Harmonic index of dense graphs*

Jianxi Liu†

Department of Applied Mathematics, School of Informatics

Guangdong University of Foreign Studies, Guangzhou 510006, PR China

Abstract

The harmonic weight of an edge is defined as reciprocal of the average degree of its end-vertices. The harmonic index of a graph G is defined as the sum of all harmonic weights of its edges. In this work, we give the minimum value of the harmonic index for any $n-$vertex connected graphs with minimum degree δ at least $k(\geq n/2)$ and show the corresponding extremal graphs have only two degrees, i.e., degree k and degree $n-1$, and the number of vertices of degree k is as close to $n/2$ as possible.

Key words: Harmonic index; dense graphs; minimum value

MR Subject Classification: 05C07; 05C35; 90C35

*Research supported by the National Natural Science Foundation of China(No.11101097) and Foundation for Distinguished Young Talents in Higher Education of Guangdong, China(No.LYM11061, No.2013LYM0027).

†Corresponding author: Tel. +86 13580599268. E-mail: liujianxi2001@gmail.com.

1 Introduction

All graphs considered in the following will be simple. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $G(n,k)$ be the set of connected simple $n-$vertex graphs with minimum vertex degree at least k. The Randić index $R(G)$ of a graph G is defined as: $R(G) = \sum_{uv\in E(G)} \left(d(u)d(v)\right)^{-\frac{1}{2}}$, where $d(u)$ denotes the degree of a vertex u. It is also known as connectivity index or branching index. Randić [23] in 1975 proposed this index for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. There is also a good correlation between the Randić index and several physicochemical properties of alkanes: boiling points, surface areas, energy levels, etc. In [7] Fajtlowitcz mentioned that Bollobás and Erdős asked for the minimum value on the Randić index among $G(n,k)$ for every $1 \leq k \leq n-1$. This problem turn out to be difficult and has been resolved completely by efforts of many scholars, see references [1, 2, 16, 18, 21, 22] and references within. For a comprehensive survey of the mathematical properties of the Randić index, see the book of Li and Gutman [15] and the book of Gutman and Furtula [10], or a survey of Li and Shi [19]. See also the books of Kier and Hall [11, 12] for chemical properties of this index.

With motivation from the Randić index, the sum-connectivity index $\chi(G)$ and the general sum-connectivity index $\chi^{\alpha}(G)$ were recently proposed by Zhou and Trinajstić in [26, 27] and defined as $\chi(G) = \sum\limits_{uv}(d(u)+d(v))^{-\frac{1}{2}}$, and $\chi^{\alpha}(G) = \sum\limits_{uv}(d(u)+d(v))^{\alpha}$, where α is a real number. It has been found that the (general) sum-connectivity index and the Randić index correlate well between themselves and with the π-electronic energy of benzenoid hydrocarbons [17, 20]. Some mathematical properties of the (general) sum-connectivity index on trees, molecular trees, unicyclic graphs and bicyclic graphs were given in [3, 4, 5, 6, 26, 27].

In this work, we consider another variant of the Randić index, named the harmonic index. For a graph G, the harmonic weight of an edge is defined as reciprocal of the average degree of its end-vertices. The harmonic index of a graph G is defined as the sum of all harmonic weights of its edges, i.e., $H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}$. Note that $H(G) = 2\chi^{-1}(G)$ and $H(G) \leq R(G)$ for any graph G and the equality holds for the last formula if and only if each component of G is regular. Thus, lower bounds of the harmonic index are lower bounds of the Randić index and upper bounds of the Randić index are upper bounds of the Harmonic index.

For two vertex disjoint graphs, G and F, let $G+F$ denote their join, i.e., the graph obtained by joining edges from every vertex of G to all vertices of F. We shall also denote by $\overline{K}_n$ the complement of K_n, which consists of n isolated vertices. To our knowledge, this index first appeared in [8]. Favaron et al.[9] considered the relation between harmonic index and the eigenvalues of graphs. Similar to the Randić index, it is meaningful to ask for the minimum value of the harmonic index among $G(n,k)$ for every $1 \leq k \leq n-1$. Zhong [25] found the minimum values of the harmonic index among $G(n,1)$ and the corresponding graph is the star, i.e., $\overline{K}_{n-1} + K_1$. Recently, Wu et al. [24] found the minimum value of the harmonic index for $G(n,2)$ and the corresponding graph is $\overline{K}_{n-2} + K_2$. In this work, we will solve this problem for any $n-$vertex connected graphs with minimum degree at least k for any $k \geq n/2$ and show the corresponding extremal graphs have only two degrees, i.e., degree k and degree $n-1$, and the number of vertices of degree k is as close to $n/2$ as possible. Thus, we have solved the problem of finding the minimum value of the harmonic index among $G(n,k)$ for $k = 1, 2$ and at least $n/2$.

2 A nonlinear programming model for the Harmonic index

Before we go forwards to investigate the relationship between the Harmonic index and the minimum degree $\delta(G)$ of graphs, we give a nonlinear programming model for the Harmonic index in this section, which is vital in sequel. Let G be an n−vertex connected graphs with degree $\delta(G) \geq k$. Denote by $x_{i,j}$ ($x_{i,j} \geq 0$), the number of edges joining the vertices of degrees i and j. Denote by n_i the number of vertices of degree of i. Let $x_{i,j} = n_i n_j - y_{i,j}$ for $i \neq j$ and $x_{i,j} = \binom{n_i}{2} - y_{i,j}$ for $i = j$. Denote by $w(i,j) = \frac{1}{i} + \frac{1}{j} - \frac{4}{i+j}$. It is easy to see that $w(i,j)$ is decreasing in i and increasing in j for $k \leq j < j \leq n-1$. In [13] we have

$$H(G) = \frac{n}{2} - \frac{1}{2} \sum_{k \leq i < j \leq n-1} w(i,j) x_{i,j}. \tag{2.1}$$

Then the problem on the minimum value of $H(G)$ in $G(n,k)$ can be formulated as the problem (P):

$$\max\ \gamma = \sum_{k \leq i < j \leq n-1} w(i,j) n_i n_j - \sum_{k \leq i < j \leq n-1} w(i,j) y_{ij}$$

$$\text{s.t.} \quad \sum_{\substack{k \leq j \leq n-1 \\ j \neq i}} y_{i,j} + 2y_{i,i} = (n-1-i)n_i \quad \text{for} \quad k \leq i \leq n-1; \tag{2.2}$$

$$n_k + n_{k+1} + \ldots + n_{n-1} = n; \tag{2.3}$$

$$n_i \geq 0 \quad \text{for} \quad k \leq i \leq n-1; \tag{2.4}$$

$$y_{i,j} \geq 0 \quad \text{for} \quad k \leq i \leq j \leq n-1; \tag{2.5}$$

$$y_{i,j},\ n_i \in \mathbb{N},\ k \leq i < j \leq n-1. \tag{2.6}$$

Denote $\gamma_1 = \sum_{k \leq i < j \leq n-1} w(i,j) n_i n_j$ and $\gamma_2 = -\sum_{k \leq i < j \leq n-1} w(i,j) y_{ij}$. Then $\max\ \gamma \leq \max\ \gamma_1 + \max\ \gamma_2$, where the maxima are subject to (2.2)–

(2.6). It is evident that max $\gamma_2 = 0$ and it is achieved for $y_{i,j} = 0, k \leq i < j \leq n-1$ and $y_{i,i} = (n-1-i)n_i/2, k \leq i \leq n-1$.

In next section we shall consider the problem of maximizing γ_1 and all sequences $(n_k, \cdots, n_{n-1})$ reaching the maximum values of γ_1 will be found and so will be sequences for some second maximum of γ_1. These sequences which have graphical realizations also reach the maximum of γ_2, hence the maximum of γ. In order to find max γ_1 we can neglect constraints (2.2) and (2.5), because for γ_1 only constraints (2.3) and (2.4) are relevant. In sequel, we will consider the problem of elements which maximizing γ_1 and the corresponding graphical realizations in D.

3 A technical lemma

Denote $\phi(n) = \max_{\ell \in \mathbb{N}} \ell(n-\ell)$; it follows that $\phi(n) = \frac{n^2}{4}$ for n even ($\ell = \frac{n}{2}$) and $\phi(n) = \frac{n^2-1}{4}$ for n odd ($\ell = \frac{n\pm1}{2}$). If $1 \leq k \leq \delta \leq \Delta \leq n-1$ ($\delta, \Delta \in \mathbb{N}$), consider the function $f(x) = \sum_{k \leq i < j \leq n-1} w(i,j)x_i x_j$ and the domains $D = \{x = (x_k, x_{k+1}, \ldots, x_{n-1}) | x_k + x_{k+1} + \ldots + x_{n-1} = n, n_i \in \mathbb{N},\ k \leq i \leq n-1\}$ and $D_1 = D \backslash \{(\frac{n}{2}, 0, \ldots, 0, \frac{n}{2})\}$ for n even.

Lemma 3.1 *(1) If $n \geq 4$ is even, then $\max_D f(x)$ is reached for $(n/2, 0, \ldots, 0, n/2)$ and $\max_{D_1} f(x)$ for $(\frac{n-2}{2}, 0, \ldots, 0, \frac{n+2}{2})$ and $(\frac{n+2}{2}, 0, \ldots, 0, \frac{n-2}{2})$;*
(2) If $n \geq 5$ is odd, then $\max_D f(x)$ is reached for $(\frac{n-1}{2}, 0, \ldots, 0, \frac{n+1}{2})$ and $(\frac{n+1}{2}, 0, \ldots, 0, \frac{n-1}{2})$.

Proof. First we shall determine the maximum of $f(x)$ in the domain D. If $x_{k+1} = \ldots = x_{n-2} = 0$ then $f(x) = w(k, n-1)x_k x_{n-1}$ and the result is obvious since $x_k + x_{n-1} = n$. Otherwise, denote by $i(k+1 \leq i \leq n-1)$ the smallest index such that $x_i \geq 1$ and by $j(k+1 \leq j \leq n-1)$

the greatest index such that $x_j \geq 1$; obviously $i \leq j$. Denote by $\mathbf{O}_1$ and $\mathbf{O}_2$ the operations consisting of replacing $x = (x_k, \ldots, x_{n-1}) \in D$ by $x' = (x_k, 0, \ldots, 0, x_i, \ldots, x_{j-1}, x_j - 1, 0, \ldots, 0, x_{n-1} + 1) \in D$ and by $x'' = (x_k + 1, 0, \ldots, 0, x_i - 1, x_{k+1}, \ldots, x_j, 0, \ldots, 0, x_{n-1}) \in D$, respectively. We have $f(x') - f(x) = w(j, n-1)(x_j - x_{n-1} - 1) + x_k\Big(w(k, n-1) - w(k, j)\Big) + \sum_{p=i}^{j-1} \Big(w(p, n-1) - w(p, j)\Big)x_p$. Since $w(p, n-1) - w(p, j) > w(j, n-1)$ for $p = k, i, i+1, \ldots, j-1$, we have

$$f(x') - f(x) \geq w(j, n-1)(x_k + x_i + x_{i+1} + \ldots + x_j - x_{n-1} - 1) \qquad (3.7)$$

and this inequality is strict if at least one of $x_k, x_i, x_{i+1}, \ldots, x_{j-1}$ is different from zero. Similarly, we have $f(x'') - f(x) = w(k, i)(x_i - x_k - 1) + x_{n-1}\Big(w(k, n-1) - w(i, n-1)\Big) + \sum_{p=i}^{j-1} \Big(w(k, p) - w(i, p)\Big)x_p$. Since $w(k, p) - w(i, p) > w(k, i)$ for $p = i+1, \ldots, j, n-1$, we have

$$f(x'') - f(x) \geq w(k, i)(x_i + x_{i+1} + \ldots + x_j + x_{n-1} - x_k - 1) \qquad (3.8)$$

and this inequality is strict if at least one of $x_k, x_i, x_{i+1}, \ldots, x_{j-1}$ is different from zero. If $i = j$ we get

$$f(x') - f(x) \geq w(i, n-1)(x_k + x_i - x_{n-1} - 1) \qquad (3.9)$$

the inequality being strict if $x_k \geq 1$ and

$$f(x'') - f(x) \geq w(k, i)(x_i + x_{n-1} - x_k - 1) \qquad (3.10)$$

the inequality being strict if $x_{n-1} \geq 1$. We shall prove that at least one of the differences $f(x') - f(x)$ and $f(x'') - f(x)$ is greater than zero, which implies that all sequences $x \in D$ realizing maximum of $f(x)$ satisfy $x_{k+1} = \ldots = x_{n-2} = 0$. Consider first the case when $i = j$. It is clear that if $x_k = x_{n-1}$ then $f(x) = 0$ which implies that $(0, \ldots, 0, n_i, 0, \ldots, 0)$ cannot maximize f. Otherwise, suppose that $x_k \geq 1$. If $x_k + x_i - x_{n-1} - 1 \geq 0$ then

(3.9) is strict and it follows that $f(x') > f(x)$ and x cannot maximize f on D. Otherwise, $x_k + x_i - x_{n-1} - 1 \leq -1$. In this case $x_{n-1} \geq x_k + x_i$, hence $x_{n-1} + x_i - x_k - 1 \geq 2x_i - 1 \geq 1$, which implies $f(x'') > f(x)$ and x cannot maximize f. If $x_{n-1} \geq 1$ the same conclusion follows since (3.10) is strict. Suppose $i < j$. In this case $x_i > 0, x_j > 0$ and both inequalities (3.7) and (3.8) are strict. If $x_k + x_i + x_{i+1} + \ldots + x_j - x_{n-1} - 1 \geq 0$ then from (3.7) it follows that $f(x') > f(x)$. Otherwise, $x_{n-1} \geq x_k + x_i + x_{i+1} + \ldots + x_j$ and $x_i + x_{i+1} + \ldots + x_j + x_{n-1} - x_k - 1 \geq 2(x_k + x_{k+1} + \ldots + x_j) - 1 > 0$, which implies $f(x'') > f(x)$ from (3.8). Consequently, all sequences maximizing f have the form $(x_k, 0, \ldots, 0, x_{n-1})$ where $x_k + x_{n-1} = n$; in this case $f(x_k, 0, \ldots, 0, x_{n-1}) = w(k, n-1)x_k x_{n-1} \leq w(k, n-1)\phi(n)$ and the conclusion follows.

Second, we consider that $x \in D_1$ when n is even. If $x_{k+1} = \ldots = x_{n-2} = 0$ then $f = w(k, n-1)x_k x_{n-1}$ and the result is obvious since $x_k + x_{n-1} = n$. If $x_{k+1} + \ldots + x_{n-2} \geq 2$ then we have seen that by an operation $\mathbf{O}_1$ or $\mathbf{O}_2$ we can find sequence $y \in D_1$ such that $f(x) < f(y)$, hence x cannot maximize f on D_1. Now, we consider the case when $x_{k+1} + \ldots + x_{n-2} = 1$. We shall prove that x cannot maximize f if $x_{k+1} + \ldots + x_{n-2} = 1$, i.e., there exists an index $i, k+1 \leq i \leq n-2$ such that $n_i = 1$ and $n_j = 0$ for every $k+1 \leq j \leq n-2$ and $j \neq i$. Let $q = \min(x_k, x_{n-1}) \leq \frac{n-2}{2}$. Without loss of generality suppose that $q = x_k$. If $q = \frac{n-2}{2}$, then $x_k = \frac{n-2}{2}$ and

$F \in \mathbf{F}(\frac{n}{2}, k-\frac{n}{2})$ for k even and $n = 0 \ (mod\ 4)$, k odd and $n = 2 \ (mod\ 4)$; $H(G) = \frac{n}{2} - \frac{n^2-4}{8} w(k, n-1)$ and $G \cong F + K_{\frac{n+2}{2}}$, where $F \in \mathbf{F}(\frac{n-2}{2}, k - \frac{n+2}{2})$ if $k \geq \frac{n+2}{2}$ for k even and $n = 2 \ (mod\ 4)$, or $H(G) = \frac{n}{2} - \frac{n^2-4}{8} w(k, n-1)$ and $G \cong F + K_{\frac{n-2}{2}}$, where $F \in \mathbf{F}(\frac{n+2}{2}, k - \frac{n-2}{2})$ for k even and $n = 2 \ (mod\ 4)$.

(2) If n is odd, then: $H(G) = \frac{n}{2} - \frac{n^2-1}{8} w(k, n-1)$ and $G \cong F + K_{\frac{n+1}{2}}$, where $F \in \mathbf{F}(\frac{n-1}{2}, k - \frac{n+1}{2})$ if $k \geq \frac{n+1}{2}$ for k even, k odd and $n = 1 \ (mod\ 4)$, or $H(G) = \frac{n}{2} - \frac{n^2-1}{8} w(k, n-1)$ and $G \cong F + K_{\frac{n-1}{2}}$, where $F \in \mathbf{F}(\frac{n+1}{2}, k - \frac{n-1}{2})$ for k even, k odd and $n = 3 \ (mod\ 4)$.

Proof. If n is even, from Lemma 3.1 we deduce that $\max \gamma_1$ is reached only for $(n_k, \ldots, n_{n-1}) = (\frac{n}{2}, 0, \ldots, 0, \frac{n}{2})$. It follows that the vertices of degree $n-1$ induce a complete subgraph $K_{\frac{n}{2}}$ and the remaining vertices, of degree k, a subgraph which is regular of degree $k - \frac{n}{2}$. This subgraph exists only if $\frac{n}{2}(k - \frac{n}{2})$ is an even number, i.e., when k is even and $n = 0(mod\ 4)$, k is odd and $n = 0(mod\ 4)$ or when k is odd and $n = 2(mod\ 4)$. In these cases any extremal graph is of the form $F + K_{\frac{n}{2}}$, where F belongs to $\mathbf{F}(\frac{n}{2}, k - \frac{n}{2})$ since $\max \gamma_2 = 0$ ($x_{i,j} = n_i n_j$ for $n/2 \leq i < j \leq n-1$).

In the remaining case (k even and $n = 2(mod\ 4)$) there is no graphical realization for $(\frac{n}{2}, 0, \ldots, 0, \frac{n}{2})$ and we shall consider the second maximum value for γ_1. This value is reached only for $x^1 = (\frac{n}{2} - 1, 0, \ldots, 0, \frac{n}{2} + 1)$ and $x^2 = (\frac{n}{2} + 1, 0, \ldots, 0, \frac{n}{2} - 1)$. In the case of x^1 we deduce that there exists a graphical realization, namely $F + K_{\frac{n}{2}+1}$, where $F \in \mathbf{F}(\frac{n}{2} - 1, k - \frac{n}{2} - 1)$ if $k \geq \frac{n}{2} + 1$ and k even and $n = 2(mod\ 4)$. In this case also $\max \gamma_2 = 0$, hence this second extremal value of γ_1 is also the second extremal value of γ. If the second extremal point is x^2, there also exists a graphical realization $F + K_{\frac{n}{2}-1}$, where $F \in \mathbf{F}(\frac{n}{2} + 1, k - \frac{n}{2} + 1)$ if k even and $n = 2(mod\ 4)$. Note that for $k = \frac{n}{2}$ it is not possible to have k even and $n = 2(mod\ 4)$,

$x_{n-1} = \frac{n}{2}$. Applying operation $\mathbf{O}_1$ we deduce that

$$
\begin{aligned}
& f(\frac{n-2}{2},0,\ldots,0,\frac{n+2}{2}) - f(\frac{n-2}{2},0,\ldots,0,1,0,\ldots,0,\frac{n}{2}) \\
&= \frac{n-2}{2}\Big(w(k,n-1)-w(k,i)\Big) - \frac{n}{2}w(i,n-1) \\
&= \frac{(n-2)(i-k)(n-1-i)(n-1+k+2i)}{j(k+i)(i+n-1)(k+n-1)} - \frac{(n-1-i)^2}{i(n-1)(i+n-1)} \\
&\geq \frac{n-1-i}{i(i+n-1)}\left(\frac{(n-2)(i-k)(n-1+k+2i)}{2(n-2)(k+n-1)} - \frac{n-1-k-1}{n-1}\right) \\
&> \frac{n-1-i}{i(i+n-1)}\left(\frac{1}{2} - \frac{n-\frac{n}{2}-2}{n-1}\right) > 0,
\end{aligned}
$$

where the first and second inequalities hold because $\frac{n}{2} \leq k < i \leq n-2$. Thus, x cannot maximize f if $x_{k+1} + \ldots + x_{n-2} = 1$. If $q = x_k \leq \frac{n}{2} - 2$. It follows that $x_{n-1} \geq \frac{n+2}{2}$ and applying an operation $\mathbf{O}_2$ we deduce from (3.10) that $f(x'') - f(x) \geq w(k,i)(x_i + x_{n-1} - x_k - 1) > 0$. In this case $f(x'') \leq \frac{n-2}{2} \cdot \frac{n+2}{2} \cdot w(k,n-1) < \max_D f(x)$, which implies that x cannot maximize f on D_1. If $\min(x_k, x_{n-1}) = x_{n-1}$ we deduce that only $(\frac{n+2}{2},0,\ldots,0,\frac{n-2}{2})$ maximize $f(x)$ on D_1 by similar arguments. ∎

4 Main results

In this section, we shall prove that every graph with minimum value of harmonic index in $G(n,k)$ is a join between a regular graph and a complete graph. Let $\mathbf{F}(n,r)$ be the set of $r-$regular graphs of order n. We have $\mathbf{F}(n,r) \neq \emptyset$ if and only if $r \leq n-1$ and nr is even (see for example [14]).

Theorem 4.1 *If G is a graph with the minimum value of harmonic index in $G(n,k)$ for $k \geq n/2$, then G is a join between a regular graph and a complete graph, namely:*

(1) If n is even, then: $H(G) = \frac{n}{2} - \frac{n^2}{8}w(k,n-1)$ and $G \cong F + K_{n/2}$, where

the extremal graph is only $\overline{K}_{\frac{n}{2}} + K_{\frac{n}{2}}$.

If n is odd, then $\max \gamma_1$ is reached for $x^3 = (\frac{n-1}{2}, 0, \ldots, 0, \frac{n+1}{2})$ or $x^4 = (\frac{n+1}{2}, 0, \ldots, 0, \frac{n-1}{2})$. x^3 correspond to a graphical realization $F + K_{\frac{n+1}{2}}$, where $F \in \mathbf{F}(\frac{n-1}{2}, k - \frac{n+1}{2})$ if $k \geq \frac{n+1}{2}$ for k even and $n = 1(mod\ 4)$, k even and $n = 3(mod\ 4)$ and k is odd and $n = 3(mod\ 4)$. For x^4 we obtain $F + K_{\frac{n-1}{2}}$, where $F \in \mathbf{F}(\frac{n+1}{2}, k - \frac{n-1}{2})$ if k is even and $n = 1(mod\ 4)$, k is even and $n = 3(mod\ 4)$ or k is odd and $n = 3(mod\ 4)$. All the corresponding extremal values are immediate by (2.1). ∎

References

[1] C. Delorme, O. Favaron, D. Rautenbach, On the Randić index, Discrete Math. **257**(1)(2002), 29-38.

[2] T. Divnić, L. Pavlović, Proof of the first part of the conjecture of Aouchiche and Hansen about the Randić index, Discrete Appl. Math. **161**(2013), 953-960.

[3] Z. Du, B. Zhou, On sum-connectivity index of bicyclic graphs, Bull. Malays. Math. Sci. Soc. (2) **35**(1)(2012), 101-117.

[4] Z. Du, B. Zhou, N. Trinajstić, Minimum general sum-connectivity index of unicyclic graphs, J. Math. Chem. **48**(2010), 697-703.

[5] Z. Du, B. Zhou, N. Trinajsti Minimum sum-connectivity indices of trees and unicyclic graphs of a given matching number, J. Math. Chem. **47**(2010), 842855.

[6] Z. Du, B. Zhou, N. Trinajstić, On the general sum-connectivity index of trees, Appl. Math. Lett. **24**(2011), 402-405.

[7] S. Fajtlowicz, Written on the wall, Conjectures derived on the basis of the program Galatea Gabriella Graffiti, University of Houston, 1998.

[8] S. Fajtlowicz, On conjectures of Graffiti-II, Congr. Numer. **60**(1987), 187-197.

[9] O. Favaron, M. Mahó, J.F. Saclé, Some eigenvalue properties in graphs (conjectures of Graffiti-II), Discrete Math. **111**(1993), 197-220.

[10] I. Gutman, B. Furtula(Eds.), Recent Results in the Theory of Randić Index, Mathematical Chemistry Monographs No.6, Univ. Kragujeva, Kragujevac, 2008.

[11] L.B. Kier, L.H. Hall, Molecular Connectivity in Chemistry and Drug Research, Academic Press, New York, 1976.

[12] L.B. Kier, L.H. Hall, Molecular Connectivity in Structure-Activity Analysis, Research Studies Press-Wiley, Chichester(UK), 1986.

[13] J. Liu, On harmonic index and diameter of graphs, Journal of Applied Mathematics and Physics, **1**(2013), 5-6.

[14] L. Lovász, Combinatorial problems and exercises, Akadémiai Kiadó, Budapest, 1979.

[15] X. Li, I. Gutman, Mathematical Aspects of Randić-Type Molecular Structure Descriptors, Mathematical Chemistry Monographs No.1, Kragujevac, 2006, pp.VI+330.

[16] X. Li, B. Liu, J. Liu, Complete solution to a conjecture on Randć index, Europ. J. Oper. Res. **200**(1)(2010), 9-13.

[17] B. Lučić, S. Nikolić, N. Trinajstić, B. Zhou, S. I. Turk, Sum-connectivity index, in: I. Gutman, B. Furtula (Eds.), Novel Molecular

Structure Descriptors-Theory and Applications I, Univ. Kragujevac, Kragujevac, 2010, pp. 101-136.

[18] B. Liu, L. Pavlović, T. Divnić, J. Liu, M. Stojanović, On the Conjecture of Aouchiche and Hansena bout the Randić index, Discrete Math. **313**(2013), 225-235.

[19] X. Li, Y.T. Shi, A survey on the Randić index, MATCH Commun. Math. Comput. Chem. **59**(1)(2008), 127-156.

[20] B. Lučić, N. Trinajstić, B. Zhou, Comparison between the sum-connectivity index and product-connectivity index for benzenoid hydrocarbons, Chem. Phys. Lett. **475**(2009), 146-148.

[21] L. Pavlović, On the conjecture of Delorme, Favaron and Rautenbach about the Randić index, Europ. J. Oper. Res. **180**(1)(2007), 369-377.

[22] L. Pavlović, T. Divnić, A quadratic programming approach to the Randić index, Europ. J. Oper. Res. **176**(1)(2007), 435-444.

[23] M. Randić, On characterization of molecular branching, J. Amer. Chem. Soc. **97**(1975), 6609-6615.

[24] R. Wu, Z. Tang, H. Deng, A lower bound for the harmonic index of a graph with minimum degree at least two, Filomat, **27**(1)(2013), 51-55.

[25] L. Zhong, The harmonic index for graphs, Appl. Math. Lett. **25**(2012), 561-566.

[26] B. Zhou, N. Trinajstić, On a novel connectivity index, J. Math. Chem. **46**(2009), 1252-1270.

[27] B. Zhou, N. Trinajstić, On general sum-connectivity index, J. Math. Chem. **47**(2010), 210-218.