The s-degenerate r-Lah numbers

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Abstract

Recently, Belbachir and Belkhir gave some recurrence relations for the r-Lah numbers. In this paper, we give other properties for the r-Lah numbers, we introduce and study a restricted class of these numbers.

Keywords. r-Lah numbers; s-degenerate r-Lah numbers; recurrence relations.
AMS classification: 11B37; 05A19; 11B83; 11B75.

1 Introduction

For $r \in \mathbb{N}$, the r-Lah number $\left\lfloor {n \atop k} \right\rfloor_r$ play an important role in combinatorics and counts the number of partitions of a n-set into k ordered blocks such that the r first elements are in different blocks. Recall that the exponential generating function of r-Lah numbers is to be

$$(1) \qquad \sum_{n \geq k} \left\lfloor {n+r \atop k+r} \right\rfloor_r \frac{t^n}{n!} = \frac{1}{k!} \frac{t^k}{(1-t)^{k+2r}}.$$

These numbers satisfy the following triangular recurrence relation

$$(2) \qquad \left\lfloor {n \atop k} \right\rfloor_r = \left\lfloor {n-1 \atop k-1} \right\rfloor_r + (n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor_r, \quad n \geq k \geq r.$$

and have the explicit expression

$$(3) \qquad \left\lfloor {n \atop k} \right\rfloor_r = \frac{(n-r)!}{(k-r)!} \binom{n+r-1}{k+r-1} = \frac{(n+r-1)!}{(k+r-1)!} \binom{n-r}{k-r}, \quad n \geq k \geq r.$$

Recently, Belbachir and Belkhir gave some recurrence relations for the r-Lah numbers, see also [3]. In this paper, we give other properties for these numbers and we study a restricted class of them by giving their generating function and some combinatorial recurrence relations.

Definition 1 *Let s be a positive integer and r be a non-negative integer. The s-degenerate r-Lah numbers, denoted by $\left\lfloor {n \atop k} \right\rfloor_r^{s\uparrow}$, count the number of partitions of a n-set into k ordered blocks, each one has a length at least s elements and such that the first r elements must be in different blocks.*

2 Some properties for the r-Lah numbers

Two recurrence relations for the r-Lah numbers are given by the following proposition on using in (3) the fact that $\binom{n+r-1}{k+r-1} = \binom{n+r-2}{k+r-1} + \binom{n+r-2}{k+r-2}$ and $\binom{n-r}{k-r} = \binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1}$.

Proposition 2 *For $r < k$ we have*

$$\begin{aligned}\left\lfloor {n \atop k} \right\rfloor_r &= (n-r)\left\lfloor {n-1 \atop k} \right\rfloor_r + \frac{n-r}{k-r}\left\lfloor {n-1 \atop k-1} \right\rfloor_r, \\ \left\lfloor {n \atop k} \right\rfloor_r &= (n+r-1)\left\lfloor {n-1 \atop k} \right\rfloor_r + \frac{n+r-1}{k+r-1}\left\lfloor {n-1 \atop k-1} \right\rfloor_r.\end{aligned}$$

Note that for $r = 0$ or 1 we obtain Corollary 2.1 given in [2].

Now, it is known from [1] that the polynomial $P_n(x) = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_0 x^k$ has only real roots, distinct and non-positive. The following proposition shows a similar result for the r-Lah numbers.

Proposition 3 *The polynomial $P_n(x;r) := \sum_{k=0}^{n} \left\lfloor {n+r \atop k+r} \right\rfloor_r x^k$ has only non-positive real roots.*

Proof. Set $T_n(x;r) := x^n \exp(x) P_n(x;r)$. The recurrence relation (2) shows that $T_n(x;r) = x^2 \frac{d}{dx} T_{n-1}(x;r)$. So, by induction on n, the function $T_n(x;r)$ vanishes on $]-\infty, 0]$. ∎

On using Newton's inequality given by Hardy et al. [4, p. 52] we may state that the sequence $\left(\left\lfloor {n \atop k} \right\rfloor_r ; 0 \le k \le n\right)$ is strongly log-concave.

Similarly to a result of Ahuja and Enneking [1] on the Lah numbers, on using (2), one can prove by induction on n, the following proposition.

Proposition 4 *We have*

$$D^n\left(\frac{1}{x^{2r}}\exp\left(\frac{1}{x}\right)\right) = \frac{(-1)^n}{x^{n+2r}}\exp\left(\frac{1}{x}\right)\sum_{k=0}^{n}\left\lfloor {n+r \atop k+r} \right\rfloor_r \frac{1}{x^k}.$$

3 The s-degenerate r-Lah numbers

From the definition, the s-degenerate r-Lah numbers are zero when $n < sk$ or $k < r$. We start this section by giving the following theorem.

Theorem 5 *For $n > sk$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r^{s\uparrow} = \frac{(n-r)!}{(k-r)!}\sum_{n_1+\cdots+n_k=n-sk}(n_1+s)\cdots(n_r+s).$$

Proof. To partition the set $[n] := \{1, 2, \ldots, n\}$ into k ordered blocks such that each block has a length $\geq s$ and the elements of the set $[r]$ are in different blocks, we construct k subsets $B_1, \ldots, B_k$ for which we insert the elements of the set $[r]$ be in $B_1, \ldots, B_r$ (one element in each subset). To construct the k subsets, there is $\frac{1}{(k-r)!}\binom{n-r}{n_1, \ldots, n_k}$ ways to choose $n_1, \ldots, n_k$ in $\{r+1, \ldots, n\}$ such that $n_i + 1 \geq s$, n_i elements are in B_i for $i = 1, \ldots, r$, and, $n_i \geq s$, n_i elements are in B_i for $i = r+1, \ldots, k$.
Now, to form the k ordered blocks, we permute the elements of each subset. The number of permutations is equal to

$$(|B_1|!) \cdots (|B_r|!) (|B_{r+1}|!) \cdots (|B_k|!) = (n_1 + 1)! \cdots (n_r + 1)! n_{r+1}! \cdots n_k!.$$

By meaning M for the set of the vectors $(n_1, \ldots, n_k)$ such that

$$n_1 + \cdots + n_k = n - r, \ (n_1 + 1, \ldots, n_r + 1, n_{r+1}, \ldots, n_k) \in \{s, s+1, \ldots\}^k,$$

the total number of such partitions is

$$\begin{aligned} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} &= \frac{1}{(k-r)!} \sum_{(n_1, \ldots, n_k) \in M} \binom{n-r}{n_1, \ldots, n_k} \prod_{i=1}^{r} (n_i + 1)! \prod_{i=r+1}^{k} n_i! \\ &= \frac{(n-r)!}{(k-r)!} \sum_{(n_1, \ldots, n_k) \in M} (n_1 + 1) \cdots (n_r + 1) \\ &= \frac{(n-r)!}{(k-r)!} \sum_{n_1 + \cdots + n_k = n - sk} (n_1 + s) \cdots (n_r + s) \end{aligned}$$

which complete the proof. ■
As a consequence of Theorem 5, we may state:

Corollary 6 *The s-degenerate r-Lah numbers have generating function*

$$\sum_{n \geq k} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow} \frac{t^n}{n!} = \frac{1}{k!} \frac{t^{sk+(s-1)r}}{(1-t)^{k+2r}} (1 + (s-1)(1-t))^r.$$

Proof. For $|t| < 1$, Theorem 5 shows that we have

$$\begin{aligned} &\sum_{n \geq k} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow} \frac{t^n}{n!} \\ = &\frac{1}{k!} \sum_{n \geq sk+(s-1)r} t^n \sum_{(n_1+s)+\cdots+(n_{k+r}+s)=n+r} (n_1 + s) \cdots (n_r + s) \\ = &\frac{1}{k!} \sum_{n \geq sk+(s-1)r} t^n \sum_{j_1+\cdots+j_{k+r}=n+r \quad j_1 \geq s, \ldots, j_{k+r} \geq s} j_1 \cdots j_r \\ = &\frac{1}{k!} \sum_{n \geq sk+(s-1)r} \sum_{j_1+\cdots+j_{k+r}=n+r, \quad j_1 \geq s, \ldots, j_{k+r} \geq s} j_1 \cdots j_r t^{j_1+\cdots+j_{k+r}-r} \end{aligned}$$

and this is equal to

$$\begin{aligned}
&\frac{1}{k!}\sum_{j_1\geq s,\ldots,j_{k+r}\geq s}\left(j_1t^{j_1-1}\right)\cdots\left(j_rt_r^{j_r-1}\right)\left(t^{j_{1+r}}\right)\cdots\left(t^{j_{k+r}}\right)\\
=&\ \frac{1}{k!}\left(\sum_{j\geq s}jt^{j-1}\right)^r\left(\sum_{j\geq s}t^j\right)^k\\
=&\ \frac{1}{k!}\left(\frac{d}{dt}\frac{t^s}{1-t}\right)^r\left(\frac{t^s}{1-t}\right)^k\\
=&\ \frac{1}{k!}\frac{t^{sk+(s-1)r}}{(1-t)^{k+2r}}\left(1+(s-1)(1-t)\right)^r
\end{aligned}$$

which complete the proof. ■

Corollary 7 *For $r \leq k$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} = \frac{(n-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\binom{n+j-(s-1)k-1}{k+j-1}(s-1)^{r-j}$$

and satisfy for $r < k$ the recurrence relations

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} &= (n-r)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{s\uparrow} + \frac{s!}{k-r}\binom{n-r}{s}\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_r^{s\uparrow},\\
\frac{1}{k-r}\begin{bmatrix} n \\ k \end{bmatrix}_{r+1}^{s\uparrow} &= \frac{s-1}{n-r}\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} + \frac{(n-r-1)!\,(k-r+1)}{(n+s-r)!}\begin{bmatrix} n+s \\ k+1 \end{bmatrix}_r^{s\uparrow}.
\end{aligned}$$

Proof. On using Corollary 6, we get

$$\sum_{n\geq sk+(s-1)r}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r^{s\uparrow}\frac{t^n}{n!} = \sum_{j=0}^{r}\binom{r}{j}\frac{t^{(s-1)(k+r)+j}}{k!}\frac{(s-1)^j\,t^{k-j}}{(1-t)^{k-j+2r}},$$

and on using (1) and (3) to express the factor $\frac{t^{k-j}}{(1-t)^{k-j+2r}}$ as

$$(k-j)!\sum_{n\geq k-j}\left\lfloor \begin{matrix} n+r \\ k-j+r \end{matrix}\right\rfloor_r\frac{t^n}{n!} = \sum_{n\geq k-j}\binom{n+2r-1}{k-j+2r-1}t^n,$$

the first identity follows by identification. The recurrence relations can be obtained by applying the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ to the coefficients $\binom{n+j-(s-1)k-1}{k+j-1}$ and $\binom{r}{j}$ in the first identity. ■

Remark 8 *From Corollary 6 and (1) we get*

$$\sum_{n\geq k}\left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow} \frac{t^n}{n!} = (s-(s-1)\,t)^r\, t^{(s-1)(k+r)} \sum_{n\geq k}\left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r \frac{t^n}{n!}$$

which proves that we have

$$\left\lfloor \begin{matrix} N \\ k \end{matrix} \right\rfloor_r^{s\uparrow} = \frac{(N-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\binom{n+j-1}{k+r-1}s^{r-j}(1-s)^j,\ N=n+(s-1)\,k.$$

The s-degenerate r-Lah numbers possess a probabilistic interpretation given by the following proposition.

Proposition 9 *Let λ be a real number with $0<\lambda<1$, $\{X_n\}$ and $\{Y_n\}$ two independent sequences of independent random variables with*

$$P(X_n=j)=\lambda(1-\lambda)^j,\ P(Y_n=j)=\frac{\lambda^2(j+s)}{1+(s-1)\lambda}(1-\lambda)^j,\quad j\geq 0,$$

and let $S_{k,r}=X_1+\cdots+X_k+Y_1+\cdots+Y_r$. Then, we have

$$P(S_{k,r}=n)=\frac{\lambda^{k+2r}(1-\lambda)^n}{(1+(s-1)\lambda)^r}\frac{k!}{(n+sk+(s-1)\,r)!}\left\lfloor \begin{matrix} n+s(k+r) \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow}.$$

Proof. The proof follows from the moment's generating function of $S_{k,r}$ given by $\mathrm{E}\left(t^{S_{k,r}}\right)=\left(\mathrm{E}\left(t^{X_1}\right)\right)^k\left(\mathrm{E}\left(t^{Y_1}\right)\right)^r$. ■

4 Combinatorial recurrence relations

In this section, we establish some combinatorial recurrence relations.

Proposition 10 *For $n>sk\geq sr\geq 1$ we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} = (n+k-1)\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{s\uparrow} + s!\binom{n-r-1}{s-1}\left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor_r^{s\uparrow} + s!r\binom{n-r-1}{s-2}\left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor_{r-1}^{s\uparrow}.$$

Proof. To partition the set $[n]$ into k ordered blocks such that each block has a length $\geq s$ and the first r elements are in different blocks, we separate the element n and proceed as follows: (a) if the element n is in a block of length $\geq s+1$, there is $\left\lfloor \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rfloor_r^{s\uparrow}$ ways to partition the set $[n-1]$ into k ordered blocks such that each block has a length $\geq s$ and the first r elements are in different blocks. The element n (not really used) can be inserted in the k blocks with $n-1+k$ ways because it can be inserted with $l+1$ ways in the ordered block $\{a_1, \ldots, a_l\}$ after a_i $(i = 1, \ldots, l)$ or before a_1. So, we count in this case $(n-1+k)\left\lfloor {n-1 \atop k} \right\rfloor_r^{s\uparrow}$ ways. **(b)** if the element n is in a block of length s and this block no contains any element of the first r elements, there is $\binom{n-1-r}{s-1}$ ways to choose $s-1$ elements to be with this element in the same subset, $s!$ ways to permute the elements of this subset and the remaining $n-s$ elements can be partitioned into $k-1$ ordered blocks (such that each block has a length $\geq s$ and the first r elements must be in different blocks) in $\left\lfloor {n-s \atop k-1} \right\rfloor_r^{s\uparrow}$ ways. The number of ways in this case is $s!\binom{n-1-r}{s-1}\left\lfloor {n-s \atop k-1} \right\rfloor_r^{s\uparrow}$. **(c)** if the element n is in a block of length s and this block contains one element of the first r elements (this case is considered only when $s \geq 2$), there is $\binom{n-1-r}{s-2}$ ways to choose $s-2$ elements to be with this element in the same subset, r ways to choose an element between the first r elements to be with the element n in the same subset, $s!$ ways to permute the elements of this subset and the remaining $n-s$ elements can be partitioned into $k-1$ ordered blocks (such that each block has a length $\geq s$ and the first $r-1$ elements must be in different blocks) in $\left\lfloor {n-s \atop k-1} \right\rfloor_{r-1}^{s\uparrow}$ ways. The number of ways in this case is $s!r\binom{n-r-1}{s-2}\left\lfloor {n-s \atop k-1} \right\rfloor_{r-1}^{s\uparrow}$. So, the number of all partitions is $\left\lfloor {n \atop k} \right\rfloor_r^{s\uparrow} = (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor_r^{s\uparrow} + s!\binom{n-1-r}{s-1}\left\lfloor {n-s \atop k-1} \right\rfloor_r^{s\uparrow} + s!r\binom{n-r-1}{s-2}\left\lfloor {n-s \atop k-1} \right\rfloor_{r-1}^{s\uparrow}$. ∎

Proposition 11 *We have*

$$\left\lfloor {n+r \atop k+r} \right\rfloor_r^{s\uparrow} = \sum_{j\geq 0}\binom{n}{j}\left\lfloor {n-j \atop k} \right\rfloor_0^{s\uparrow}\left\lfloor {j+r \atop r} \right\rfloor_r^{s\uparrow}.$$

Proof. For fixed j, there is $\binom{n}{n-j}\left\lfloor {n-j \atop k} \right\rfloor_0^{s\uparrow}$ ways to form the k ordered blocks (each one is of length $\geq s$) with using only the remaining $n-j$ elements of the set $\{r+1, \ldots, n+r\}$. To form the r remaining blocks, there is $\left\lfloor {j+r \atop r} \right\rfloor_r^{s\uparrow}$ ways to form r ordered blocks (each one is of length $\geq s$) on using the j elements of the set $\{r+1, \ldots, n+r\}$ and the elements of the set $[r]$ (not really used) such that each block contains one element of the set $[r]$. Then, the number of ordered partitions is given by $\left\lfloor {n+r \atop k+r} \right\rfloor_r^{s\uparrow} = \sum_{j\geq 0}\binom{n}{n-j}\left\lfloor {n-j \atop k} \right\rfloor_0^{s\uparrow}\left\lfloor {j+r \atop r} \right\rfloor_r^{s\uparrow}$. ∎

Proposition 12 *For $n \geq sk \geq sr \geq 1$ we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} = \sum_{j \geq s} j! \binom{n-r}{j-1} \left\lfloor \begin{matrix} n-j \\ k-1 \end{matrix} \right\rfloor_{r-1}^{s\uparrow}.$$

Proof. To partition the set $[n]$ into k ordered blocks such that each block has a length $\geq s$ and the first r elements are in different blocks, the element r can be in a block of length $j \geq s$ in $j!\binom{n-r}{j-1}\left\lfloor \begin{smallmatrix} n-j \\ k-1 \end{smallmatrix} \right\rfloor_{r-1}^{s\uparrow}$ ways: (a) $\binom{n-r}{j-1}$ is the number of ways for choosing $j-1$ elements between $(n-1)-(r-1)$ elements (none is of the first $r-1$ elements) to be in the same subset with the element r (b) $j!$ is the number of ways to permute the elements of this subset and (c) $\left\lfloor \begin{smallmatrix} n-j \\ k-1 \end{smallmatrix} \right\rfloor_{r-1}^{s\uparrow}$ is the number of ways to partition the remaining $n-j$ elements into $k-1$ blocks such that each block has a length $\geq s$ and the first $r-1$ elements are in different blocks. ■

For $s = 1$ in Proposition 12 we get Theorem 3 given in [2].

Proposition 13 *For $r \geq 1$ we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} - \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1}^{s\uparrow} = r \sum_{j \geq s} j! \binom{n-r-1}{j-2} \left\lfloor \begin{matrix} n-j \\ k-1 \end{matrix} \right\rfloor_{r-1}^{s\uparrow}.$$

Proof. The number of partitions of the set $[n]$ into k ordered blocks such that each block has a length $\geq s$ and the first r elements are in different blocks but not the element $r+1$ there is $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r^{s\uparrow} - \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_{r+1}^{s\uparrow}$ ways. This number can be obtained by considering the element $r+1$ to be in one of the r subsets which contain the first r elements. The element $r+1$ can be with the element i, $(i = 1, ..., r)$, in a subset of cardinality j in $j!r\binom{n-r-1}{j-2}\left\lfloor \begin{smallmatrix} n-j \\ k-1 \end{smallmatrix} \right\rfloor_{r-1}^{s\uparrow}$ ways: (a) $\binom{n-r-1}{j-2}$ is the number of ways to choose $j-2$ elements between the last $n-r-1$ elements to be in the same subset with i and $r+1$, (b) $j!$ is the number of ways to permute the elements of this subset, (c) $\left\lfloor \begin{smallmatrix} n-2-j \\ k-1 \end{smallmatrix} \right\rfloor_{r-1}^{s\uparrow}$ is the number of ways to partition the remaining $n-j$ into $k-1$ ordered blocks such that each block has a length $\geq s$ and the elements of the set $[r]-\{i\}$ are in different blocks and (d) r is the number of ways to choose i between the first r elements to be with the element $r+1$ in the same block. Then, the number of all partitions is $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r^{s\uparrow} - \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_{r+1}^{s\uparrow} = r\sum_{j \geq s} j!\binom{n-r-1}{j-2}\left\lfloor \begin{smallmatrix} n-j \\ k-1 \end{smallmatrix} \right\rfloor_{r-1}^{s\uparrow}$. ■

We note that for $s = 1$, Theorem 5 gives

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r := \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{1\uparrow} = \frac{(n-r)!}{(k-r)!} \sum_{n_1 + \cdots + n_k = n-k} (n_1+1)\cdots(n_r+1),$$

Corollary 7 gives (3) and Proposition 13 gives

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r - \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1} = r\sum_{j\geq 1} j! \binom{n-r-1}{j-2} \left\lfloor \begin{matrix} n-j \\ k-1 \end{matrix} \right\rfloor_{r-1}.$$

Also, for $r = 0$ in Proposition 10 and Corollary 6, the s-degenerate Lah numbers $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor^{s\uparrow} := \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_0^{s\uparrow}$ satisfy

$$\begin{aligned} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{s\uparrow} &= (n+k-1) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor^{s\uparrow} + s! \binom{n-1}{s-1} \left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor^{s\uparrow}, \\ \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{s\uparrow} &= \frac{n!}{k!} \binom{n-1-(s-1)k}{k-1}, \quad n \geq sk > 0. \end{aligned}$$

5 Acknowledgments

The authors would like to acknowledge the support from the PNR project 8/U160/3172 and the RECITS's laboratory and thank the anonymous referee for his/her careful reading and valuable suggestions.

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