is not adjacent to any vertex in X, then u has degree at most $(n-4)$ in G by (2.1), a contradiction. Thus, since X is a minimum dominating set of G',there exists a vertex $y_{ij} \notin N_{G'}[x_i] \cup N_{G'}[x_j]$ in G' for $1 \leq i \neq j \leq 3$. Let $Y = \{y_{12}, y_{23}, y_{13}\}$. Then $|Y| = 3$ since the vertex x_i has degree $(n-3)$ in G' by (2.1) for each $i = 1, 2, 3$.

Since G is $(n-3)$-regular, $\{y_{12}, y_{13}\}$, $\{y_{12}, y_{23}\}$, and $\{y_{23}, y_{13}\}$ are the only two vertices not adjacent to x_1, x_2 and x_3 in G, respectively. Similarly, $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_1\}$ are the only two vertices not adjacent to y_{12}, y_{23} and y_{13} in G, respectively. Also since G is $(n-3)$-regular, $G[Y] = K_3$. Note that any vertex in X can dominate all vertices in G except for some two vertices in Y. If $G[Y] - B$ contains a vertex, say y_{23}, of degree two, then $\{x_1, y_{23}\}$ is a dominating set of $G - B$, a contradiction. Therefore, $G[Y] - B$ contains no vertices of degree two, which means that

$$|E(G[Y]) \cap B| \geq 2. \tag{2.2}$$

Let $S = V(G) \setminus (X \cup Y)$. Then $S \neq \emptyset$ since $n \geq 7$. By (2.2) and $|B| \leq n-4$, we have that $|E_G(Y, S) \cap B| \leq n - 6$. Then there is some $s \in S$ such that $|E_G(s, Y) \cap B| \leq 1$. Without loss of generality, assume that two edges sy_{12} and sy_{13} are both not in B. Then $\{s, x_1\}$ is a dominating set of G' since s can dominate two vertices y_{12} and y_{13} and x_1 can dominate all vertices except for y_{12} and y_{13}, which implies that $\gamma(G') \leq 2$, a contradiction. ∎

Lemma 2.4 *Let G be an $(n-3)$-regular graph of order $n \geq 7$ and B be a minimum bondage set of G. If there exists a vertex $x \in V(G)$ such that $E_G(x) \cap B = \emptyset$, then $|B| = n - 3$.*

Proof. By Lemma 2.2, we only need to prove that $|B| \geq n - 3$. Let $x \in V(G)$ such that $E_G(x) \cap B = \emptyset$, let y, z be the only two vertices not adjacent to x in G, and $G' = G - B$. Then $\gamma(G') \geq 3$ by Lemma 2.1.

If $yz \notin E(G)$, then both y and z are adjacent to each vertex $s \in V_1 = V(G) \setminus \{x, y, z\}$ in G. Thus at least one edge in $\{sy, sz\}$ belongs to B otherwise $\{x, s\}$ is a dominating set of G'. Then

$$|B| \geq |E_G(\{y, z\}, V_1) \cap B| \geq |V_1| = n - 3.$$

If $yz \in E(G)$, then $yz \in B$ otherwise $\{x, y\}$ is a dominating set of G'. Let p and q be the only vertex except x not adjacent to y and z in G, respectively. Then both y and z are adjacent to any vertex $t \in V_2 = V(G) \setminus \{x, y, z, p, q\}$ in G. Thus at least one of $\{ty, tz\}$ belongs to B otherwise $\{x, t\}$ is a dominating set of G'. Thus,

$$|E_G(\{y, z\}, V_2) \cap B| \geq |V_2| = \begin{cases} n-4 & \text{if } p = q; \\ n-5 & \text{if } p \neq q. \end{cases} \tag{2.3}$$

If $p = q$ then, by (2.3), we have that

$$|B| \geq |\{yz\}| + |E_G(\{y, z\}, V_2) \cap B| \geq n - 3.$$

If $p \neq q$, then $|(E_G(p) \cup E_G(q)) \cap B| \geq 1$ since $E_G(x) \cap B = \emptyset$ and there are at most two vertices $u, v \in V(G)$ such that $E_G(u) \cap B = E_G(v) \cap B = \emptyset$ by Lemma 2.3. Thus, by (2.3), we have that

$$|B| \geq |\{yz\}| + |E_G(\{y, z\}, V_2) \cap B| + |(E_G(p) \cup E_G(q)) \cap B| \geq n - 3.$$

The lemma follows. ∎

Theorem 2.1 *$b(G) = n-3$ for any $(n-3)$-regular graph G of order $n \geq 4$.*

Proof. We first consider $n \in \{4, 5, 6\}$. If $n = 4$, then $G = K_2 + K_2$, so $b(G) = 1$. If $n = 5$, then $G = C_5$, thus $b(G) = 2$. Assume $n = 6$ below.

Let x be a vertex and y, z be the only two vertices not adjacent to x in G. It is easy to verify that

$$G = \begin{cases} C_3 \times K_2 & \text{if } yz \in E(G); \\ K_{3,3} & \text{if } yz \notin E(G), \end{cases}$$

and so $b(G) = 3$.

We now assume $n \geq 7$ in the following discussion.

Let B be a minimum bondage set of G and $G' = G - B$. Then $\gamma(G') = 3$ by Lemma 2.1. If there exists a vertex $x \in V(G)$ such that $E_G(x) \cap B = \emptyset$, then $|B| = n - 3$ by Lemma 2.4. We now assume that $E_G(x) \cap B \neq \emptyset$ for every vertex $x \in V(G)$. By Lemma 2.2, $|B| \leq n - 3$. Next, we prove that $|B| \geq n-3$. Then there exists a vertex $x \in V(G)$ such that $|E_G(x) \cap B| = 1$.

Let $xw \in B$, y and z be the only two vertices not adjacent to x in G. Let p and q be the only two vertices not adjacent to w in G. We claim that for any vertex $x' \in V(G) \setminus \{x, y, z, w\}$,

$$|E_G(\{w, y, z\}, x') \cap B| \geq 1 \text{ if } \{wx', yx', zx'\} \subseteq E(G). \qquad (2.4)$$

To see this, note that if $|E_G(\{w, y, z\}, x') \cap B| = 0$, then $\{x, x'\}$ is a dominating set of G' since w, y and z can be dominated by x' and others can be dominated by x in G', a contradiction.

We now prove that $|B| \geq n-3$ by considering the following three cases.

Case 1 $\{y, z\} = \{p, q\}$.

In this case, $yz \in E(G)$ and x' is adjacent to every vertex in $\{w, y, z\}$ for any vertex $x' \in V_1 = V(G) \setminus \{x, y, z, w\}$ in G. By (2.4), $|E_G(\{w, y, z\}, x') \cap B| \geq 1$, and so $|E_G(\{w, y, z\}, V_1) \cap B| \geq |V_1| = n - 4$. Thus

$$\begin{aligned} |B| &\geq |\{xw\}| + |E_G(\{w, y, z\}, V_1) \cap B| \\ &\geq n - 3. \end{aligned}$$

Case 2 $|\{y, z\} \cap \{p, q\}| = 1$. Without loss of generality, let $p = y$.

In this case, $yz, wz \in E(G)$ and hence $|E_G(z, \{y, w\}) \cap B| \geq 1$, for otherwise $\{x, z\}$ is a dominating set of G' since $\{y, w\}$ can be dominated by z and others can be dominated by x in G'. Let r be the only vertex except x not adjacent to z in G. Thus, x' is adjacent to every vertex in $\{w, y, z\}$ for any vertex $x' \in V_2 = V(G) \setminus \{x, y, z, w, q, r\}$ in G. By (2.4), $|E_G(\{w, y, z\}, x') \cap B| \geq 1$, and so $|E_G(\{w, y, z\}, V_2) \cap B| \geq |V_2| = n - 6$. Thus

$$\begin{aligned} |B| &\geq |\{xw\}| + |E_G(\{w, y, z\}, V_2) \cap B| \\ &\quad + |E_G(z, \{y, w\}) \cap B| + |(E_G(q) \cup E_G(r)) \cap B| \\ &\geq n - 3. \end{aligned}$$

Case 3 $\{y, z\} \cap \{p, q\} = \emptyset$.

In this case, $wy, wz \in E(G)$.

subcase 3.1 $yz \notin E(G)$.

In this case, $|E_G(w, \{y, z\}) \cap B| \geq 1$, for otherwise $\{x, w\}$ is a dominating set of G'. Note that x' is adjacent to every vertex in $\{w, y, z\}$ for any vertex $x' \in V_3 = V(G) \setminus \{x, y, z, w, p, q\}$ in G. By (2.4), $|E_G(\{w, y, z\}, x') \cap B| \geq 1$, and so $|E_G(\{w, y, z\}, V_2) \cap B| \geq |V_2| = n - 6$. Thus

$$\begin{aligned} |B| &\geq |\{xw\}| + |E_G(\{w, y, z\}, V_3) \cap B| \\ &\quad + |E_G(w, \{y, z\}) \cap B| + |(E_G(p) \cup E_G(q)) \cap B| \\ &\geq n - 3. \end{aligned}$$

Subcase 3.2 $yz \in E(G)$.

In this case, $|E(G[\{w,y,z\}]) \cap B| \geq 2$, since otherwise x and one vertex in $\{w,y,z\}$ consist of a dominating set of G'. Let r and s be the only vertex except x not adjacent to y and z in G, respectively. Note that x' is adjacent to every vertex in $\{w,y,z\}$ for any vertex $x' \in V_4 = V(G) \setminus \{x,y,z,w,p,q,r,s\}$ in G. By (2.4), $|E_G(\{w,y,z\},x') \cap B| \geq 1$, and so

$$|E_G(\{w,y,z\},V_4) \cap B| \geq \begin{cases} n-6 & \text{if } |\{r,s\} \cup \{p,q\}| = 2; \\ n-7 & \text{if } |\{r,s\} \cup \{p,q\}| = 3; \\ n-8 & \text{if } |\{r,s\} \cup \{p,q\}| = 4. \end{cases}$$

Thus

$$\begin{aligned} |B| \quad &\geq |\{xw\}| + |E_G(\{w,y,z\},V_4) \cap B| \\ &+|E(G[\{w,y,z\}]) \cap B| \\ &+|(E_G(p) \cup E_G(q) \cup E_G(r) \cup E_G(s)) \cap B| \\ &\geq n-3. \end{aligned}$$

The theorem follows. ∎

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