Domination in lexicographic product digraphs*

Juan Liu[a,b], Xindong Zhang[a], Jixiang Meng[b†]

a. College of Maths-physics and Information Sciences, Xinjiang Normal University Urumqi, Xinjiang, 830054, P.R.China

b. College of Mathematics and System Sciences, Xinjiang University Urumqi, Xinjiang, 830046, P.R.China

Abstract

In this paper, we consider the domination number, the total domination number, the restrained domination number, the total restrained domination number and the strongly connected domination number of lexicographic product digraphs.

Keywords: Lexicographic product; Total domination number; Restrained domination number

1 Introduction

Throughout this article, a digraph $G = (V(G), E(G))$ always means a finite directed graph without loops and multiple arcs, where $V = V(G)$ is the vertex set and $E = E(G)$ is the arc set. Given two vertices u and v in G, we say u dominates v if $u = v$ or $uv \in E$. For a vertex $v \in V$, $N_G^+(v)$ and $N_G^-(v)$ denote the set of out-neighbors and in-neighbors of v, $d_G^+(v) = |N_G^+(v)|$ and $d_G^-(v) = |N_G^-(v)|$ denote the out-degree and in-degree of v in G, $\delta^+(G) = \min\{d_G^+(v) : \forall\, v \in V\}$ and $\delta^-(G) = \min\{d_G^-(v) : \forall\, v \in$

*This research is supported by NSFC (10971255), the research fund for the doctoral program of Xinjiang Normal University(xjnubs0908), the Key Project of Chinese Ministry of Education (208161), Program for New Century Excellent Talents in University, and The Project-sponsored by SRF for ROCS, SEM.

†Corresponding author. E-mail: liujuan1999@126.com

$V\}$ denote the minimum out-degree and in-degree of G, respectively. Let $N_G^+[v] = N_G^+(v) \cup \{v\}$. A vertex v dominates all vertices in $N_G^+[v]$. A set $D \subseteq V$ is a *dominating set* of G if D dominates $V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set D is called a $\gamma(G)$-set of G if $|D| = \gamma(G)$. Note that each dominating set of digraph G contains all vertices with in-degree 0 in G. Let D^0 be the vertex set of vertices with in-degree 0 in G, and set $|D^0| = i(G)$. Let D and U be two vertex sets of V, U is called a *monitor set* of D if there exists a vertex $v \in U$ different from u such that $vu \in E$ for each vertex $u \in D \setminus D^0$. The *monitor number* of D, denoted by $\iota(D)$, is the minimum cardinality of a monitor set of D. Set $\iota(G) = \min\{\iota(D) : \text{D is a } \gamma(G)\text{-set of G}\}$.

A set $D \subseteq V$ is a *total dominating set* (**TDS**) if every vertex in V has at least one in-neighbor in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a **TDS** of G. A **TDS** D is called a $\gamma_t(G)$-set of G if $|D| = \gamma_t(G)$. Clearly, $\gamma(G) \leq \gamma_t(G)$. It is easy to verify that $\gamma_t(G)$-set exists for a loopless digraph G if and only if $\delta^-(G) \geq 1$. A set $D \subseteq V$ is a *restrained dominating set* (**RDS**) if every vertex not in D has at least one in-neighbor in D and at least one in-neighbor in $V \setminus D$. Every digraph has a **RDS**, since $D = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** D is called a $\gamma_r(G)$-set of G if $|D| = \gamma_r(G)$. Clearly, $\gamma(G) \leq \gamma_r(G)$. A set $D \subseteq V$ is a *total restrained dominating set* (**TRDS**) if every vertex in $V \setminus D$ has at least one in-neighbor in D and at least one in-neighbor in $V \setminus D$, and every vertex in D has at least one in-neighbor in D. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** D is called a $\gamma_{tr}(G)$-set of G if $|D| = \gamma_{tr}(G)$. Clearly, $\gamma(G) \leq \gamma_{tr}(G)$. A dominating set D of G is called a *strongly connected domination set* (**CDS**) if the induced subdigraph $\langle D \rangle$ is strongly connected. The *strongly connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a **CDS** of G. A **CDS** D is called a $\gamma_c(G)$-set of G if $|D| = \gamma_c(G)$. Clearly, $\gamma(G) \leq \gamma_c(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs, where $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ and $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. The *lexicographic product* $G_1[G_2]$ of G_1 and G_2 has vertex set $V_1 \times V_2$ and $(x_i, y_j)(x_{i'}, y_{j'}) \in E(G_1[G_2])$ if and only if either $x_i x_{i'} \in E_1$, or $x_i = x_{i'}$ and $y_j y_{j'} \in E_2$. The subdigraph $G_2^{x_i}$ is the digraph with vertex set $\{(x_i, y_j) : \forall\, y_j \in V_2\}$ and arc set $\{(x_i, y_j)(x_i, y_{j'}) : \forall\, y_j y_{j'} \in E_2\}$. Clearly, $G_2^{x_i} \cong G_2$ for all $x_i \in V_1$. From the definition of lexicographic product, it is easy to see that $G_1[G_2]$ can be obtained from G_1 by replacing each vertex of G_1 with a copy of G_2, in such a way that for every arc $x_i x_j$ in G_1, contains all possible arcs from $G_2^{x_i}$ to $G_2^{x_j}$.

There are many research articles on the domination number of undirected graphs. However, to date only few results have been done on this concept for digraphs (See [2]-[7] and the related references). In this paper, we will consider the domination number, the total domination number, the restrained domination number, the total restrained domination number and the strongly connected domination number of lexicographic product digraphs.

Terminologies not given here are referred to [1].

2 Main results

Clearly, for any two digraphs G_1 and G_2, if G_1 is an isolated vertex, then $G_1[G_2] \cong G_2$, if G_2 is an isolated vertex, then $G_1[G_2] \cong G_1$. Hence we consider that G_1 and G_2 are two digraphs with at least two vertices.

First, we consider the domination number of $G_1[G_2]$.

Theorem 2.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\gamma(G_2) = 1$, then $\gamma(G_1[G_2]) = \gamma(G_1)$.*

Proof. Clearly, $\gamma(G_1[G_2]) \geq \gamma(G_1)$. Now we prove that $\gamma(G_1[G_2]) \leq \gamma(G_1)$. Let D_1 be a $\gamma(G_1)$-set of G_1, and let $D_2 = \{y_1\}$ be a $\gamma(G_2)$-set of G_2. Set $D = D_1 \times \{y_1\} \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1$.

If $y = y_1$, then $(x, y) \in D$. If $y \neq y_1$, then $y_1 y \in E_2$ for $\gamma(G_2) = 1$. Thus, $(x, y_1)(x, y) \in E(G_1[G_2])$ and $(x, y_1) \in D$.

Case 2. $x \notin D_1$.

There exists a vertex $x_i \in D_1$ such that $x_i x \in E_1$. Thus, $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$.

Therefore, every vertex in $V(G_1[G_2]) \setminus D$ has at least one in-neighbor in

D, D is a dominating set of $G_1[G_2]$. Hence,

$$\gamma(G_1[G_2]) \leq |D| = |D_1| = \gamma(G_1).$$

From above we have $\gamma(G_1[G_2]) = \gamma(G_1)$.

□

Theorem 2.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\gamma(G_2) \geq 2$, then $\gamma(G_1) \leq \gamma(G_1[G_2]) \leq i(G_1)(\gamma(G_2) - 1) + \gamma(G_1) + \iota(G_1)$.*

Proof. Clearly, $\gamma(G_1[G_2]) \geq \gamma(G_1)$. Now we prove that $\gamma(G_1[G_2]) \leq i(G_1)(\gamma(G_2) - 1) + \gamma(G_1) + \iota(G_1)$. Let D_1^0 be the vertex set of vertices with in-degree 0 in G_1 and $|D_1^0| = i(G_1)$. Let D_1 be a $\gamma(G_1)$-set such that there exists a minimum monitor set U_1 of D_1 with $|U_1| = \iota(D_1) = \iota(G_1)$, and such that $|U_1 \cap D_1|$ is as small as possible. Let D_2 be a $\gamma(G_2)$-set of G_2. Take two vertices $y_1, y_2 \in D_2$ and set $D = ((D_1 \setminus D_1^0) \times \{y_1\}) \cup (U_1 \times \{y_2\}) \cup (D_1^0 \times D_2) \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1^0$.

If $y \in D_2$, then $(x, y) \in D$. If $y \notin D_2$, then there exists a vertex $y_i \in D_2$ such that $y_i y \in E_2$. Thus, $(x, y_i)(x, y) \in E(G_1[G_2])$ and $(x, y_i) \in D$.

Case 2. $x \in D_1 \setminus D_1^0$.

If $y = y_1$, then $(x, y) \in D$. We consider the case that $y \neq y_1$. If there exists a vertex $x_i \in D_1$ such that $x_i x \in E_1$, then $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. Otherwise, there exists a vertex $x_j \in U_1$ such that $x_j x \in E_1$, since each vertex in $D_1 \setminus D_1^0$ has at least one in-neighbor and $|U_1 \cap D_1|$ is as small as possible. Thus, $(x_j, y_2)(x, y) \in E(G_1[G_2])$ and $(x_j, y_2) \in D$.

Case 3. $x \notin D_1$.

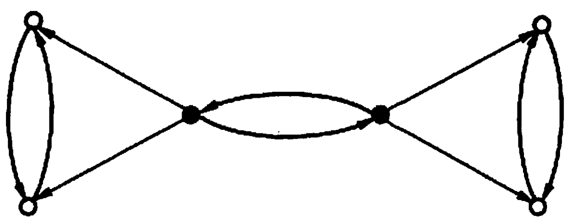
There exists a vertex $x_i \in D_1$ such that $x_i x \in E_1$. Thus, $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$.

Therefore, every vertex in $V(G_1[G_2]) \setminus D$ has at least one in-neighbor in D, D is a dominating set of $G_1[G_2]$. Hence,

$$\begin{aligned}
\gamma(G_1[G_2]) &\leq |D| = |D_1 \setminus D_1^0| + |U_1| + |D_1^0||D_2| \\
&= \gamma(G_1) - i(G_1) + \iota(G_1) + i(G_1)\gamma(G_2) \\
&= i(G_1)(\gamma(G_2) - 1) + \gamma(G_1) + \iota(G_1)
\end{aligned}$$

Therefore, we have $\gamma(G_1) \leq \gamma(G_1[G_2]) \leq i(G_1)(\gamma(G_2) - 1) + \gamma(G_1) + \iota(G_1)$. $\square$

Remark: The lower bound and upper bound in Theorem 2.2 are sharp. Let $\overrightarrow{P_4}$ denote the directed path with four vertices, and $\overrightarrow{C_3}$ denote the directed cycle with three vertices. Clearly, $\gamma(\overrightarrow{C_3}) = 2, \gamma(\overrightarrow{P_4}) = 2, \iota(\overrightarrow{P_4}) = 1, i(\overrightarrow{P_4}) = 1$, we have $\gamma(\overrightarrow{P_4}[\overrightarrow{C_3}]) = i(\overrightarrow{P_4})(\gamma(\overrightarrow{C_3})-1)+\gamma(\overrightarrow{P_4})+\iota(\overrightarrow{P_4}) = 1\times(2-1)+2+1 = 4$. Thus, the domination number of $\overrightarrow{P_4}[\overrightarrow{C_3}]$ achieves the upper bound. Let G_0 be the digraph in Figure 1, then $\gamma(G_0[\overrightarrow{C_3}]) = \gamma(G_0) = 2$. Thus, the domination number of $G_0[\overrightarrow{C_3}]$ achieves the lower bound.



G_0

Figure 1.

We study the total domination number of $G_1[G_2]$ in the following. Since $\gamma_t(G_1[G_2])$-set exists if and only if $\delta^-(G_1[G_2]) \geq 1$. Thus, in order to make $\gamma_t(G_1[G_2])$-set exist, we have $\delta^-(G_1) \geq 1$ or $\delta^-(G_2) \geq 1$.

Theorem 2.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\delta^-(G_1) \geq 1$, then $\gamma_t(G_1[G_2]) \leq \gamma_t(G_1)$.*

Proof. Since $\delta^-(G_1) \geq 1$, let D_1^t be a $\gamma_t(G_1)$-set of G_1. Set $D = D_1^t \times \{y_1\} \subseteq V(G_1[G_2])$ for some vertex $y_1 \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$. Then there exists a vertex $x_i \in D_1^t$ such that $x_i x \in E_1$. Therefore $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. Thus, every vertex in $V(G_1[G_2])$ has at least one in-neighbor in D, D is a total dominating set of $G_1[G_2]$. Hence,

$$\gamma_t(G_1[G_2]) \leq |D| = |D_1^t| = \gamma_t(G_1).$$

$\square$

Theorem 2.4. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\delta^-(G_1) = 0$ and $\delta^-(G_2) \geq 1$, then $\gamma_t(G_1[G_2]) \leq \gamma(G_1)\gamma_t(G_2)$.*

Proof. Let D_1 be a $\gamma(G_1)$-set of G_1 and let D_2^t be a $\gamma_t(G_2)$-set of G_2. Set $D = D_1 \times D_2^t \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1$.

There exists a vertex $y_i \in D_2^t$ such that $y_i y \in E_2$ since D_2^t is a $\gamma_t(G_2)$-set of G_2. Thus, $(x, y_i)(x, y) \in E(G_1[G_2])$ and $(x, y_i) \in D$.

Case 2. $x \notin D_1$.

There exists two vertices $x_j \in D_1$ and $y_i \in D_2^t$ such that $x_j x \in E_1$ and $y_i y \in E_2$. Thus, $(x_j, y_i)(x, y) \in E(G_1[G_2])$ and $(x_j, y_i) \in D$.

Therefore, every vertex in $V(G_1[G_2])$ has at least one in-neighbor in D, D is a total dominating set of $G_1[G_2]$. Hence,

$$\gamma_t(G_1[G_2]) \leq |D| = |D_1||D_2^t| = \gamma(G_1)\gamma_t(G_2).$$

□

Remark: The upper bounds in Theorem 2.3 and Theorem 2.4 are sharp. Let G_0 be the digraph in Figure 1, and G_2 be any digraph with at least two vertices, then $\gamma(G_0[G_2]) = \gamma_t(G_0) = 2$. Thus, the total domination number of $G_0[G_2]$ achieves the upper bound in Theorem 2.3. Let $\overrightarrow{K_n}$ $(n \geq 2)$ and $\overrightarrow{P_m}$ $(m \geq 2)$ denote a complete digraph of order n and a directed path of order m, respectively. Then $\gamma_t(\overrightarrow{K_n}) = 2$, $\gamma(\overrightarrow{P_m}) = \lceil \frac{m}{2} \rceil$ and $\gamma_t(\overrightarrow{P_m})$ does not exist(see [6]). Therefore, $\gamma_t(\overrightarrow{P_m}[\overrightarrow{K_n}]) = 2\lceil \frac{m}{2} \rceil$, the total domination number of $\gamma_t(\overrightarrow{P_m}[\overrightarrow{K_n}])$ achieves the upper bound in Theorem 2.4.

Next, we study the restrained domination number of $G_1[G_2]$.

Theorem 2.5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\gamma(G_2) = 1$ and $\gamma_r(G_2) \neq |V_2|$, then $\gamma_r(G_1[G_2]) \leq \gamma(G_1) + i(G_1)(\gamma_r(G_2) - 1)$.*

Proof. Let $D_2 = \{y_1\}$ be a $\gamma(G_2)$-set and let D_2^r be a $\gamma_r(G_2)$-set. Let D_1^0 be the vertex set of vertices with in-degree 0 in G_1 and $|D_1^0| = i(G_1)$. Let D_1 be a $\gamma(G_1)$-set. Set $D = ((D_1 \setminus D_1^0) \times \{y_1\}) \cup (D_1^0 \times D_2^r) \subseteq V(G_1[G_2])$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1^0$.

If $y \in D_2^r$, then $(x, y) \in D$. If $y \notin D_2^r$, then there exist two vertices $y_i \in D_2^r$ and $y_j \notin D_2^r$ such that $y_iy, y_jy \in E_2$. Thus, $(x, y_i)(x, y) \in E(G_1[G_2])$ and $(x, y_i) \in D$, $(x, y_j)(x, y) \in E(G_1[G_2])$ and $(x, y_j) \notin D$.

Case 2. $x \in D_1 \setminus D_1^0$.

If $y = y_1$, then $(x, y) \in D$. If $y \neq y_1$, then $y_1y \in E_2$, $(x, y_1)(x, y) \in E(G_1[G_2])$ and $(x, y_1) \in D$. Since $x \in D_1 \setminus D_1^0$, x has at least one in-neighbor x_i in G_1, we find that there exists at least one vertex (x_i, y_s) not in D since $\gamma_r(G_2) \neq |V_2|$. Therefore, $(x_i, y_s)(x, y) \in E(G_1[G_2])$ and $(x_i, y_s) \notin D$.

Case 3. $x \notin D_1$.

There exists a vertex $x_i \in D_1$ such that $x_ix \in E_1$, and there must exist two vertices $(x_i, y_l) \in D$ and $(x_i, y_t) \notin D$ since $\gamma_r(G_2) \neq |V_2|$. Thus, $(x_i, y_l)(x, y) \in E(G_1[G_2])$ and $(x_i, y_l) \in D$, $(x_i, y_t)(x, y) \in E(G_1[G_2])$ and $(x_i, y_t) \notin D$.

Therefore, every vertex in $V(G_1[G_2]) \setminus D$ has at least one in-neighbor in D and at least one in-neighbor in $V(G_1[G_2]) \setminus D$, D is a restricted dominating set of $G_1[G_2]$. Hence,

$$\begin{aligned}
\gamma(G_1[G_2]) &\leq |D| = |D_1 \setminus D_1^0| + |D_1^0||D_2^r| \\
&= \gamma(G_1) - i(G_1) + i(G_1)\gamma_r(G_2) \\
&= \gamma(G_1) + i(G_1)(\gamma_r(G_2) - 1)
\end{aligned}$$

Therefore, we have $\gamma_r(G_1[G_2]) \leq \gamma(G_1) + i(G_1)(\gamma_r(G_2) - 1)$.

□

Theorem 2.6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If $\gamma(G_2) \geq 2$ and $\gamma_r(G_2) \neq |V_2|$, then $\gamma_r(G_1[G_2]) \leq \gamma(G_1) + i(G_1)(\gamma_r(G_2) - 1) + \iota(G_1)$.*

Proof. We claim that $|V_2| \geq 3$ since $\gamma(G_2) \geq 2$ and $\gamma_r(G_2) \neq |V_2|$. Let D_1^0 be the vertex set of vertices with in-degree 0 and $|D_1^0| = i(G_1)$. Let D_1 be a $\gamma(G_1)$-set such that there exists a minimum monitor set U_1 of D_1 with $|U_1| = \iota(D_1) = \iota(G_1)$, and such that $|U_1 \cap D_1|$ is as small as possible. Let D_2^r be a $\gamma_r(G_2)$-set of G_2. Take two vertices $y_1, y_2 \in D_2^r$ and set $D = (D_1 \setminus D_1^0 \times \{y_1\}) \cup (U_1 \times \{y_2\}) \cup (D_1^0 \times D_2^r) \subseteq V(G_1[G_2])$. By Theorem 2.2, we known that D is a dominating set of $G_1[G_2]$. It is easy to see that $|D \cap V(G_2^x)| \leq |V_2| - 1$ for each vertex $x \in V_1$. Let (x, y) be an arbitrary vertex in $V(G_1[G_2]) \setminus D$.

If $x \in D_1^0$, then $y \notin D_2^r$, there exists a vertex $y_i \notin D_2^r$ such that $y_i y \in E_2$ since D_2^r is a $\gamma_r(G_2)$-set of G_2. Thus, $(x, y_i)(x, y) \in E(G_1[G_2])$ and $(x, y_i) \notin D$.

If $x \notin D_1^0$, then x has at least one in-neighbor x_i in G_1. Therefore, there exists a vertex $(x_i, y_t) \notin D$ such that $(x_i, y_t)(x, y) \in E(G_1[G_2])$ since $\gamma_r(G_2) \neq |V_2|$. Hence, D is a restrained dominating set of $G_1[G_2]$. Thus,

$$\begin{aligned} \gamma(G_1[G_2]) &\leq |D| = |D_1 \setminus D_1^0| + |U_1| + |D_1^0||D_2^r| \\ &= \gamma(G_1) - i(G_1) + \iota(G_1) + i(G_1)\gamma(G_2) \\ &= i(G_1)(\gamma(G_2) - 1) + \gamma(G_1) + \iota(G_1) \end{aligned}$$

Therefore, we have $\gamma_r(G_1[G_2]) \leq \gamma(G_1) + i(G_1)(\gamma_r(G_2) - 1)$.

□

We will discuss the total restrained domination number of $G_1[G_2]$ in the following theorem 2.7.

Theorem 2.7. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. If there exists a $\gamma_{tr}(G_1)$-set of G_1, then $\gamma_{tr}(G_1[G_2]) \leq \gamma_{tr}(G_1)$.*

Proof. Let D_1^{tr} be a $\gamma_{tr}(G_1)$-set of G_1. Set $D = D_1^{tr} \times \{y_1\} \subseteq V(G_1[G_2])$ for some vertex $y_1 \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$.

Case 1. $x \in D_1^{tr}$.

There exists a vertex $x_i \in D_1^{tr}$ such that $x_i x \in E_1$. If $y = y_1$, then $(x, y) \in D$. We have $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$. If $y \neq y_1$, then $(x, y) \notin D$. We have $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$, $(x_i, y)(x, y) \in E(G_1[G_2])$ and $(x_i, y) \notin D$.

Case 2. $x \notin D_1^{tr}$.

Clearly, $(x, y) \notin D$. Therefore there exist two vertices $x_i \in D_1^{tr}$ and $x_j \notin D_1^{tr}$ such that $x_i x, x_j x \in E_1$. We have $(x_i, y_1)(x, y) \in E(G_1[G_2])$ and $(x_i, y_1) \in D$, $(x_j, y)(x, y) \in E(G_1[G_2])$ and $(x_j, y) \notin D$. Thus, every vertex in $V(G_1[G_2]) \setminus D$ has at least one in-neighbor in D and at least one in-neighbor in $V(G_1[G_2]) \setminus D$, and every vertex in D has at least one in-neighbor in D, D is a total restricted dominating set of $G_1[G_2]$. Hence,

$$\gamma_{tr}(G_1[G_2]) \leq |D| = |D_1^{tr}| = \gamma_{tr}(G_1).$$

□

Finally, we consider the strongly connected domination number of $G_1[G_2]$. Note that if $\gamma_c(G_1) = 1$ and $\gamma(G_2) \geq 2$ and there does not exist strongly connected dominating set with at least two vertices in G_1, then $\gamma_c(G_1[G_2])$-set does not exist.

Theorem 2.8. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs with at least two vertices. Then $\gamma_c(G_1[G_2]) = \gamma_c(G_1)$, if one of the following conditions holds:*

(i) $\gamma(G_2) = 1$ and $\gamma_c(G_1) \geq 1$,

(ii) $\gamma(G_2) \geq 2$ and $\gamma_c(G_1) \geq 2$.

Proof. Clearly, $\gamma_c(G_1[G_2]) \geq \gamma_c(G_1)$.

Case 1. $\gamma(G_2) = 1$ and $\gamma_c(G_1) \geq 1$.

Let D_1^c be a $\gamma_c(G_1)$-set of G_1 and $D_2 = \{y_1\}$ be a $\gamma(G_2)$-set of G_2. Set $D = D_1^c \times \{y_1\} \subseteq V(G_1[G_2])$. We know that D is a dominating set of $G_1[G_2]$ from the proof of Theorem 2.1. Since $\langle D_1^c \rangle$ is strongly connected, $\langle D \rangle$ is also strongly connected. Thus, D is a strongly connected dominating set of $G_1[G_2]$. Hence,

$$\gamma_c(G_1[G_2]) \leq |D| = |D_1^c| = \gamma_c(G_1).$$

Case 2. $\gamma(G_2) \geq 2$ and $\gamma_c(G_1) \geq 2$.

Let D_1^c be a $\gamma_c(G_1)$-set of G_1. Set $D = D_1^c \times \{y_j\} \subseteq V(G_1[G_2])$ for some vertex $y_j \in V_2$. Let (x, y) be an arbitrary vertex of $G_1[G_2]$. Since D_1^c is a strongly connected dominating set of G_1, there exists a vertex $x_i \in D_1^c$ such that $x_i x \in E_1$. Thus $(x_i, y_j)(x, y) \in E(G_1[G_2])$ and $(x_i, y_j) \in D$. Hence, D is a dominating set of $G_1[G_2]$. Since $\langle D_1^c \rangle$ is strongly connected, $\langle D \rangle$ is also strongly connected. Thus, D is a strongly connected dominating set of $G_1[G_2]$. Hence,

$$\gamma_c(G_1[G_2]) \leq |D| = |D_1^c| = \gamma_c(G_1).$$

□

References

[1] G. Chartrand, L. Lesniak, Graphs and Digraphs, fourth ed., Chapman and Hall, Boca Raton, FL, 2005.

[2] K. Carlson, M. Develin, On the bondage number of planar and directed graphs, Discrete Math. 306 (2006) 820-826.

[3] G. Chartrand, F. Harary, B.Yue, On the out-domination and in-domination numbers of a digraph, Discrete Math. 197/198 (1999) 179-183.

[4] J. Ghoshal, R. Laskar and D. Pillone, Topics on domination in directed graphs. In: T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Editors, Domination in Graphs: Advanced Topics, Marcel Dekker, New York (1998), pp. 401-437.

[5] C. Lee, Domination in Digraphs, J.Korean Math.Soc. 35(4) (1998) 843-853.

[6] J. Huang, J.M. Xu, The total domination and bondage numbers of extended de Bruijn and Kautz digraphs, Comput. Math. Appl. 53 (2007) 1206-1213.

[7] J. Huang, J.W. Wang, J.M. Xu, Reinforcement numbers of digraphs, Discrete Appl. Math. (2009) doi:10.1016/j.dam.2009.01.002.

Optimal orientations of $P_3 \times K_5$ and $C_8 \times K_3$

R. Lakshmi
Department of Mathematics
Annamalai University
Annamalainagar - 608 002
Tamilnadu, India.
mathlakshmi@gmail.com

Abstract. For a graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. *Orientation number* of G, denoted by $\vec{d}(G)$, is defined as $\min\{ d(D) \,|\, D \in \mathcal{D}(G) \}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, we prove that $\vec{d}(P_3 \times K_5) = 4$ and $\vec{d}(C_8 \times K_3) = 6$, where $\times$ is the tensor product of graphs.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity*, denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\{d_G(v,x) \,|\, x \in V(G)\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{ e_G(v) \,|\, v \in V(G) \}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, arcs with same tail and same head). For $v \in V(D)$, the notions $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $x, y \in V(D)$, we write $x \to y$ or $y \leftarrow x$ if $(x,y) \in A(D)$. For sets $X, Y \subseteq V(D)$, $X \to Y$ denotes $\{(x,y) \in A(D) : x \in X \text{ and } y \in Y\}$. For distinct vertices $v_1, v_2, \ldots, v_k$, $v_1 \to v_2 \to \ldots \to v_k$ represents the directed path in D with arcs $v_1 \to v_2, v_2 \to v_3, \ldots, v_{k-1} \to v_k$. For subsets $V_1, V_2, \ldots, V_k$ of V, we write $V_1 \to V_2 \to \ldots \to V_k$ for the set of all directed paths of length $k-1$ whose ith vertex is in V_i, $1 \le i \le k$.

For graphs G and H, the *tensor product*, $G \times H$, of G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{ (u,v)(x,y) : ux \in$