A note on $k-$tridiagonal matrices with the balancing and Lucas-balancing numbers

Fatih YILMAZ,* Emrullah KIRKLAR
Department of Mathematics, Polatlı Art and Science Faculty, Gazi University,06900
Ankara, Turkey

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Abstract

In this note, we consider one type of $k-$tridiagonal matrix family whose permanents and determinants are specified to the balancing and Lucas-balancing numbers. Moreover, we provide some properties between Chebyshev polynomial properties and the given number sequences.

1 Introduction

The n-square k-tridiagonal matrix [2] is defined as follows:

$$T_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & & & & a_2 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & & & d_{n-k} & & & & a_{n-k} \\ b_{k+1} & & & & \ddots & & & 0 \\ 0 & b_{k+2} & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & d_{n-1} & 0 \\ 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n \end{pmatrix}.$$

By definition of the matrix family, it is readily seen that $k-$tridiagonal matrices are a generalization of some special kind of matrices such as tridiagonal, pentadiagonal, etc. This kind of matrices have applications in many fields of science (see [2, 3, 4, 5, 6]).

*e-mail adresses: fatihyilmaz@gazi.edu.tr, e.kirklar@gazi.edu.tr

In literature, there are several kinds of integer sequences and their generalizations. Most of them are defined as a result of an event in nature or to give a mathematical model to a problem. Integer sequences appear in huge amount fields of science and they have many applications in mathematics, biology and engineering.

Recently there is great interest to balancing numbers, introduced by Behera and Panda in [8] with following recurrence relation for $n \geq 2$

$$B_{n+1} = 6B_n - B_{n-1}, \quad \text{with } B_1 = 1, \; B_2 = 6.$$

The balancing numbers [8] are obtained as a solution of Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$ calling $r \in \mathbb{Z}^+$, the balancer corresponding to the balancing number n and it is proved that a positive integer n is a balancing number if and only if n^2 is a triangular number or $8n^2+1$ is a perfect square. In other words, the following statements are equivalent:

· n is a balancing number

· n^2 is a triangular number

· $8n^2+1$ is a perfect square.

Moreover [8] Behera and Panda showed that $\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3+\sqrt{8}$.

The Lucas-balancing numbers [11] are defined with the following recurrence relations for $n \geq 2$:

$$C_{n+1} = 6C_n - C_{n-1}$$

with $C_1 = 3$, $C_2 = 17$.

First six values of balancing (B_n) and Lucas-balancing (C_n) numbers are given in the table, below:

n	1	2	3	4	5	6
B_n	1	6	35	204	1189	6930
C_n	3	17	99	577	3363	19601

The determinant [7] of an $n \times n$ matrix $A = (a_{ij})$ may be given by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n represents the symmetric group of degree n. Analogously, the permanent [7] of A is

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Brualdi and Gibson [1] give a new method, which is called as *contraction* method, to compute permanents of matrices. Let $A = (a_{ij})$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A is *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a integer matrix and B is a contraction of A, then

$$\operatorname{per} A = \operatorname{per} B. \tag{1}$$

A matrix A is called convertible if there exists an $n \times n$ $(1,-1)$–matrix H such that $det(A \circ H) = perA$, where $A \circ H$ is well known Hadamard product of A and H. Here H is called as a converter of A [1].

Let us define n-square symmetric k-tridiagonal matrices $H_{n,k}$ and $U_{n,k}$ of the form

$$H_{n,k} = \begin{cases} -1, & h_{i,i+k} = h_{i+k,i} \text{ for } i = 1,2,\ldots,n-k \\ 6, & h_{i,i} \text{ for } i = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$U_{n,k} = \begin{cases} -1, & u_{i,i+k} = u_{i+k,i} \text{ for } i = 1,2,\ldots,n-k \\ 3, & u_{i,i} \text{ for } i = 1,2,\ldots,k \\ 6, & u_{i,i} \text{ for } i = k+1,k+2,\ldots,n \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

here, we assume that there exists m such that $n = mk$.

2 On k–tridiagonal matrices and the balancing and Lucas-balancing numbers

Let us consider k-tridiagonal matrices given with (2) and (3). Then, we have the following theorem.

Theorem 1 *Let $\{B_n\}$ and $\{C_n\}$ are nth balancing and Lucas-balancing numbers, respectively, then it follows that*

$$\det H_{n,k} = B^k_{m+1} \tag{4}$$

and

$$\det U_{n,k} = C^k_m. \tag{5}$$

Proof. By using fast block diagonalization method for $k-$tridiagonal matrices given by [2], a block diagonal matrix will be obtained and by determinants of tridiagonal matrices of the form

$$f_i = \begin{vmatrix} d_1 & a_1 & & & \\ b_2 & d_2 & a_2 & & \\ & b_3 & \ddots & \ddots & \\ & & \ddots & d_{i-1} & a_{i-1} \\ & & & b_i & d_i \end{vmatrix}$$

which satisfies a three-term recurrence

$$f_i = d_i f_{i-1} - b_i a_{i-1} f_{i-2}$$

given by [14], the proof can be seen easily. ■

By using Hadamard matrix multiplication property given above, we consider $n-$square matrices $T_{n,k}$ and $V_{n,k}$ as below:

$$T_{n,k} = \begin{cases} -1, & t_{i,i+k} \text{ for } i = 1,2,\ldots,n-k \\ 1, & t_{i+k,i} \text{ for } i = 1,2,\ldots,n-k \\ 6, & t_{i,i} \text{ for } i = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases},$$

and

$$V_{n,k} = \begin{cases} -1, & v_{i,i+k} \text{ for } i = 1,2,\ldots,n-k \\ 1, & v_{i+k,i} \text{ for } i = 1,2,\ldots,n-k \\ 3, & v_{i,i} \text{ for } i = 1,2,\ldots,k \\ 6, & v_{i,i} \text{ for } i = k+1,k+2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

here $n = mk$. Then we have the following theorem.

Theorem 2

$$perT_{n,k} = B_{m+1}^k \tag{6}$$

and

$$perV_{n,k} = C_m^k. \tag{7}$$

Proof. Since the matrices have exactly two non-zero elements at the last columns, using consecutive contraction steps on the last columns, the result can be seen, readily. ■

3 Balancing Q-Matrices

The Q-matrix was first studied by King [13] in 1960. Motivated by this, Ray introduced [12] balancing Q-matrix which is given by

$$Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$$

and obtained that

$$Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$$

here B_n denotes n^{th} balancing number. Moreover, Ray gave

$$detQ_B^n = [B_n^2 - B_{n-1}B_{n+1}] = 1.$$

Let us consider n-square matrix

$$\mathbb{Q}_B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 6 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{8}$$

then we have the following theorem.

Theorem 3 *Let $\mathbb{Q}_B$ be n-square matrix given by (8). Then*

$$\begin{aligned} \mathbb{Q}_B^n &= \begin{pmatrix} 0 & -B_n & 0 & B_{n-1} \\ B_n & 0 & B_{n+1} & 0 \\ 0 & B_{n+1} & 0 & -B_n \\ B_{n-1} & 0 & B_n & 0 \end{pmatrix} \text{ if } n \text{ odd integer} \\ \mathbb{Q}_B^n &= \begin{pmatrix} -B_{n-1} & 0 & -B_n & 0 \\ 0 & B_{n+1} & 0 & -B_n \\ B_n & 0 & B_{n+1} & 0 \\ 0 & B_n & 0 & -B_{n-1} \end{pmatrix} \text{ if } n \text{ even integer} \end{aligned}$$

here n is positive integer and B_n is nth balancing number.

Proof. It holds for $n = 1$ and $n = 2$. Suppose it verifies for $n = k$, then by definition of the sequence

$$\mathbb{Q}_B^{k+1} = \begin{pmatrix} 0 & -B_n & 0 & B_{n-1} \\ B_n & 0 & B_{n+1} & 0 \\ 0 & B_{n+1} & 0 & -B_n \\ B_{n-1} & 0 & B_n & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 6 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$Q_B^{k+2} = \begin{pmatrix} -B_{n-1} & 0 & -B_n & 0 \\ 0 & B_{n+1} & 0 & -B_n \\ B_n & 0 & B_{n+1} & 0 \\ 0 & B_n & 0 & -B_{n-1} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 6 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

by induction, the theorem holds for all positive integers. ∎

Corollary 4

$$\det Q_B^n = [B_n^2 - B_{n-1}B_{n+1}]^2 = 1.$$

4 Combining Chebyshev polynomials with balancing and Lucas-balancing numbers

We know [9] that

$$Q_n(x) = \begin{vmatrix} x & 1 & & \\ a & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & a & x \end{vmatrix} = (\sqrt{a})^n U_n(\frac{x}{2\sqrt{a}}) \qquad (9)$$

here $U_n(x)$ is the second kind Chebyshev polynomial which satisfies the following recurrence relation:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$ for all $n = 1, 2, \ldots$.

From (2) for $k = 1$ and (9) for $a = 1$ and $x = 6$, we have

$$B_{n+1} = U_n(3).$$

Moreover, from [10] we can write

$$U_n(x) = U_l(x)U_{n-l}(x) - U_{l-1}(x)U_{n-l-1}(x)$$

with $1 \leq l \leq n$. So we have

$$B_{n+1} = B_{l+1}B_{n-l+1} - B_lB_{n-l}$$

which is given in ([8],Theorem 5.1).

It is known that [7]

$$T_n(x) = \begin{vmatrix} x & 1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{vmatrix}$$

here $T_n(x)$ denotes Chebyshev polynomial of first kind. It is also well-known that

$$2T_n(x) = U_n(x) - U_{n-2}(x) \tag{10}$$

From (3) for $k = 1$, (9) and (10), it is seen, readily

$$2C_{n-1} = B_n - B_{n-2}.$$

Consequently, by exploiting properties of Chebyshev polynomials of first and second kind, some new properties can be verified for balancing and Lucas-balancing numbers. In [15], the author obtained some results for Fibonacci and Pell numbers by using similar method.

5 Illustrative examples

Let us consider the results given with (4) and (6) for balancing numbers, for some k and m taken below.

For $k = 2$ and $m = 3$, then

$$H_{6,2} = \begin{pmatrix} 6 & 0 & -1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -1 & 0 & 0 \\ -1 & 0 & 6 & 0 & -1 & 0 \\ 0 & -1 & 0 & 6 & 0 & -1 \\ 0 & 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & 0 & -1 & 0 & 6 \end{pmatrix}$$

$$\det H_{6,2} = B_4^2$$

and for $k = 3$ and $m = 2$

$$T_{6,3} = \begin{pmatrix} 6 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \\ -1 & 0 & 0 & 6 & 0 & 0 \\ 0 & -1 & 0 & 0 & 6 & 0 \\ 0 & 0 & -1 & 0 & 0 & 6 \end{pmatrix}$$

$$\text{per}T_{6,3} = B_3^3.$$

Taking into account (5) and (7) for Lucas-balancing numbers, for some k and m given below, we give determinant and permanents for Lucas-balancing numbers.

$$U_{8,2} = \begin{pmatrix} 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 6 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 6 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 6 \end{pmatrix} \quad \text{if } k = 2 \text{ and } m = 4$$

$$\det U_{8,2} = C_4^2$$

$$V_{8,4} = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \text{if } k = 4 \text{ and } m = 2$$

$$\text{per}V_{8,4} = C_2^4.$$

6 Conclusion

In this paper, we give determinants and permanents of $k-$tridiagonal matrices with balancing and Lucas-balancing numbers. Moreover, we show that these sequences can be obtained with Chebyshev polynomials. So, they verify properties of Cheyshev polynomials of first and second kind.

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