On the interior H-points of H-polygons

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Abstract

An H-polygon is a simple polygon whose vertices are H-points, which are points of the set of vertices of a tiling of $\mathbb{R}^2$ by regular hexagons of unit edge. Let $G(v)$ denote the least possible number of H-points in the interior of a convex H-polygon K with v vertices. In this paper we prove that $G(8) = 2$, $G(9) = 4$, $G(10) = 6$ and $G(v) \geq \left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1$ for all $v \geq 11$, where $\lceil x \rceil$ denotes the minimal integer more than or equal to x.

Keywords: lattice points; H-points; H-polygon; interior points.

1 Introduction

Let $\vec{u}$, $\vec{v}$ be two linearly independent real vectors in $\mathbb{R}^2$. The set of all points $X = m\vec{u} + n\vec{v}$ with integral m, n is called a *general lattice* Λ generated by $\vec{u}$ and $\vec{v}$. Specially, if $\vec{u}$ and $\vec{v}$ are mutually orthogonal unit vectors, the lattice Λ is called an *integral lattice* $\mathbb{Z}^2$ and a point of the integral lattice is called an *integral lattice point*. Without confusion, an integral lattice point is called a lattice point for short in this paper.

A simple polygon in the plane with vertices in the integral lattice $\mathbb{Z}^2$ is called a *lattice polygon*. For a lattice polygon P, we use $v = v(P)$, $b = b(P)$ and $i = i(P)$ to denote the number of vertices, the number of boundary

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lattice points and the number of interior lattice points of P respectively. The problem of finding the relationships between the numbers v, b and i is of great interest and has been investigated by many authors and in different setting (not only for the integral lattice), see among others ([2][5][7][8]). Rabinowitz [7] has obtained many relationships between the numbers v and i. Following Rabinowitz we define the function: $g(v) = \min\{i(K) : v(K) = v\}$, where K is a convex lattice polygon in the plane. In [1][7] the property of the function $g(v)$ is studied. It is known that $g(3) = g(4) = 0$, $g(5) = g(6) = 1$, $g(7) = g(8) = 4$, $g(9) = 7$ and $g(10) = 10$.

Geometrically, the integral lattice $\mathbb{Z}^2$ is the set of corners of a tiling of $\mathbb{R}^2$ by unit squares. In this paper we are motivated to investigate analogous properties of polygons with vertices in the set H consisting of all corners of a monohedral tiling $\mathcal{H}$ of $\mathbb{R}^2$ by regular hexagons with unit edge. Let H be the set of corners of tiling $\mathcal{H}$. A point of H is called an *H-point*. Completely analogously, we can define an *H-polygon* P as a simple polygon in $\mathbb{R}^2$ whose vertices lie in H. To avoid confusion, in this paper for an H-polygon P we use $v_H(P)$, $b_H(P)$ and $i_H(P)$ to denote the number of vertices, the number of boundary H-points and the number of interior H-points of P respectively.

Let K be a convex H-polygon in the plane. Similarly, we define the function

$$G(v) = \min\{i_H(K) : v_H(K) = v\}.$$

Trivially, it is known that $G(3) = G(4) = G(5) = G(6) = 0$. The first results about $G(v)$ appeared in [3] and it is only shown that $G(7) = 2$. As yet little is known about the function $G(v)$. In this paper we discuss the property of the function $G(v)$. In section 2, some basis denotations and lemmas are presented. In section 3, It is shown that $G(8) = 2$, $G(9) = 4$, $G(10) = 6$. Furthermore, a lower bound of $G(v)$, $\left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1$, is presented in this paper.

2 Preliminaries

In fact, the set H can be regarded as a disjoint union of two sets H^+ and H^-, where all points in H^+ have three tiling edges leaving the points in the same three directions, and all points in H^- have edges which leave in the opposite three directions, as shown in Figure 1. A point of H^+ (resp. H^-) is called an *H^+-point* (resp. *H^--point*). Let C denote the set of all centers of the hexagonal tiles which determine $\mathcal{H}$. A point of C is said to be a *C-point*.

Without loss of generality, we build a cartesian coordinate system of $\mathbb{R}^2$ with the origin at an H^+-point and the x-axis lying along one edge of a

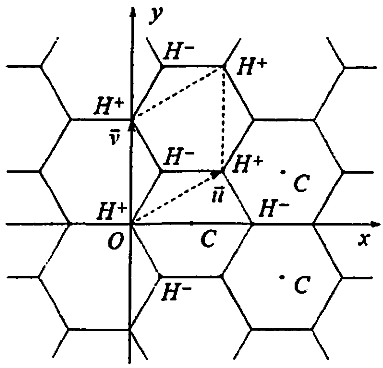

Figure 1: H^+-points and H^--points

regular hexagonal tile, as shown in Figure 1. Let $\vec{u} = (\frac{3}{2}, \frac{\sqrt{3}}{2})$, $\vec{v} = (0, \sqrt{3})$. It is easy to see that the set H^+ is a general lattice generated by the two non-collinear vectors $\vec{u}$ and $\vec{v}$. That is to say,

$$H^+ = \{s\vec{u} + t\vec{v} \mid s, t \in \mathbb{Z}\}. \tag{2.1}$$

It is not difficult to see that the set H^- and C can be considered to be a general lattice translated by the set H^+, hence the set H^- and C can be represented as

$$H^- = \{(-1,0) + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{Z}\}, C = \{(1,0) + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{Z}\}.$$

Furthermore, it is clear that $H \cup C$ is a general lattice generated by two vectors $\vec{u_0} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\vec{v_0} = (1, 0)$. $H \cup C$ is called a *triangular lattice* T. A point of T is called a *T-point*. Similarly, a simple polygon in $\mathbb{R}^2$ with all vertices in T (resp. C) is called a *T-polygon* (resp. *C-polygon*). For an T-polygon P, we use $v_T(P)$, $b_T(P)$ and $i_T(P)$ to denote the number of vertices, the number of boundary T-points and the number of interior T-points of P respectively.

Clearly an H-polygon or a C-polygon is also a T-polygon. Particularly, a triangle with three vertices in C is called a *C-triangle*. A C-triangle Δ is said to be *primitive* if $\Delta \cap C$ consists only of the three vertices of Δ. A segment with endpoints in C is called a *C-segment*. In [3] [9] the property of C-triangle and C-segment has been investigated. Here the following two lemmas are very useful.

Lemma 2.1 ([9]). *Any C-segment contains zero or an even number of H-points.*

Lemma 2.2 ([3], [9]). *Let $\triangle$ be a primitive C-triangle. If there exists one side of $\triangle$ which contains H-points, $\triangle$ contains at least 2 H-points, otherwise $\triangle$ contains at least one H-point in its interior.*

Lemma 2.3. *Let $Y = \{c_1, c_2, \cdots, c_n\}$ be a set of n C-points with $n \geqslant 3$ such that not all elements are collinear. Then Y can span at least $n-2$ pairwise interior-disjoint primitive C-triangles.*

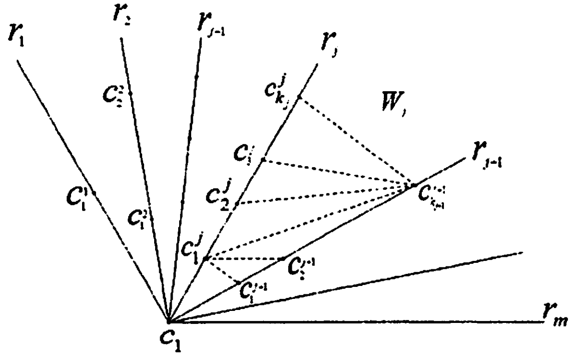

Figure 2: C-points span primitive C-triangles

Proof. Let $c_j = (x_j, y_j)$ ($j = 1, 2, \cdots, n$). We may assume without loss of generality that $y_1 = \min\{y_i : i = 1, 2, 3, \cdots, n\}$, $x_1 = \min\{x_i : y_i = y_1, i = 1, 2, 3, \cdots, n\}$. Sort $Y \setminus \{c_1\}$ radially around c_1 (clockwise), to produce rays $r_1, r_2, \cdots, r_m$ such that $r_j \cap (Y \setminus c_1) \neq \emptyset$, $Y \setminus \bigcup_{j=1}^{m} r_j = \emptyset$, where $j = 1, 2, 3, \cdots, m$. Now we renumber all the C-points in Y as follows. If $|r_j \cap (Y \setminus c_1)| = k_j$, we renumber those k_j C-points by $c_1^j, c_2^j, \cdots, c_{k_j}^j$ such that $|c_1 c_1^j| < |c_1 c_2^j| < \cdots < |c_1 c_{k_j}^j|$, as shown in Figure 2. Now let W_j denote the wedge formed by r_j and r_{j+1}, where $j = 1, 2, \cdots, m-1$. It is easy to see that line segments $c_1^j c_1^{j+1}$, $c_1^j c_2^{j+1}, \cdots, c_1^j c_{k_{j+1}}^{j+1}$, $c_{k_{j+1}}^{j+1} c_2^j$, $c_{k_{j+1}}^{j+1} c_3^j, \cdots, c_{k_{j+1}}^{j+1} c_{k_j}^j$ form $k_j + k_{j+1} - 1$ pairwise interior-disjoint C-triangles in W_j. Therefor Y can span $\sum_{j=1}^{m-1}(k_j + k_{j+1} - 1) = \sum_{j=1}^{m} k_j + \sum_{j=2}^{m-1}(k_{j+1} - 1) - 1 = n - 2 + \sum_{j=2}^{m-1}(k_{j+1} - 1) \geqslant n - 2$ pairwise interior-disjoint C-triangles. Since each non-primitive C-triangle contains at least one primitive C-triangle, the proof is complete. □

Let K be a convex lattice polygon in the plane. Denote by $H(K)$ the *interior hull* of K, that is, the convex hull of the lattice points lying in the interior of K. Notice that $H(K)$ might degenerate into a segment, a point, or even the empty set. The following lemmas from [7] and [2] will be useful.

Lemma 2.4 ([7]). *Let K be a convex lattice polygon with interior hull $H(K)$. If $v(K) \geqslant 7$, then $v(H(K)) \geqslant \lceil \frac{1}{2}v(K) \rceil$; if $v(K) \geqslant 9$, then $b(H(K)) \geqslant \lceil \frac{2}{3}v(K) \rceil$.*

Lemma 2.5 ([2]). *A convex lattice nonagon can have 7 interior lattice points or 10 interior lattice points, but it cannot have either 8 or 9 interior lattice points.*

Lemma 2.6 ([2]). *A convex lattice decagon can have 10 interior lattice points or 13 interior lattice points, but it cannot have either 11 or 12 interior lattice points.*

Lemma 2.7 ([2]). *If $H(K)$ is the interior hull of a convex lattice decagon with 10 interior lattice points, then $v(H(k)) = 6$, $b(H(K)) = 8$ and $i(H(K)) =$ 2.*

Remark 1. Since there is a linear transformation mapping $\tau : x' = x - \frac{\sqrt{3}}{3}y, y' = \frac{2\sqrt{3}}{3}y$. Under this mapping τ the integral lattice $\mathbb{Z}^2$ transforms into the triangular lattice T. It is easy to check that under this mapping τ all the statements in Lemma 2.4 to Lemma 2.7 still hold for corresponding T-polygons.

3 Main Results

In this section we determine some values of the function $G(v)$.

Theorem 3.1. $G(8) = 2$.

Proof. Firstly, we'll prove that $G(8) \geqslant 2$. Take a convex H-octagon K and embed it into the triangular lattice T. Then clearly K is a convex T-octagon. By Lemma 2.4 and Remark 1, the interior hull of K has at least four vertices. Thus there exist at least four interior T-points, say, a_1, a_2, a_3, a_4 in K such that $\text{conv}\{a_1, a_2, a_3, a_4\}$ is a quadrangle. There are three cases to consider.

(i) If $|H \cap \{a_1, a_2, a_3, a_4\}| \geqslant 2$, then $G(8) \geqslant 2$.

(ii) If $|H \cap \{a_1, a_2, a_3, a_4\}| = 1$, we may assume without loss of generality that $a_4 \in H$, $a_1, a_2, a_3 \in C$. Trivially, $a_4 \notin \Delta = \text{conv}\{a_1, a_2, a_3\}$. Furthermore, we may assume that the triangle Δ is a primitive C-triangle (if not, there must be a primitive C-triangle contained in Δ). By Lemma 2.2, Δ

contains at least one H-point. Therefore there are at least two H-points lying in K, namely, $G(8) \geqslant 2$.

(iii) If $|H \cap \{a_1, a_2, a_3, a_4\}| = 0$, then by Lemma 2.3 these four non-collinear C-points can span at least 2 interior-disjoint primitive C-triangles. By Lemma 2.2, K must contain at least 2 interior H-points, which means $G(8) \geqslant 2$.

Thus we prove that $G(8) \geqslant 2$.

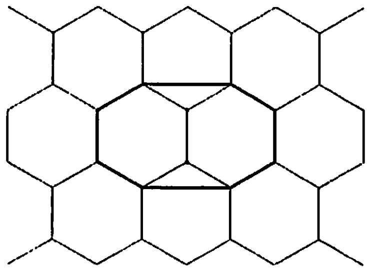

Figure 3: A convex H-octagon with two interior H-points

Furthermore, we construct a convex H-octagon with exactly 2 interior H-points, as shown in Figure 3. That is to say, $G(8) \leqslant 2$.

Combining the above discussions, we have $G(8) = 2$. The proof is complete. □

Theorem 3.2. $G(9) = 4$.

Proof. Let K be a convex H-nonagon and we embed it into the triangular lattice T. By Lemma 2.5 and Remark 1, $i_T(K) \geqslant 7$, $i_T(K) \neq 8, 9$. Furthermore, by Lemma 2.4 and Remark 1, for the convex hull $H(K)$ of the interior T-points in K, we have $v_T(H(K)) \geqslant 5$, $b_T(H(K)) \geqslant 6$. Since $g(5) = 1$, $i_T(H(K)) \geqslant 1$. We firstly prove that $G(9) \geqslant 4$. According to the number of T-points in the interior of K, there are two cases to consider.

Case 1. $i_T(K) = 7$.

Denote these seven T-points by $a_1, \cdots, a_7$. If $|H \cap \{a_1, \cdots, a_7\}| \geqslant 4$, we are done. Otherwise, notice that $G(9) \geqslant G(8) = 2$, there are two subcases to consider.

Subcase 1.1. $i_H(K) = 2$. Since $T = H \cup C$, it is easy to see that there are 5 non-collinear interior C-points in K, which can span at least 3 primitive pairwise interior-disjoint C-triangles (by Lemma 2.3). By Lemma 2.2, these three primitive C-triangles contain at least 3 H-points, which contradicts the condition $i_H(K) = 2$.

Subcase 1.2. $i_H(K) = 3$. Then we may assume that $a_1, a_2, a_3 \in H$ and $a_4, a_5, a_6, a_7 \in C$. Recalling that $v_T(H(K)) \geqslant 5$, $b_T(H(K)) \geqslant 6$,

$i_T(H(K)) \geqslant 1$ and $i_T(K) = 7$, we know that $H(K)$ is either a convex pentagon with one interior T-point and six boundary T-points, or a convex hexagon with one interior T-point. Clearly, a_4, a_5, a_6, a_7 are not collinear. Let $Q = \text{conv}\{a_4, a_5, a_6, a_7\}$. By Lemma 2.3, Q can span at least 2 interior-disjoint primitive C-triangles Δ_1 and Δ_2, and therefore contains at least 2 H-points (by Lemma 2.2). In order to prove $G(9) \geqslant 4$, we only need to prove the following Claim.

Claim 1. $|Q \cap \{a_1, a_2, a_3\}| \leqslant 1$.

Proof of Claim 1. (i) If $|Q \cap \{a_1, a_2, a_3\}| = 3$, namely, Q contains all a_1, a_2, a_3, then $H(K) = Q$. Notice that $v_T(Q) \leqslant 4$, which contradicts the condition $v_T(H(K)) \geqslant 5$.

(ii) If $|Q \cap \{a_1, a_2, a_3\}| = 2$, we can assume that Q contains two H-points a_1 and a_2. Firstly, if Q contains both a_1 and a_2 in its interior, this contradicts the condition $i_T(H(K)) = 1$, which is impossible. Secondly, if a_1 and a_2 are both boundary points of Q, Lemma 2.1 implies that a_1 and a_2 are lying on the same side of Q. By Lemma 2.2, we can see that primitive C-triangles Δ_1 and Δ_2 must contain at least one H-point other than a_1 and a_2, this contradicts $i_H(K) = 3$. At last, if a_1 is a interior points and a_2 is a boundary points of Q. Notice that $Q = \text{conv}\{a_4, a_5, a_6, a_7\}$, we can assume that a_2 is lying on the C-segment $\overline{a_4a_5}$. By Lemma 2.1, the C-segment $\overline{a_4a_5}$ must contain at least one H-point other than a_2, which also contradicts $i_H(K) = 3$.

By Claim 1, Q contains at most one of three H-points a_1, a_2, a_3. Thus K must contain at least 4 interior H-points, namely, $G(9) \geqslant 4$.

Case 2. $i_T(K) \geqslant 10$.

We suppose the contrary that $i_H(K) \leqslant 3$. Then $i_C(K) \geqslant 7$. Notice that $v_T(H(K)) \geqslant 5$, $b_T(H(K)) \geqslant 6$ and $i_T(H(K)) \geqslant 1$. It is easy to see that these C-points are not collinear. Combining with Lemma 2.2 and Lemma 2.3, we can conclude that there exist at least 5 interior H-points in K, which is a contradiction. Therefore, $G(9) \geqslant 4$.

Further, the left figure in Figure 4 illustrates that there exists a convex H-nonagon with exactly 4 interior H-points. That is to say, $G(9) \leqslant 4$.

Combining the above discussions, we have $G(9) = 4$. The proof is complete. □

Theorem 3.3. $G(10) = 6$.

Proof. On the basis of Lemma 2.7, Lemma 2.6 and $g(10) = 10$, we can prove that $G(10) \geqslant 6$ by an analogous method used in the proof of Theorem 3.2, and the details are omitted here. Furthermore, the right figure in Figure 4 illustrates that $G(10) \leqslant 6$. Therefore, we can obtain that $G(10) = 6$. □

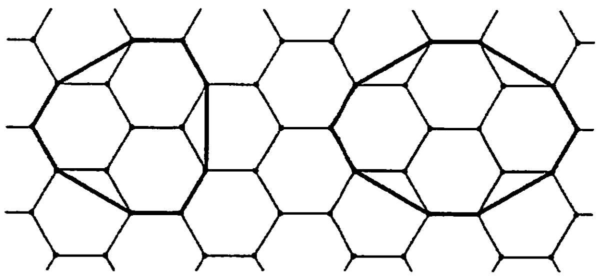

Figure 4: convex H-polygons with interior H-points

Recall that $g(v)$ is the least possible number of lattice points in the interior of a convex lattice v-gon. Rabinowitz gave a lower bound of $\lceil \frac{v^3}{8\pi^2} - \frac{v}{2} + 1 \rceil$ for $g(v)$ in [8]. Next we present bound for $G(v)$ on the basis of the result for $g(v)$. Firstly, the following lemma is fundamental for proving the result.

Lemma 3.4. *If K is a convex H-polygon with $v \geqslant 11$ vertices and contains at least 3 interior C-points, then not all of the C-points in the interior of K are collinear.*

Proof. Let K be a convex H-polygon which contains at least 3 interior C-points, and $V(K)$ be the set of vertices of K, where $|V(K)| \geqslant 11$. Similarly, we use $i_C(K)$ to denote the number of interior C-points of K, then $i_C(K) \geqslant 3$.

We suppose the contrary that all interior C-points in K are collinear, and denote the line determined by these C-points by l_0. The line l_0 divides K into two polygons, saying K_1 and K_2. Notice that $|V(K)| \geqslant 11$, then $|K_1 \cap V(K)| \geqslant 6$ or $|K_2 \cap V(K)| \geqslant 6$. Without loss of generality, we assume that $|K_1 \cap V(K)| \geqslant 6$.

Let $a_0 \in K_1 \cap V(K)$ farthest vertex to line l_0 in K_1. With the similar method used in the proof of Lemma 2.2, translating the line l_0 towards a_0, we have the family of parallel lines

$$\mathcal{L} = \{l_i | l_i \parallel l_0, l_i \cap T \cap K_1 \neq \emptyset, i = 1, 2, \cdots, s\}.$$

Clearly $a_0 \in l_s$, and the distance between l_i and l_{i+1} is the same for every i. Since $|K_1 \cap V(K)| \geqslant 6$, it is obvious that $s \geqslant 3$. There are two cases to consider.

Case 1. $l_0 \cap H \neq \emptyset$.

In this case, we know each line of the family parallel lines $\mathcal{L}$ contains three types of T-points. H^--points, H^+-points and C-points appear periodically on each line. Furthermore, the distances between consecutive

T-points along these lines are the same, which denoted by d_1. Notice that $i_C(K) \geqslant 3$, we may suppose that $c_1, c_2, c_3 \in \text{int}K$ are consecutive C-points along the line l_0. From Lemma 2.1 it follows that each C-segment $\overline{c_1c_2}$ and $\overline{c_2c_3}$ must contain exactly two H-points in their relative interiors. By the

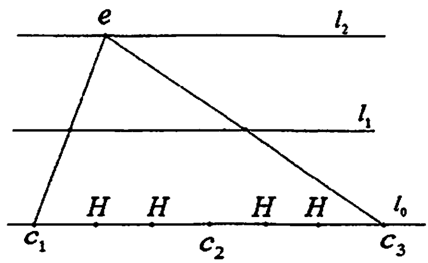

Figure 5: T-triangles

property of T-points, we know $|l_0 \cap K| > |\overline{c_1c_3}| = 6d_1$. Since $s \geqslant 3$, thus the line l_2 contains a T-point e such that $e \in K$, as shown in Figure 5. Let $\Delta_T = \text{conv}\{e, c_1, c_3\}$, it is easy to see $\Delta_T \setminus \{e\} \subset \text{int}K$. Geometrically, we know $|l_1 \cap K| > |l_1 \cap \Delta_T| = 3d_1$. Thus $|l_1 \cap \text{int}K \cap T| > 3$, that is to say, K must contain at least one interior C-point which is not lying on the line l_0, which is a contradiction.

Case 2. $l_0 \cap H = \emptyset$.

In this case, we know that each line in the family of parallel lines $\mathcal{L}$ contains one and only one type of T-points, namely, H^--points, H^+-points or C-points. If $i \equiv 0 \pmod 3$, then the line l_i contains only infinitely many C-points; if $i \not\equiv 0 \pmod 3$, then the line l_i contains only infinitely many H-points. Furthermore, the distances between consecutive T-points along these lines are the same. We denote the distance by d_2. Notice that $i_C(K) \geqslant 3$, thus $|l_0 \cap K| > 2d_2$. By the property of T-points, we have the following three statements.

(i). If $i \geqslant 7$, then $l_i \cap K \cap H = \emptyset$. If $l_i \cap K \cap H \neq \emptyset$, that is to say the line l_i contains H-points in K. Notice that $|l_0 \cap K| > 2d_2$ and $i \geqslant 7$. Geometrically, $|l_3 \cap K| > \frac{i-3}{i} \cdot 2d_2 = \frac{2(i-3)d_2}{i} = 2d_2 - \frac{6d_2}{i} > d_2$. Therefore, we have $l_3 \cap \text{int}K \cap C \neq \emptyset$, which contradicts all interior C-points in K are collinear.

(ii). $|l_4 \cap K \cap H| \leqslant 1$. If $|l_4 \cap K \cap H| \geqslant 2$, namely, $|l_4 \cap K| \geqslant d_2$. Notice that $|l_0 \cap K| > 2d_2$, thus $|l_3 \cap K| > d_2$. Then we also have $l_3 \cap \text{int}K \cap C \neq \emptyset$, which contradicts the condition.

(iii). $|l_5 \cap K \cap H| \leqslant 1$. The statement can be reached by similar method.

Notice that $|K_1 \cap V(K)| \geqslant 6$, we can suppose that $a_1, a_2, \cdots, a_6 \in K_1 \cap V(K)$ are H-points such that $P = \text{conv}\{a_1, a_2, a_3, \cdots, a_6\}$ is an

H-hexagon, that is to say, these six H-points a_1, a_2, a_3, $\cdots$, a_6 are the vertices of P, denote them by $V(P)$. By above three statements we have $|l_1 \cap V(P) \cap H| = |l_2 \cap V(P) \cap H| = 2$, $|l_4 \cap V(P) \cap H| = |l_5 \cap V(P) \cap H| = 1$. Similarly, notice that $g(6) = 1$, by Remark 1 we know that P contains at least one interior T-point f. If $f \in C$, this contradicts the condition. Thus $f \in H$, it is not difficult to see the H-point f is lying on the line l_2. Therefore, $|l_2 \cap K \cap H| \geqslant 3$, namely, $|l_2 \cap K| \geqslant 2d_2$. Geometrically, $|l_3 \cap K| \geqslant d_2$, that is to say, $l_3 \cap \text{int}K \cap C \neq \emptyset$, which contradicts all interior C-points in K are collinear.

The proof is complete. □

Theorem 3.5. *$G(v) \geqslant \left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1$, where v is the number of vertices of any convex H-polygon.*

Proof. Let K be a convex H-polygon with v vertices. It is easy to verify that $G(v) \geqslant \left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1$ holds for $v \leqslant 10$. Now we assume that $v \geqslant 11$. Similarly, embed K into the triangular lattice T. By Remark 1, we have

$$i_T(K) \geqslant \left\lceil \frac{v^3}{8\pi^2} - \frac{v}{2} + 1 \right\rceil,$$

where $i_T(K)$ is the number of interior T-points in K. Notice that $T = H \cup C$, thus

$$i_H(K) + i_C(K) \geqslant \left\lceil \frac{v^3}{8\pi^2} - \frac{v}{2} + 1 \right\rceil.$$

If $i_C(K) \leqslant 2$, then we have $i_H(K) \geqslant \left\lceil \frac{v^3}{8\pi^2} - \frac{v}{2} + 1 \right\rceil - 2 > \left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1$. If $i_C(K) \geqslant 3$, Combining Lemma 3.4, Lemma 2.3 and Lemma 2.2, it is clear that $i_H(K) \geqslant i_C(K) - 2$. Then we have $i_H(K) \geqslant \frac{1}{2} \left\lceil \frac{v^3}{8\pi^2} - \frac{v}{2} + 1 \right\rceil - 1$. Noticing that $G(v)$ is an integer, we obtain

$$G(v) \geqslant \left\lceil \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right\rceil - 1.$$

□

Acknowledgement

The third author gratefully acknowledges financial supports by NNSF of China (11071055); NSF of Hebei Province (A2012205080, A2013205189); Program for New Century Excellent Talents in University, Ministry of Education of China (NCET-10-0129); the project of Outstanding Experts' Overseas Training of Hebei Province.

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