An alternative approach to the classification of the regular near hexagons with parameters $(s, t, t_2) = (2, 11, 1)$

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Abstract

Based on some results of Shult and Yanushka [7], Brouwer [1] proved that there exists a unique regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$, namely the one related to the extended ternary Golay code. His proof relies on the uniqueness of the Witt design $S(5, 6, 12)$, Pless's characterization of the extended ternary Golay code G_{12} and some properties of $S(5, 6, 12)$ and G_{12}. It is possible to avoid all this machinery and to give an alternative more elementary and self-contained proof for the uniqueness. It was only observed recently by the author that such an alternative proof is implicit in the literature: it can be obtained by combining some results from the papers [1], [4] and [7]. This survey paper has the aim to bring this fact to the attention of the mathematical community. We describe the parts of the above papers which are relevant for this alternative proof of the classification. The alternative proof also requires that we prove a number of extra facts which are not explicitly contained in any of the three above papers. The present paper can also been seen as an addendum to Section 6.5 of the book [3] where the uniqueness of the near hexagon was not proved.

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1 Introduction

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there exists a unique point on L nearest to p. Here, distances are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles.

A near hexagon $\mathcal{S}$ is called *regular* with *parameters* (s,t,t_2) if every line is incident with precisely $s+1$ points, every point is incident with precisely $t+1$ lines and if every two points at distance 2 have precisely t_2+1 common neighbours.

Let $\mathbb{F}_3^{12}$ denote the 12-dimensional vector space over the field $\mathbb{F}_3$ of order 3 whose vectors are row matrices of length 12 with entries in $\mathbb{F}_3$. The 6 rows of the matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

generate a 6-dimensional subspace G_{12} of $\mathbb{F}_3^{12}$ which is called the *extended ternary Golay code*. By deleting one coordinate position, one gets a code (a subspace of $\mathbb{F}_3^{11}$) which was discovered by Golay [5]. Let $\mathbb{E}_1$ be the point-line geometry whose points are all the cosets of G_{12} and whose lines are all the triples of the form $\{\bar{v}+G_{12}, \bar{v}+\bar{e}_i+G_{12}, \bar{v}-\bar{e}_i+G_{12}\}$, with incidence being containment. Here, $\bar{v}$ is some vector of $\mathbb{F}_3^{12}$ and $\bar{e}_i$, $i \in \{1,\ldots,12\}$, denotes the row matrix all of whose entries are 0 except for the i-th one which is equal to 1. Shult and Yanushka [7, pp. 30–33] proved that $\mathbb{E}_1$ is a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$ (see also [3, Theorem 6.59]).

Let Π_∞ be a hyperplane of the projective space $PG(6,3)$. For every set $\mathcal{K}$ of points of Π_∞, let $T_5^*(\mathcal{K})$ denote the point-line geometry whose points are the points of $PG(6,3)\setminus\Pi_\infty$ and whose lines are the lines of $PG(6,3)$ not contained in Π_∞ which intersect Π_∞ in a point of $\mathcal{K}$ (natural incidence). After fixing some reference system in Π_∞, the 12 columns of the matrix M define a set $\mathcal{K}^*$ of 12 points of Π_∞. This set $\mathcal{K}^*$ of 12 points of Π_∞ satisfies several nice properties, see Coxeter [2]. By [3, Theorem 6.62(b)], $T_2^*(\mathcal{K}^*)$ is a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$.

Using some results of Shult and Yanushka [7], Brouwer [1] proved the following.

Theorem 1.1 *[1, 7] Up to isomorphism, there is a unique regular near hexagon with parameters* $(s,t,t_2) = (2,11,1)$.

We briefly discuss the proof of Theorem 1.1 as it occurs in [1] and [7]. Suppose $\mathcal{S}$ is a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$.

A subspace X of S is called a *cube* if the lines of S which are contained in X give X the structure of a $(3 \times 3 \times 3)$-cube.

(1) Assuming that every three distinct lines through a given point are contained in a unique cube, Shult and Yanushka [7, pp. 34–39] proved that every two distinct cubes which contain a pair $\{x, y\}$ of opposite points have a unique third point in common, namely the unique point of C_1 (or C_2) at distance 3 from x and y. They also proved that S admits a point-regular elementary abelian 3-group of automorphisms, see [7, pp. 39–40].

(2) Brouwer [1, Section 2] succeeded in proving that any three distinct lines through a given point are indeed contained in a unique cube. Using the results of Shult and Yanushka [7], he then succeeded in proving that S is isomorphic to the near hexagon arising from the extended ternary Golay code, see [1, Sections 3, 4 and 5]. The latter proof makes use of the the uniqueness of the Witt design $S(5, 6, 12)$ [8], Pless's characterization of the extended ternary Golay code G_{12} [6] and some properties of $S(5, 6, 12)$ and G_{12}.

It is possible to avoid all the above results regarding the Witt design $S(5, 6, 12)$ and the extended ternary Golay code G_{12}, making the (total) proof more elementary, self-contained and also shorter. We can achieve this by first extending some results of Shult and Yanushka [7] (see Step 2 below) and subsequently applying the results described in De Bruyn and De Clerck [4, Section 8]. This alternative approach to the classification was only observed recently by the author.

The aim of this survey paper is to bring this alternative approach to the attention of the mathematical community. We indeed give a survey of those parts of the papers [1], [4], [7] which are relevant for this alternative approach. The alternative proof for Theorem 1.1 which arises by combining these parts then consists of the following three steps:

Step 1. In the first step, it is proved that any three distinct lines through a given point are contained in a unique cube. As mentioned above, this result is due to Brouwer [1]. Since we give complete proofs for the results involved in the two other steps (for the reasons mentioned there), we have opted to include also a proof here, see Section 2.2.

Step 2. In the second step, it is proved that every two distinct cubes C_1 and C_2 which contain a pair $\{x, y\}$ of opposite points have a unique third point in common. This was proved by Shult and Yanushka [7, pp. 34–39] by means of a lengthy case by case analysis. We will give an alternative shorter argument in the proof of Lemma 2.3. Shult and Yanushka [7] also proved that S admits a point-regular elementary abelian 3-group of automorphisms. This result will also be discussed in Section 2.3. Besides the two above-mentioned results of [7], we need a few other facts in order to be able to start with Step 3. The fact that $\mathcal{S}$ admits a point-regular elementary abelian 3-group of automorphisms implies that the point set of $\mathcal{S}$ can be given the structure of an affine space $\mathcal{A} \cong \mathrm{AG}(6,3)$. We prove that the line set of $\mathcal{S}$ can be partitioned into 12 spreads and that each such spread corresponds to a parallel class of lines of the affine space $\mathcal{A}$. These two facts are easily proved by relying on [7]. (A *spread* in this context is a set of lines of $\mathcal{S}$ partitioning the point set.)

Step 3. By Step 2, $\mathcal{S} \cong T_5^*(\mathcal{K})$ for some set $\mathcal{K}$ of points of Π_∞. In Section 2.4, some properties of $\mathcal{K}$ are derived which allow us to prove that $\mathcal{K}$ is projectively equivalent with $\mathcal{K}^*$. This proves that $\mathcal{S} \cong T_5^*(\mathcal{K}^*)$. The third step of the proof is based on De Bruyn and De Clerck [4, Section 8]. We will however give a shorter and more elegant version of the original argument.

The present paper can also be seen as an addendum to Section 6.5 of [3] where the uniqueness of the near hexagon was not proved (due to the large amount of work which then seemed to be necessary to give a complete self-contained proof based on the results of [6] and [8]).

2 The alternative approach

In this section, $\mathcal{S}$ denotes a regular near hexagon with parameters $(s,t,t_2) = (2,11,1)$. If x_1 and x_2 are two points of $\mathcal{S}$, then $\mathrm{d}(x_1,x_2)$ denotes the distance between x_1 and x_2. Suppose x is a point, X a set of points and $i \in \mathbb{N}$. Then $\Gamma_i(x)$ denotes the set of points at distance i from x, $\mathrm{d}(x,X) := \min\{\mathrm{d}(x,y) \,|\, y \in X\}$ and $\Gamma_i(X)$ denotes the set of points at distance i from X.

2.1 Basic properties of $\mathcal{S}$

The following properties of $\mathcal{S}$ immediately follow from the structure theory of general near polygons (see e.g. [3, Sections 1.7 and 1.8], [7, Section 2.2]).

(A) Every two points x_1 and x_2 at distance 2 are contained in a unique convex subspace $\langle x_1, x_2\rangle$ of diameter 2 which has the structure of a (3×3)-subgrid. These convex subspaces are called *quads*. Every two intersecting lines L_1 and L_2 are contained in a unique quad $\langle L_1, L_2\rangle$. If x_1 and x_4 are two points at distance 2 from each other and if x_1, x_2, x_3, x_4 is a path of length 3 connecting x_1 and x_4, then $x_2, x_3 \in \langle x_1, x_4\rangle$. (Notice that, since $\mathrm{d}(x_1,x_3) \leq 2$ and $\mathrm{d}(x_1,x_4) = 2$, x_1 has distance 1 from a unique point of $x_3x_4 \setminus \{x_4\}$ which is necessarily contained in $\langle x_1, x_4\rangle$. It follows that x_3 and hence also $x_2 \in \Gamma_1(x_1) \cap \Gamma_1(x_3)$ are contained in $\langle x_1, x_4\rangle$.)

(B) Let Q be a quad and x a point not contained in Q. If $\mathrm{d}(x,Q)=1$, then x is collinear with a unique point $\pi_Q(x)$ of Q and $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_Q(x),y)$ for every $y \in Q$. If $\mathrm{d}(x,Q)=2$, then $\Gamma_2(x)\cap Q$ is an *ovoid* of Q, i.e. a set of three mutually noncollinear points of Q.

(C) If Q is a quad and L is a line of $\mathcal{S}$, then one of the following cases occurs: (i) $L\subseteq Q$; (ii) $|L\cap Q|=1$; (iii) $L\subseteq\Gamma_1(Q)$; (iv) $|L\cap\Gamma_1(Q)|=1$ and $|L\cap\Gamma_2(Q)|=2$; (v) $L\subseteq\Gamma_2(Q)$. If case (iii) occurs, then $\{\pi_Q(x)\,|\,x\in L\}$ is a line of Q. If case (iv) occurs with $L=\{x,x_1,x_2\}$ where $x\in\Gamma_1(Q)$, then $\Gamma_2(x_1)\cap Q$ and $\Gamma_2(x_2)\cap Q$ are the two ovoids of Q through $\pi_Q(x)$. If case (v) occurs, then $\{\Gamma_2(x)\cap Q\,|\,x\in L\}$ is a partition of Q into ovoids.

2.2 The existence of cubes

The aim of this subsection is to prove Lemma 2.2 which guarantees the existence of many cubes. This subsection is only a slight rephrasing of Section 2 of Brouwer [1].

Suppose Q and Q' are two quads such that Q' contains a point at distance 2 from Q. Then applying Property (C) of Section 2.1 to the lines of Q', we see that one of the following cases occurs.

(1) Q' contains a unique point x^* of Q, $\Gamma_1(x^*)\cap Q'\subseteq\Gamma_1(Q')$ and $\Gamma_2(x^*)\cap Q'\subseteq\Gamma_2(Q)$.

(2) There exists a point x^* in Q' such that $\{x^*\}\cup(\Gamma_1(x^*)\cap Q')\subseteq\Gamma_1(Q)$ and $\Gamma_2(x^*)\cap Q'\subseteq\Gamma_2(Q)$.

(3) $Q'\cap\Gamma_1(Q)$ is a line L and $\Gamma_2(Q)\cap Q'=Q'\setminus L$.

(4) $Q'\subseteq\Gamma_2(Q)$.

(5) $Q'\subseteq\Gamma_1(Q)\cup\Gamma_2(Q)$ and $Q'\cap\Gamma_1(Q)$ is a set of $i\in\{1,2,3\}$ mutually noncollinear points of Q'.

Lemma 2.1 *If case* (5) *occurs, then* $i=2$.

Proof. Suppose $i=3$, put $Q'\cap\Gamma_1(Q)=\{x_1,x_2,x_3\}$ and let x be a point collinear with x_1 and x_2. Now, $\{\pi_Q(x_1),\pi_Q(x_2),\pi_Q(x_3)\}$ is an ovoid of Q such that $\pi_Q(x_1)$ and $\pi_Q(x_2)$ are contained in $\Gamma_2(x)\cap Q$. Hence, also $\pi_Q(x_3)\in\Gamma_2(x)\cap Q$ and x is collinear with x_3, a contradiction.

Suppose $i=1$. Let x_1,x_2,x_3,x_4,x_1 be a closed path in Q' which determines an ordinary quadrangle and such that the third point x on the line x_1x_2 belongs to $\Gamma_1(Q)$. By (C-v), each of the three pairs $\{\Gamma_2(x_2)\cap Q,\Gamma_2(x_3)\cap Q\}$, $\{\Gamma_2(x_3)\cap Q,\Gamma_2(x_4)\cap Q\}$, $\{\Gamma_2(x_4)\cap Q,\Gamma_2(x_1)\cap Q\}$ consist of two disjoint ovoids of Q. Hence, the ovoids $\Gamma_2(x_1)\cap Q$ and $\Gamma_2(x_2)\cap Q$ of Q must be equal or disjoint. But this is impossible, since by (C-iv), $\Gamma_2(x_1)\cap Q$ and $\Gamma_2(x_2)\cap Q$ are the two ovoids of Q through $\pi_Q(x)$. ∎

For every point x of $\Gamma_2(Q)$ and every $i \in \{1,2,3,4\}$, let $n_i(x)$ denote the number of quads Q' through x for which case (i) above occurs. For every point x of $\Gamma_2(Q)$ and every $i \in \{5,6,7\}$, let $n_i(x)$ denote the number of quads Q' through x for which case (5) occurs with $|\Gamma_1(x) \cap Q' \cap \Gamma_1(Q)| = i - 5$. By counting, we obtain

$$\begin{cases} n_1(x) = 3; \\ n_1(x) + n_2(x) + n_3(x) + n_4(x) + n_5(x) + n_6(x) + n_7(x) = 66; \\ 2n_2(x) + n_3(x) = 12; \\ n_4(x) + n_5(x) = 15; \\ n_1(x) + n_2(x) + n_7(x) = 15; \\ 2n_1(x) + 3n_2(x) + 2n_3(x) + 2n_5(x) + n_6(x) = 66. \end{cases}$$

The first equation expresses that every quad of Type (1) through x is equal to $\langle x, y \rangle$ where y is one of the 3 points of $\Gamma_2(x) \cap Q$. The second equation counts the total number of quads through x. The third equation counts the number of pairs (y, L) where $L \subseteq \Gamma_1(Q)$ and $y \in \Gamma_1(x) \cap L$. The fourth equation counts unordered pairs of lines through x which are contained in $\Gamma_2(Q)$. The fifth equation counts unordered pairs of lines through x which meet $\Gamma_1(Q)$. The sixth equation counts the number of points of $\Gamma_1(Q) \cap \Gamma_2(x)$. There are 66 such points, 6 which are collinear with some point of $\Gamma_2(x) \cap Q$ and 60 which are collinear with some point of $\Gamma_3(x) \cap Q$.

By the above equations, we have $n_5(x) = 6 - \frac{1}{2}n_2(x)$ and $n_6(x) = 24 + 2n_2(x)$. Now, let $\mathcal{Q}$ denote the set of quads Q' which have Type (5) with respect to Q. By standard double counting, we have $|\mathcal{Q}| = \sum_{x \in \Gamma_2(Q)} n_5(x) = \sum_{x \in \Gamma_2(Q)} (6 - \frac{1}{2}n_2(x))$ and $4 \cdot |\mathcal{Q}| = \sum_{x \in \Gamma_2(Q)} n_6(x) = \sum_{x \in \Gamma_2(Q)} (24 + 2n_2(x))$. Hence, $\sum_{x \in \Gamma_2(x)} n_2(x) = 0$, i.e. $n_2(x) = 0$ for every $x \in \Gamma_2(Q)$.

Lemma 2.2 (1) *There exists a unique cube through any three distinct lines L_1, L_2 and L_3 which meet in a point.*

(2) *Let x and y be two points at distance 3 from each other and let L be an arbitrary line through y. Then $L \cup \{x\}$ is contained in a unique cube.*

Proof. (1) Let Q denote the unique quad through L_1 and L_2. Let R_i, $i \in \{1,2\}$, denote the unique quad through L_3 and L_i, let u be an arbitrary point of $L_3 \setminus (L_1 \cap L_2)$, let M_i, $i \in \{1,2\}$, denote the unique line of R_i through u distinct from L_3 and let Q' denote the unique quad through M_1 and M_2. Since $L_1 \cap L_2$ is not contained in Q', Q' is disjoint from Q. Since $n_2(x) = 0$ for every point $x \in \Gamma_2(Q)$, $Q' \subseteq \Gamma_1(Q)$. Every point x of Q is contained in a unique line which meets Q' in a point x'. We denote by x'' the unique point on that line distinct from x and x'. Every line L of Q is contained in a unique quad which meets Q' in a line L'. We denote by L'' the unique line of that quad disjoint from $L \cup L'$. Clearly, the points x''

$(x \in Q)$ and the lines L'' (L a line of Q) determine a subgrid. So, the set $Q'' := \{x'' \,|\, x \in Q\}$ is a quad. It is now clear that $\{x, x', x'' \,|\, x \in Q\}$ is a cube and that it is the unique cube containing the set $L_1 \cup L_2 \cup L_3$.

(2) Let z denote the unique point of L at distance 2 from x and let L' and L'' denote the two lines through z contained in the quad $\langle z, x \rangle$. The cubes through $\{x\} \cup L$ are precisely the cubes through $L \cup L' \cup L''$. There is only one such cube by (1). ∎

2.3 The existence of an affine space on the point set of $\mathcal{S}$

If x_1 and x_2 are two points of $\mathcal{S}$ at distance 3 from each other, then by Lemma 2.2(2), x_1 and x_2 are contained in precisely 4 cubes. On pp. 35–39 of Shult and Yanushka [7], it was proved that these 4 cubes have besides x_1 and x_2 a third point in common, and that this third point necessarily is the unique point in each of these cubes at distance 3 from x_1 and x_2. We now give an alternative shorter argument for these claims.

Lemma 2.3 *Let x_1 and x_2 be two points of $\mathcal{S}$ at distance 3 from each other, let C_1 and C_2 be two cubes through x_1 and x_2, and let x_3 be the unique point of C_1 at distance 3 from x_1 and x_2. Then $C_1 \cap C_2 = \{x_1, x_2, x_3\}$.*

Proof. Let $i \in \{1, 2\}$. If u were a point of $C_1 \cap C_2 \setminus \{x_i\}$ at distance at most 2 from x_i, then any line through x_i contained in $\langle x_i, u \rangle$ would be contained in $C_1 \cap C_2$, clearly a contradiction. Hence, $C_1 \cap C_2 \subseteq \{x_1, x_2, x_3\}$. So, we still must prove that $x_3 \in C_2$.

Let $y_1 \in \Gamma_1(C_1)$, let $x \in \Gamma_1(y_1) \cap C_1$ and let $L = xy_1 = \{x, y_1, y_2\}$. The collinearity relation of $\mathcal{S}$ induces a $(2\times2\times2)$-cube on the set $Z := \Gamma_3(x) \cap C_1$. We prove: (i) $\Gamma_1(y_1) \cap C_1 = \{x\}$; (ii) $\Gamma_2(y_1) \cap Z$ and $\Gamma_2(y_2) \cap Z$ partition the set Z in two subsets, each one of which consists of 4 points at mutual distance 2. Let $i \in \{1, 2\}$. Considering a quad through x and y, we see that $\mathrm{d}(y_i, y) = 1 + \mathrm{d}(x, y)$ for every $y \in C_1$ with $\mathrm{d}(x, y) \leq 2$. If z_1 and z_2 are two collinear points of Z and $z_3 \in C_1 \cap \Gamma_2(x)$ denotes the third point on the line z_1z_2, then since $\mathrm{d}(y_i, z_3) = 3$, precisely one of $\{z_1, z_2\}$ lies at distance 2 from y_i and the other lies at distance 3 from y_i. If z_j, $j \in \{1, 2\}$, would lie at distance 2 from both y_1 and y_2, then z_j would be collinear with x, a contradiction. So, precisely one of z_1, z_2 has distance 2 from y_1 and the other has distance 2 from y_2. The above claims (i) and (ii) follow.

In the sequel, $L = \{x_1, y_2, y_3\}$ denotes an arbitrary line of C_2 through x_1. We suppose that $y_2 \in \Gamma_2(x_2)$. There are 3 quads through L meeting C_1 in a line. Let $\mathcal{L}_i$, $i \in \{2, 3\}$, denote the set of 8 lines through y_i which are not contained in any of these quads. There are 4 quads through y_i which contain a point of $\Gamma_2(y_i) \cap \Gamma_3(x_1) \cap C_1$. If K is a line contained in one of these 4 quads, then since K contains a point w collinear with a point of $Z_1 := \Gamma_3(x_1) \cap C_1$, w cannot be collinear with a point of $C_1 \cap \Gamma_1(x_1)$, implying that $K \subseteq \mathcal{L}_i$. We prove that these 4 quads determine a partition of $\mathcal{L}_i$. If this were not the case, then there exists a line $M \in \mathcal{L}_i$ which is contained in two of these quads, say Q_1 and Q_2. Let u'_j, $j \in \{1,2\}$, be the unique point of $Q_j \cap Z_1$, and let u_j denote the unique point of M collinear with u'_j. Now, $\mathrm{d}(u'_1, u'_2) = 2$ and either u'_1, u_1, u'_2 (if $u_1 = u_2$) or u'_1, u_1, u_2, u'_2 (if $u_1 \neq u_2$) is a path connecting u'_1 and u'_2. This implies that all the points of this path belong to the convex subspace $\langle u'_1, u'_2 \rangle \subseteq C_1$, which is clearly impossible.

In order to prove that $x_3 \in C_2$, it suffices to prove that x_3 lies at distance 1 from any (of the two) line(s) $K_3 \subseteq C_2$ through y_3 distinct from L. Consider the quad $R = \langle L, K_3 \rangle$ and let $K_2 \subseteq \langle y_2, x_2 \rangle$ denote the line of R through y_2 distinct from L. Since $K_3 \in \mathcal{L}_3$, K_3 is contained in a quad R_3 which meets Z_1 in a point u'_3. Let u_3 denote the unique point of K_3 collinear with u'_3. Let u_2 denote the unique point of K_2 collinear with x_2. It suffices to prove that $u'_3 = x_3$. Suppose this is not the case. Then either $\mathrm{d}(u'_3, x_2) = 1$ and $\mathrm{d}(u_2, u_3) = 2$ or $\mathrm{d}(u'_3, x_2) = 1$ and $\mathrm{d}(u_2, u_3) = 1$. The former case is impossible since otherwise u_2, x_2, u'_3, u_3 would be a path which is entirely contained in $\langle u_2, u_3 \rangle \subseteq R$. The latter case is also impossible since otherwise the unique third point $u_1 \notin C_1$ on the line u_2u_3 would be collinear with two distinct points of C_1, namely the point x_1 and the unique third point on the line $x_2u'_3$ (which lies at distance 2 from x_1). ■

Definition. Let x^* be a given point of $\mathcal{S}$. If L is a line through x^*, then we define $\theta(L) := L$. If L is a line not containing x^*, then x^* and L are contained in a unique subspace F which is a quad if $\mathrm{d}(x^*, L) = 1$ and a cube if $\mathrm{d}(x^*, L) = 2$. We denote by $\theta(L)$ the unique line of F through x^* parallel with L. Let $L_1, L_2, \ldots, L_{12}$ denote the 12 lines through x^*. For every $i \in \{1, \ldots, 12\}$, let S_i denote the set of all lines L for which $\theta(L) = L_i$. The following is obvious from Lemma 2.2.

Lemma 2.4 *$S_1, S_2, \ldots, S_{12}$ are spreads of $\mathcal{S}$ which determine a partition of the line set of $\mathcal{S}$.*

Definition. For every $i \in \{1, 2, \ldots, 12\}$, let x_i^* denote a given point of $L_i \setminus \{x^*\}$. For every point x of $\mathcal{S}$, we now define a permutation t_x of the point set $\mathcal{P}$ of $\mathcal{S}$. Let $y \in \mathcal{P}$ and let F denote a subspace of diameter $\mathrm{d}(x, y)$ through x and y which is either a point, a line, a quad or a cube. Then $t_x(y)$ denotes the unique point of F at distance $\mathrm{d}(x, y)$ from both x and y. If $\mathrm{d}(x, y) = 3$, then by Lemma 2.3, $t_x(y)$ is independent from the chosen cube F through x and y.

With exception of the claim that ϕ_i, $i \in \{1, \dots, 12\}$, stabilizes each line of S_i, the contents of the following lemma were already proved in Shult and Yanushka [7].

Lemma 2.5 (1) *For every point x of $\mathcal{S}$, t_x is an automorphism of $\mathcal{S}$.*

(2) *For every $i \in \{1, \dots, 12\}$, $\phi_i := t_{x_i^*} \circ t_{x^*}$ is an automorphism of $\mathcal{S}$ without fix points stabilizing each line of the spread S_i.*

(3) *For all $i, j \in \{1, \dots, 12\}$, ϕ_i and ϕ_j commute.*

(4) *The group $G := \langle \phi_i \,|\, i \in \{1, \dots, 12\} \rangle$ is an elementary abelian 3-group of acting regularly on $\mathcal{P}$. Hence, $|G| = 3^6$.*

Proof. (1) We need to prove that $t_x(L)$ is a line of $\mathcal{S}$ for every line L of $\mathcal{S}$. Let F denote the subspace of diameter $i := \mathrm{d}(x, L) + 1$ through $\{x\} \cup L$ which is either a line, a quad or a cube. Let V be an i-dimensional vector space over $\mathbb{F}_3$ with basis $\{\bar{e}_1, \dots, \bar{e}_i\}$. We can label the points of F with the vectors of V such that: (i) x has label $\bar{o}$; (ii) two points of F are collinear if and only if the difference of their labels has weight 1. The restriction of t_x to F corresponds to the map $\bar{x} \mapsto -\bar{x}$ and hence must preserve the collinearity of points in F. Hence, $t_x(L)$ is a line of $\mathcal{S}$.

(2) By part (1), ϕ_i is an automorphism of $\mathcal{S}$. Now, let L denote an arbitrary line of S_i and let F denote the subspace of diameter $j := \mathrm{d}(x^*, L) + 1$ through $\{x^*\} \cup L$ which is either a line, a quad or a cube. Let V be a j-dimensional vector space over $\mathbb{F}_3$ with basis $\{\bar{e}_1, \dots, \bar{e}_j\}$. We can label the points of F with the vectors of V such that: (i) x^* has label $\bar{o}$; (ii) x_i^* has label $\bar{e}_1$; (iii) two points of F are collinear if and only if the difference of their labels has weight 1. Then t_{x^*} corresponds to the map $\bar{x} \mapsto -\bar{x}$ and $t_{x_i^*}$ corresponds to the map $\bar{x} \mapsto -\bar{x} - \bar{e}_1$. Hence, ϕ_i corresponds with the map $\bar{x} \mapsto \bar{x} - \bar{e}_1$. It is now obvious that ϕ_i stabilizes L (without fixpoints).

(3) This holds if $i = j$. So, suppose $i \neq j$ and let k be an arbitrary element of $\{1, 2, \dots, 12\} \setminus \{i, j\}$. Let C denote the unique cube through L_i, L_j and L_k. The points of C can be labeled with the vectors of a 3-dimensional vector space V over $\mathbb{F}_3$ with basis $\{\bar{e}_i, \bar{e}_j, \bar{e}_k\}$ such that: (i) x^*, x_i^*, x_j^* and x_k^* have respective labels $\bar{o}$, $\bar{e}_i$, $\bar{e}_j$ and $\bar{e}_k$; (ii) two points of C are collinear if and only if the difference of their labels has weight 1. Then the restriction of ϕ_i (respectively ϕ_j) to the point set of C corresponds to the map $\bar{x} \mapsto \bar{x} - \bar{e}_i$ (respectively $\bar{x} \mapsto \bar{x} - \bar{e}_j$). Hence, the commutator $[\phi_i, \phi_j]$ induces the identity on C. Since k was an arbitrary element of $\{1, \dots, 12\} \setminus \{i, j\}$, $[\phi_i, \phi_j]$ fixes every point at distance at most 1 from x^*. This also implies that $[\phi_i, \phi_j]$ fixes every point y of $\Gamma_2(x^*)$ and every point z of $\Gamma_3(x^*)$ since such points y and z are uniquely determined by the sets $\Gamma_1(y) \cap \Gamma_1(x^*)$ and $\Gamma_1(z) \cap \Gamma_2(x^*)$.

(4) Since $\langle \phi_i \rangle$, $i \in \{1, \dots, 12\}$, is an cyclic group of order 3, G is an elementary abelian 3-group by part (3). By part (2), $\langle \phi_i \rangle$, $i \in \{1, \dots, 12\}$,

acts regularly on each line of S_i. So, by the connectedness of $\mathcal{S}$, G acts transitively on $\mathcal{P}$. Together with the fact that G is abelian, this implies that G acts regularly on $\mathcal{P}$. ■

Lemma 2.6 *The set $\mathcal{P}$ can be given the structure of an affine space $\mathcal{A} \cong \mathrm{AG}(6,3)$ such that every spread S_i, $i \in \{1,\ldots,12\}$, corresponds to a parallel class of lines of $\mathcal{A}$.*

Proof. Let V be a vector space of dimension 6 over $\mathbb{F}_3$ and let $\bar{x} \mapsto \phi_{\bar{x}}$ be an isomorphism between the additive group of V and G. Label the point $p \in \mathcal{P}$ with the vector $\bar{x}$ if $\phi_{\bar{x}}(x^*) = p$. In this way, $\mathcal{P}$ gets the structure of an affine space $\mathcal{A} \cong \mathrm{AG}(6,3)$. Now, let $i \in \{1,\ldots,12\}$. Let $\bar{u}_i \in V$ such that $\phi_{\bar{u}_i} = \phi_i$. Since $L_i = \{x^*, \phi_i(x^*), \phi_i^2(x^*)\}$, the points of L_i are labeled with the vectors $\bar{o}$, $\bar{u}_i$ and $-\bar{u}_i$. Now, let L be an arbitrary line of S_i. If $p \in L$, then $L = \{p, \phi_i(p), \phi_i^2(p)\}$ by Lemma 2.5(2). If $\bar{v} \in V$ such that $p = \phi_{\bar{v}}(x^*)$, then since $L = \{p, \phi_{\bar{u}_i}(p), \phi_{\bar{u}_i}^2(p)\} = \{\phi_{\bar{v}}(x^*), \phi_{\bar{v}+\bar{u}_i}(x^*), \phi_{\bar{v}-\bar{u}_i}(x^*)\}$, the labels of the points of L are $\bar{v}$, $\bar{v}+\bar{u}_i$ and $\bar{v}-\bar{u}_i$. This proves that the elements of S_i correspond to parallel lines of $\mathcal{A}$. ■

2.4 The isomorphism between $\mathcal{S}$ and $T_5^*(\mathcal{K}^*)$

As in the introduction, let Π_∞ be a hyperplane of $\mathrm{PG}(6,3)$. By Lemmas 2.4 and 2.6, there exists a set $\mathcal{K}$ of 12 points of Π_∞ such that $\mathcal{S} \cong T_5^*(\mathcal{K})$. Let V be a 6-dimensional vector space over $\mathbb{F}_3$ such that $\Pi_\infty = \mathrm{PG}(V)$. If $X \cup \{x\}$ is a set of points of Π_∞ such that $x \in \langle X \rangle$, then $i_X(x)$ denotes the smallest size of a subset $Y \subseteq X$ satisfying $x \in \langle Y \rangle$.

Lemma 2.7 *Let X be a set of 12 points of Π_∞ satisfying: (a) X generates Π_∞; (b) there are no 4 points of X which are contained in a plane; (c) if $\{x_1,x_2,x_3,x_4\}$ is a line of Π_∞ such that $x_1 \in X$ and $i_X(x_3) = i_X(x_4) = 2$, then $x_2 \in X$. Then X is projectively equivalent with $\mathcal{K}^*$.*

Proof. (1) We prove that there are no $i \in \{2,3,4,5\}$ distinct points of X which are contained in some $(i-2)$-dimensional subspace. By (b) this holds of $i \in \{2,3,4\}$. Suppose now that x_1,x_2,x_3,x_4,x_5 are 5 points which are contained in some 3-dimensional subspace. By (b), we can choose a basis $\{\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_6\}$ of V such that $x_i = \langle \bar{e}_i \rangle$, $i \in \{1,2,3,4\}$, and $x_5 = \langle \bar{e}_1+\bar{e}_2+\bar{e}_3+\bar{e}_4 \rangle$. By (b), the line through $\langle \bar{e}_1 \rangle$ and $\langle \bar{e}_2+\bar{e}_3 \rangle$ contains a unique point of X, namely $\langle \bar{e}_1 \rangle$. But this is impossible by (c) since the i_X-values of $\langle \bar{e}_2+\bar{e}_3 \rangle \in x_2x_3$ and $\langle \bar{e}_1+\bar{e}_2+\bar{e}_3 \rangle \in x_4x_5$ are equal to 2.

(2) By (a), we can choose a basis $\{\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_6\}$ of V such that $\langle \bar{e}_i \rangle \in X$ for every $i \in \{1,\ldots,6\}$. Suppose $p = \langle \epsilon_1\bar{e}_1+\epsilon_2\bar{e}_2+\cdots+\epsilon_6\bar{e}_6 \rangle$ and $p' = \langle \epsilon_1'\bar{e}_1+\epsilon_2'\bar{e}_2+\cdots+\epsilon_6'\bar{e}_6 \rangle$ are two other points of X. By (1), $\epsilon_i = 0$ for at most one $i \in \{1,\ldots,6\}$. Similarly, $\epsilon_i' = 0$ for at most one $i \in \{1,\ldots,6\}$. Let I_0,

I_+, respectively I_-, denote the set of all $i \in \{1, \dots, 6\}$ for which $\epsilon_i \cdot \epsilon'_i = 0$, $\epsilon_i \cdot \epsilon'_i = +1$, respectively $\epsilon_i \cdot \epsilon'_i = -1$. Then $|I_0| \le 2$ and $|I_0| + |I_+| + |I_-| = 6$. Putting $\{I_1, I_2\} = \{I_+, I_-\}$ such that $|I_1| \le |I_2|$, we find $|I_1| \le \frac{6-|I_0|}{2}$. Now, the points p, p' and $\langle \bar{e}_i \rangle$, $i \in I_0 \cup I_1$, are contained in some $(|I_0| + |I_1|)$-dimensional subspace. So, we have $6 \le 2 + |I_1| + |I_0| \le 2 + \frac{6+|I_0|}{2} \le 6$, i.e. $|I_0| = |I_1| = |I_2| = 2$. This implies that $\epsilon_0 \cdot \epsilon'_0 + \epsilon_1 \cdot \epsilon'_1 + \cdots + \epsilon_6 \cdot \epsilon'_6 = 0$, i.e. p and p' are orthogonal.

(3) By (2), the remaining 6 points of X have coordinates of the form $c_1 = (0, *, *, *, *, *)$, $c_2 = (*, 0, *, *, *, *)$, $c_3 = (*, *, 0, *, *, *)$, $c_4 = (*, *, *, 0, *, *)$, $c_5 = (*, *, *, *, 0, *)$ and $c_6 = (*, *, *, *, *, 0)$. Using the remaining freedom in coordinatization, we may suppose that $c_1 = (0, -1, -1, -1, -1, -1)$. Since c_1 and c_2 are orthogonal (and using the remaining freedom in coördinatization), we may suppose that $c_2 = (1, 0, 1, -1, -1, 1)$. Since c_1, c_2 and c_3 are mutually orthogonal (and using the remaining freedom in coordinatization), we may suppose that $c_3 = (1, 1, 0, 1, -1, -1)$. The remaining three points are now completely determined by the orthogonality of $c_1, \dots, c_6$. We have $c_4 = (1, -1, 1, 0, 1, -1)$, $c_5 = (1, -1, -1, 1, 0, 1)$ and $c_6 = (1, 1, -1, -1, 1, 0)$. It is now clear that X is projectively equivalent with $\mathcal{K}^*$. ∎

Lemma 2.8 *The set $\mathcal{K}$ satisfies the conditions (a), (b) and (c) of Lemma 2.7 and hence $\mathcal{K}$ and $\mathcal{K}^*$ are projectively equivalent. As a consequence, $\mathcal{S} \cong T_5^*(\mathcal{K}^*)$.*

Proof. (a) Let z be an arbitrary point of Π_∞ and let x, y be two points of $PG(6,3) \setminus \Pi_\infty$ such that $z \in xy$. Let $x = x_0, x_1, \dots, x_k = y$ be a shortest path of length $k := d(x, y)$ between x and y (in the connected geometry $T_5^*(\mathcal{K}) \cong \mathcal{S}$). Then $\{z\} = x_0x_k \cap \Pi_\infty$ is contained in $\langle x_0x_1 \cap \Pi_\infty, x_1x_2 \cap \Pi_\infty, \dots, x_{k-1}x_k \cap \Pi_\infty \rangle = \langle x_0, x_1, \dots, x_k \rangle \cap \Pi_\infty$. Hence, $\langle \mathcal{K} \rangle = \Pi_\infty$.

(b) Suppose there are 4 points of $\mathcal{K}$ which are contained in some plane. Then there are 4 points $x_1, x_2, x_3, x_4 \in \mathcal{K}$ such that $\langle x_1, x_2, x_3 \rangle$ is a plane and $x_4 \in \langle x_1, x_2, x_3 \rangle$. Now, let α be a 3-dimensional subspace of $PG(6,3)$ which intersects Π_∞ in $\langle x_1, x_2, x_3 \rangle$. The 27 points of $\alpha \setminus \Pi_\infty$ together with the 27 lines of α which intersect Π_∞ in either x_1, x_2 or x_3 determine a $(3 \times 3 \times 3)$-cube. Hence, $\alpha \setminus \Pi_\infty$ must be a cube. So, there are only 27 lines of $T_5^*(\mathcal{K}) \cong \mathcal{S}$ which are contained in $\alpha \setminus \Pi_\infty$. This is however impossible since any line of α which intersects Π_∞ in x_4 must also be such a line.

(c) Let L be a line of $PG(6,3)$ which intersects Π_∞ in the point x_1 and let $x \in \langle L, x_2 \rangle \setminus (L \cup x_1x_2)$. Let $y \in \{x_3, x_4\}$, let $y_1y_2 \in \mathcal{K}$ such that $y \in y_1y_2$ and let z denote the unique point in $xy \cap L$. Since $y_2z \cap y_1x$ is a common neighbour of x and z, $d(x, z) = 2$. So, the point x of $T_5^*(\mathcal{K})$ has distance 2 from the points $xx_3 \cap L$ and $xx_4 \cap L$ of L. Hence, x has distance 1 from the third point $xx_2 \cap L$ of L. This implies that $x_2 \in \mathcal{K}$. ∎

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